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ON JENSEN'S INEQUALITY AND THERMODYNAMICS

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1. A "GEDANKENEXPERIMENT"

In this note we want to study equalizing processes in the model "n bodies with different heat capacities" (see Alberti/Uhlmann[1]) from a more conceptional point of view. For this purpose we consider the total equalization of temperature between two of the n bodies with the following refinement:

Assume the two bodies are realized as physical systems, e.g., as ideal gas, as black radiator, etc. Following an old idea by Sommerfeld[2] and studying $\Delta S \geq 0$ in detail one obtains system-depending algebraic inequalities. For illustrating we give his example:

Let C^1, C^2 resp. T^1, T^2 be the heat capacities and the temperatures of the two bodies. Permitting equalization one gets the mixing temperature:

$$\left. \begin{aligned} T^m = \alpha^1 T^1 + \alpha^2 T^2 \quad \text{with} \quad \alpha^1 &= \frac{C^1}{C^1 + C^2} \\ \text{and} \quad \alpha^2 &= \frac{C^2}{C^1 + C^2} \end{aligned} \right\} \tag{1.1}$$

Choosing now the bodies as ideal gas*) we have

$$\Delta S = C^1 \ln \frac{T^m}{T^1} + C^2 \ln \frac{T^m}{T^2} =$$

$$= (C^1 + C^2) \ln T^m - C^1 \ln T^1 - C^2 \ln T^2 ,$$

from which after exponating it follows:

$$\left. \begin{aligned} &\alpha^1 T^1 + \alpha^2 T^2 \geq (T^1)^{\alpha^1} \cdot (T^2)^{\alpha^2} \\ &\text{with } T^i \geq 0, \ \alpha^i \geq 0, \ \Sigma \alpha^i = 1 \end{aligned} \right\} \tag{1.2}$$

Next we will prove that this procedure is not only intuitively reasonable, but also mathematically correct. We start recalling Jensen's inequality

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*) The volume and other parameters are kept constant.

$$g(\alpha^1 X^1 + \alpha^2 X^2) \geq \alpha^1 g(X^1) + \alpha^2 g(X^2) \qquad (1.3)$$

with

$X^i \geq 0$, $\alpha^i \geq 0$, $\Sigma\alpha^i = 1$

and $g(X)$ concave.

If $S=S(T)$ were a concave function, everything would be clear. But from

$$\frac{\partial S}{\partial T} = \frac{C_v}{T} > 0$$

and

$$\frac{\partial^2 S}{\partial T^2} = T^{-1}\frac{\partial C_v}{\partial T} - T^{-2} C_v \gtrless 0$$

it only follows that $S(T)$ is monotonically increasing. Nevertheless, we have

$$\left.\begin{aligned} \frac{\partial S}{\partial E} &= T^{-1} > 0 \\ &\text{and} \\ \frac{\partial^2 S}{\partial E^2} &= - T^{-2}\cdot\frac{\partial T}{\partial E} < 0 \end{aligned}\right\} \qquad (1.4)$$

Hence, $S=S(E)$ is concave and for systems with $E(T) \propto T$ - the ideal gas belongs to them - the proof is ready. But as one easily checks, Sommerfeld's method also provides correct inequalities for other $E(T)$-dependencies. This results from $\frac{\partial E}{\partial T} > 0$, i.e. $E=E(T)$ is monotone, which guarantees that from $T^1 \gtrless T^m \gtrless T^2$ there always follows $E^1 \gtrless E^m \gtrless E^2$. Taking into account this fact, the concavity of $S(E)$ can be applied. So, we find that the procedure can always be traced back - by means of an additional monotone substitution - to an inequality of the Jensen type. The essential point lies in using S, E, etc., not as arbitrary functions but in the thermodynamic context which guarantees that the necessary properties are fulfilled a priori.

2. THE CONNECTION WITH THE PARTIAL ORDER "MORE C-MIXED THAN"

We continue with (1.3):

$$(C^1 + C^2)g(X^m) \geq C^1 g(X^1) + C^2 g(X^2),$$

calling $X^1 = \frac{q^1}{C^1}$, $X^2 = \frac{q^2}{C^2}$, and $X^m = \frac{q^1+q^2}{C^1+C^2}$

one obtains

$$(C^1 + C^2)g\left(\frac{q^1+q^2}{C^1+C^2}\right) \geq C^1 g\left(\frac{q^1}{C^1}\right) + C^2 g\left(\frac{q^2}{C^2}\right),$$

which gives with $q^{1'} := \alpha^1(q^1 + q^2)$ and $q^{2'} := \alpha^2(q^1 + q^2)$ just:

$$C^1 g\left(\frac{q^{1'}}{C^1}\right) + C^2 g\left(\frac{q^{2'}}{C^2}\right) \geq C^1 g\left(\frac{q^1}{C^1}\right) + C^2 g\left(\frac{q^2}{C^2}\right).$$

Now we add to both sides (n-2) summands:

$$\left.\begin{aligned} &C^1 g\left(\frac{q^{1'}}{C^1}\right) + C^2 g\left(\frac{q^{2'}}{C^2}\right) + \sum_{i=3}^{n} C^i g\left(\frac{q^i}{C^i}\right) \geq \\ &\geq C^1 g\left(\frac{q^1}{C^1}\right) + C^2 g\left(\frac{q^2}{C^2}\right) + \sum_{i=3}^{n} C^i g\left(\frac{q^i}{C^i}\right) \end{aligned}\right\} \qquad (2.1)$$

Denoting the left q^i uniformly by $q^{i'}$ (i.e., $\boldsymbol{q} = (q^1, q^2, q^3 \ldots \ldots, q^n)$ and $\boldsymbol{q}' = (q^{1'}, q^{2'}, q^3 \ldots, q^n)$) and demanding - to forget no one $S(E)$ - the validity of (3.1) for <u>all</u> concave g we arrive at:

$$\sum_{i=1}^{n} C^i g\left(\frac{q^{i'}}{C^i}\right) \geq \sum_{i=1}^{n} C^i g\left(\frac{q^i}{C^i}\right) \qquad (2.2)$$

for all concave g. But this is nothing - as shown by Uhlmann [3], and Ruch, Schranner, and Seligmann [4] but

$$\boldsymbol{q}' \overset{c}{\lesssim} \boldsymbol{q}.$$

3. DISCUSSION

Up to one restriction - see below - this argumentation can be reversed, and one obtains the following impression: Considering equalizing processes it is essentially the same if one demands that for <u>one</u> "abstract" system the Felderhof functional for <u>all</u> concave g does increase or one demands the increasing of the one thermodynamic entopy but for <u>all</u> physical systems. Both kinds of view have the same physical content. In this sense the appearance of all concave functions in the first point of view reflects the variety of physical systems in the second one. Therefore it seems rather clear that the partial order hardly can be taken as the origin for a new thermodynamical principle as discussed occasionally - but on the contrary: in the above "Gedankenexperiment" it appears just as that set of conditions which guarantee that the second law of thermodynamics for no one physical system can be violated.

The circumstance that it would be impossible to find for an arbitrary concave g a corresponding physical system is not a serios counter-argument, since the question wheter or not a physical system "exists" is surely not the one of pure thermodynamics.

Finally, we want to mention that Lesche[5] using information theoretical arguments arrived at a similar conclusion.

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Цылька Х. E5-86-237

О неравенстве Йенсена и термодинамике

Статья представляет собой вклад в изучение соотношения между вторым законом термодинамики и частичным порядком "более чем C-смешанным" при уравнивающих процессах. С помощью мысленного эксперимента, предложенного Зоммерфельдом, найдена аргументация, согласно которой частичный порядок появляется в виде набора условий, которые гарантируют то, что второй закон термодинамики не может нарушаться ни для какой физической системы.

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On Jensen's Inequality and Thermodynamics

This note is a contribution to the discussion on the relation between the second law of thermodynamics and the partial order "more C-mixed than". Using an old Gedankenexperiment by Sommerfeld an argumentation is found, according to which the partial order appears just as the set of conditions which guarantee that for no one physical system the second law of thermodynamics can be violated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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