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B.V.Bykovskij, V.I.Inozemtsev

ELLIPTIC SOLUTIONS OF EQUATIONS OF MOTION OF THE INTERACTING PARTICLES IN AN EXTERNAL FIELD

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The problem of explicit construction of solutions to equations of motion of classical Hamilton system with N degrees of freedom and N independent involutary constant of motion can, in principle, be solved with the use of the Liouville theorem /1/. However, in practice, the Liouville reduction allows the solution of rather simple problems, and the integration of equations of motion usually requires the search for "roundabout" ways, for instance, the construction of extra constants of motion on the basis of a certain dynamic symmetry. This also concerns the class of integrable systems /2,5/ describing the motion of particles interacting in pairs through the potential $V(\xi)$ in an external field $W(\xi)$. In the particular case $W(\xi)=0$ considered in detail in /2/ effective methods have been found for integration of the equations of motion for all admissible potentials $V(\xi)$

$$V(\xi) = g^2/\xi^2 \tag{1a}$$

$$V(\xi) = g^2/\mathrm{sh}^2\xi \tag{1b}$$

$$V(\xi) = g^2\wp(\xi)\,. \tag{1c}$$

where $\wp(\xi)$ is the Weierstrassion elliptic function. In cases (1a, b) the problem was solved by several methods /2/, whereas for the potential (1c) the problem turned out to be nontrivial, and it was solved by introducing a spectral parameter into the Lax representation and further using the methods of algebraic geometry /4/.

Note that in searching for the methods of integration of equations of motion of particles with interaction of type (1) allowed the authors of /3/ to connect their trajectories with motion of poles of singular solutions of nonlinear evolution equations. This connection was used in ref. /6/ for integrating equations of motion of particles with interaction (1a,b) in the nontrivial external field

$$W(\xi) = \gamma_1\xi^4 + \gamma_2\xi^2 + \gamma_3\xi \tag{2a}$$

$$W(\xi) = \gamma_1\,\mathrm{ch}\,4\xi + \gamma_2\,\mathrm{sh}\,2\xi + \gamma_3\,\mathrm{ch}\,2\xi\,. \tag{2b}$$

However, solutions have been found only under certain initial conditions and constraints on the interaction parameters, the most important of which are $\gamma_1 < 0$ and $g^2 < 0$.

Probably, the best way for constructing a general solution is the introduction of a spectral parameter into the Lax representations for systems with interaction (1a,b -2) and application of the methods of algebraic geometry like those developed in /4/. At present it is yet unknown how to realize even the first part of the program.

In this note, we shall consider the most simple case of motion of two particles interacting through potential (1a) and being in the field of a symmetric anharmonic oscillator,

$$W(\xi) = \frac{1}{2}\left(A\xi^2 + B\right)^2. \tag{3}$$

We shall show that for arbitrary values of the parameters A, B and g^2 there exists a class of solutions of equations of motion os systems of that type containing two arbitrary constants and depending on time via the Jacobi elliptic functions: it is not of necessity to introduce the restrictions $\gamma_1 < 0$ and $g^2 < 0$ indispensable for the method of ref. /6/.

In what follows, we shall assume for convenience that (without loos of generality in virtue of (3)). The Hamiltonian of the considered system has the form

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{g^2}{(x_1 - x_2)^2} + \frac{1}{2}\left[(Ax_1^2 + B)^2 + (Ax_2^2 + B)^2\right]. \tag{4}$$

Here p_i and x_i are momenta and coordinates of particles. By using the Lax representation for system (4) it is not difficult to write the second constant of motion:

$$I^2 = |\xi_1 + i\xi_2|^2, \tag{5}$$

where

$$\xi_1 = p_1 p_2 - (Ax_1^2 + B)(Ax_2^2 + B) - \frac{g^2}{(x_1 - x_2)^2}$$

$$\xi_2 = p_1(Ax_2 + B) + p_2(Ax_1 + B).$$

From (5) it follows that

$$\frac{d}{dt}(\xi_1 + i\xi_2) = iS(\xi_1 + i\xi_2) \tag{6}$$

the factor S being real. By a direct calculation it can be easily verified that $S = 2A(x_1 + x_2)$; setting $\xi_1 + i\xi_2 = Ie^{i\theta}$ we may express the coordinate of the centre of mass of particles in terms of the derivative of angle θ with respect to time:

$$x_1 + x_2 = \frac{\dot{\theta}}{2A} \tag{7}$$

(hereafter $\dot{u} \equiv \frac{du}{dt}$). Using also the equation of motion

$\frac{d}{dt}(p_1 + p_2) = -2A[x_1(Ax_1^2 + B) + x_2(Ax_2^2 + B)]$ an the relation following from (4,5)

$$2(H + I\cos\theta) = \frac{\ddot{\theta}^2}{4A^2} + \frac{\dot{\theta}^2}{4}(x_1 - x_2)^2 \tag{8}$$

we arrive at the equation only for the angle θ

$$2\dot{\theta}\dddot{\theta} - 3\ddot{\theta}^2 + 4AB\dot{\theta}^2 + \frac{\dot{\theta}^4}{4} + 24A^2(H + I\cos\theta) = 0. \tag{9}$$

Note that all solutions of equations of motion may be constructed from solutions to eq. (9), but the inverse statement is not valid. So, eq. (9) is not a result of the reduction of equations of motion on the basis of the known integrals (4) and (5). Nevertheless, as it will be shown below, with the help of (9) it is possible to find particular solutions to these equations corresponding to a certain choice of initial conditions.

We lower the order of eq. (9) by the substitution $\dot{\theta}^2 = y(\theta)$ which gives the following equation for y

$$y\frac{d^2y}{d\theta^2} - \frac{3}{4}\left(\frac{dy}{d\theta}\right)^2 + 4ABy + \frac{y^2}{4} = -24A^2(H + I\cos\theta). \tag{10}$$

It may be verified from the right-hand side of (10) that the following solutions

$$y(\theta) = 4A^2(x_1 + x_2)^2 = \dot{\theta}^2 = \lambda + \mu\cos\theta \tag{11}$$

satisfy that equation under the condition that λ and μ obey the nonlinear system of equations

$$(\lambda - 8AB)\mu = 48A^2 I$$
$$(\lambda + 8AB)^2 - 3\mu^2 = 64A^2B^2 - 96A^2H. \tag{12}$$

Thus, the time dependence of the coordinate of the centre of mass of particles can be found by integrating equation (11) and then using (7). From (8) and (12) we may obtain also the relative coordinate as a function of the angle θ .

$$(x_1 - x_2)^2 = \frac{1}{4A^2}\left(-\frac{\lambda + 16AB}{3} + \mu\cos\theta\right). \tag{13}$$

However, as has been noted above, not each of solutions of (9) is associated with a trajectory of the considered system. This may be verified by substituting (13), (7), (11) into the equations of motion for the coordinate of the centre of mass and relative coordinate. The first of them holds valid for all values of the parameters λ and μ . The second is to be written in the form

$$u\ddot{u} - \frac{1}{2}\dot{u}^2 = 128 g^2 A^4 - u^2\left(\frac{3v+u}{4} + 4AB\right), \qquad (14)$$

where the following notation is used:

$$(4A^2)^{-1} u = (x_1-x_2)^2, \quad (4A^2)^{-1} v = (x_1+x_2)^2 = \lambda + \mu\cos\theta. \qquad (15)$$

Equality (14) takes place only in the case when the constants λ and μ are related by the condition

$$\mu^2 = \frac{1}{9}(\lambda+16AB)^2 - \frac{192 g^2 A^4}{\lambda+4AB}. \qquad (16)$$

Note that the sign of parameter μ can always be chosen positive. Since $\mu^2>0$, not all values of the parameter λ are permissible. Condition (16) restricts the set of values of the constants of motion $\{H, I\}$ for which solutions to the equations of motion have the form (11), (13). On the other hand, having constructed solutions in the form (11), (13), we may parametrize them by one constant λ instead of I and H. Note also that, according to (11), (13), the obtained trajectory satisfies the simple relation

$$x_1 x_2 = \frac{1}{12A^2}(\lambda + 4AB). \qquad (17)$$

The second constant specifying the obtained solutions should be found integrating the first-order equations (11).

Now let us determine the interval of admissible values of λ. Apart from the inequality $\frac{1}{9}(\lambda+16AB)^2 > \frac{192 g^2 A^4}{\lambda+4AB}$ following from (16) there should hold the conditions for the right-hand sides of (11) and (13) to be non-negative. For $B \geq 0$ this gives the following constraints on λ:

$$\lambda_1 < \lambda < -4AB, \qquad (18)$$

where λ_1 is a negative root of the equation $(\lambda+4AB)^2(8AB-\lambda) = 216 g^2 A^4$. This root does exist and does not exceed $-4AB$ for any values of the parameters $A, g^2, B>0$ and, consequently, the set of admissible values of $\{\lambda\}$ is not empty (at $B=0$ $\lambda_1 = -6(g^2A^4)^{1/3}$). By substituting $y = tg\,\theta/2$ equation (11) is reduced to the standard form

$$\dot{y}^2 = \frac{\lambda+\mu}{4}(1+y^2)\left(1 + \frac{\lambda-\mu}{\lambda+\mu}y^2\right). \qquad (19)$$

Values of λ from the interval (18) satisfy the condition $\mu > |\lambda|$, which allows us to write the solution to (19):

$$tg\frac{\theta}{2} = \sqrt{\frac{\mu+\lambda}{\mu-\lambda}}\, cn\left(\sqrt{\frac{\mu}{2}}(t-t_0)\,\middle|\,\frac{\mu+\lambda}{2\mu}\right), \qquad (20)$$

where $cn(u|k^2)$ is the Jacobi elliptic cosine with modulud k. Together with condition (18) expression (20) completely determines, in accordance with (11), (13), solutions to the equations of motion dependent on two parameters λ and t_0.

When $B<0$, the value for λ may also vary in the interval (18) that exists for any A, B and g^2. On the condition that

$$\frac{Ag^2}{B} < \frac{32}{243}B^2 \qquad (21)$$

there may exist another type of motion. The parameter λ is also allowed to vary inside the interval $\tilde{\lambda}_1 < \lambda < \tilde{\lambda}_2$, where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are positive roots of the equation

$$\frac{1}{9}(\lambda+16AB)^2(\lambda+4AB) = 192 g^2 A^4$$

not exceeding $-16AB$. According to (17), both the particles are localized in one of two potential wells of the anharmonic oscillator $(A\xi^2+B)^2$; since in this case $\lambda>\mu$, the solution to eq. (19) is of the form

$$tg\frac{\theta}{2} = sc\left(\sqrt{\frac{\lambda+\mu}{2}}(t-t_0)\,\middle|\,\frac{2\mu}{\lambda+\mu}\right),$$

where $sc(u|k^2) = sn(u|k^2)/cn(u|k^2)$. From (21) it is seen that this type of motion occurs only in the case when the characteristic energy of repulsion of particles in the well ($\sim Ag^2/B$) is significantly smaller than the height of the well ($\sim B^2/2$).

Thus, we have found particular solutions of the equations of motion for a two-particle system with Hamiltonian (4) dependent on two arbitrary constants. These solutions exist for all values of the Hamiltonian parameters. In conclusion we notice that a similar procedure allows also the determination of particular solutions for a more general class of Hamiltonians with two degrees of freedom connected with semisimple Lie algebras /7/:

$$H = \frac{p_1^2+p_2^2}{2} + \frac{g^2}{(x_1-x_2)^2} + \frac{g'^2}{(x_1+x_2)^2} + \frac{1}{2}\left[(Ax_1^2+B)^2 + (Ax_2^2+B)^2\right]. \qquad (22)$$

The equations of motion for Hamiltonians (22) may be written in terms of variables v and u (15) as follows:

$$
\begin{aligned}
v\ddot{v}-\tfrac{1}{2}\dot{v}^2 &= 128 g'^2 A^4 - v^2\left(\frac{3u+v}{4}+4AB\right)\\
u\ddot{u}-\tfrac{1}{2}\dot{u}^2 &= 128 g^2 A^4 - u^2\left(\frac{3v+u}{4}+4AB\right).
\end{aligned}
\tag{23}
$$

Setting in (23)

$$
u=d_1+\mu\cos\theta,\quad v=d_2+\mu\cos\theta,\quad \dot{\theta}^2=\lambda+\mu\cos\theta \tag{24}
$$

and making not difficult but rather lengthy computations we verify that (24) is a solution of the equations of motion provided that the parameters λ, μ, d_1 and d_2 are related by

$$
\begin{aligned}
\tfrac{3}{2}(d_1+d_2) &= \lambda-8AB\\
(\mu^2-d_1^2)(d_1-\lambda) &= 256 g^2 A^4\\
(\mu^2-d_2^2)(d_2-\lambda) &= 256 g'^2 A^4 .
\end{aligned}
\tag{25}
$$

Together with the positivity conditions for expressions (24) relations (25) allow, like in the case $g'^2=0$, one to establish the interval of feasible values of λ, one of the free parameters characterizing solutions to equations (23) in form (24). As $g'^2 \to 0$, these solutions pass over to the equations obtained above. The problem of explicit construction of the general solution to eqs. (23) dependent on four arbitrary constants remains still open.

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Быковский Б.В., Иноземцев В.И. E5-86-340

Эллиптические решения уравнений движения двух взаимодействующих частиц во внешнем поле

Рассматривается проблема нахождения аналитических решений уравнений движения вполне интегрируемых систем двух классических частиц на прямой во внешнем поле ангармонического осциллятора. Показано, что для всех значений параметров гамильтониана взаимодействия существуют решения, зависящие от времени посредством эллиптических функций Якоби.

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Bykovskij B.V., Inozemtsev V.I. E5-86-340

Elliptic Solutions of Equations of Motion of the Interacting Particles in an External Field

The problem of finding analytic solutions to the equations of motion of completely integrable systems of two interacting particles on a straightline in an external field of an anharmonic oscillator is considered. It is shown that for all values of parameters of the interaction Hamiltonian there exist solutions dependent on time through the Jacobian elliptic functions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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