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WHICH STATES ARE ACCESSIBLE BY CONTINUOUS C-CONCAVE PROCESSES?

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1. INTRODUCTION

By "accessibility of states" we mean the following problem: Let $\mathcal{S}$ be a state space; $q^* \in \mathcal{S}$, a distinguished state, called initial state; and $\mathcal{P}$, a class of processes. Next, we look for the set $\mathcal{Z}(q^*)$ of all states which are accessible from q^* by an allowed process.

But only a suitable simultaneous choice of $\mathcal{S}$ and $\mathcal{P}$ opens up the possibility of proving anything at all. For the subsequent pair the question above can be answered completely:

Let $\mathcal{S} \equiv \mathcal{S}_n := \{ q : q = (q^1, q^2, \ldots, q^n), q^i \geq 0,$

$\Sigma q^i = 1, \quad n \text{ finite} \}$,

i.e., the set of all finite-dimensional probability vectors,

and $\mathcal{P} \equiv \mathcal{P}_c := \{[q(t)]_{t \geq 0}$ with

(i) $[q(t)]_{t \geq 0} := R^+ \ni t \to q(t) \in \mathcal{S}_n$

(ii) $q(t'') \overset{c}{\succeq} q(t')$; $\forall t', t''$

with $0 \leq t' \leq t''$ and $c \in \mathcal{S}_n \setminus \partial \mathcal{S}_n$

(iii) $\lim_{t' \to t''} q(t') = q(t'')$; $\forall t', t'' \}$,

i.e., the set of all continuous c-concave processes in the sense of UHLMANN, where the sign "$\overset{c}{\succeq}$" symbolizes the partial order "more mixed than with respect to the state c", see [1,2], and also [3].

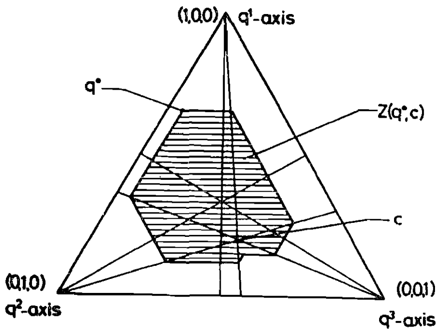
To illustrate the latter definition, we should stress that the solution of a master equation

$$(+) \qquad (d/dt)q = L q ,$$

where L is a stochastic generator fulfilling $Lc = 0$ (i.e., the fixed point serves as a reference state), does belong to $\mathcal{P}_c$ which can be found in [1], too.

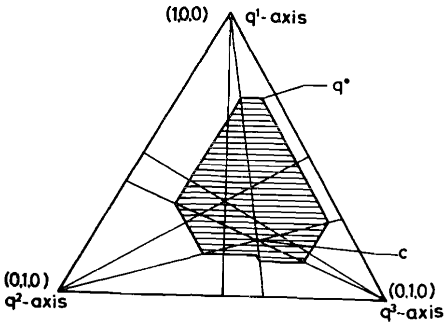
2. THE RESULT

In this note we will mainly present the solution for $n = 3$. (The cases $n = 1,2$ are trivial). Using barycentric coordinates for a convenient geometrical representation one recognizes the hatched sets in the following figures as the desired regions of accessibility. As they depend also on $\underline{c}$ we denote them more precisely by $\mathcal{Z}(q^*, \underline{c})$. In these examples the reference state should be fixed, here $\underline{c} = (4/20, 6/20, 10/20)$. By varying the initial state q^* one obtains a good impression of the appearing sets:



$$q^* = \left(\frac{14}{20}, \frac{5}{20}, \frac{1}{20}\right)$$

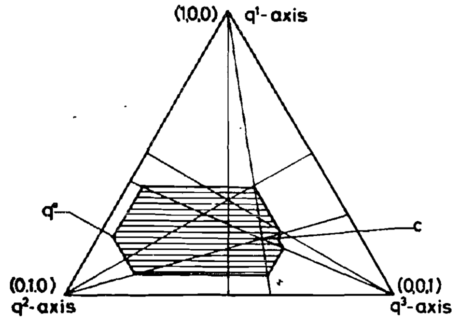
Fig. 1



$$q^* = \left(\frac{14}{20}, \frac{1}{20}, \frac{5}{20}\right)$$

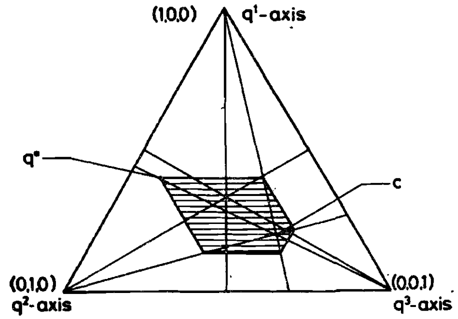
Fig. 2

The general solution ($n \geq 3$) consists in an iterative construction procedure which generates $\mathcal{Z}(q^*, c)$ after a finite number of steps. By doing this one also obtains some interesting pro-



$$q^* = \left(\frac{4}{20}, \frac{15}{20}, \frac{1}{20}\right)$$

Fig. 3



$$q^* = \left(\frac{8}{20}, \frac{10}{20}, \frac{2}{20}\right)$$

Fig. 4

perties of these sets, such as

α) $\mathcal{Z}(q^*, c)$ is a polyhedron.

β) $\mathcal{Z}(q^*, c)$ is star-like with respect to c.

γ) $\mathcal{Z}(q^*, c)$ is nonconvex, in general.

3. THE IDEA OF THE PROOF

We start with a rather physical model: One considers heat transfer between n numbered macroscopic bodies characterized by their heat capacities $c^i > 0$, $i = 1,2,\ldots n$. By identifying these c^i with the components of a state one obtains a normalization which connects an arbitrary distribution of the available amount of heat among the n bodies with an element of $\mathcal{S}_n$. Next one studies the set of states which are accessible by all possible successions of heat transfer between pairs of bodies, where naturally heat is allowed to flow only from the hotter body to the cooler one. (The temperature of the i-th body is just q^i/c^i).

Quite a good starting point for such consideration lies in a theorem by Hardy, Littlewood, and Polya[4], (Theorem 45) which connects the partial order " $\succsim$ "* with the above-mentioned heat transfer. But since it is a result on double stochastic matrices it is useful only in the case where the reference state is the equipartition or, in other words, when all heat capacities are equal. Nevertheless - as is shown in[5] it can be taken as origin for a construction procedure that leads to the set of all accessible heat distributions in this special case.

On the other hand, recently the following generalization of this theorem could be proved:

T h e o r e m

Let $q', q, c \in \mathcal{S}_n$, $c \notin \partial\mathcal{S}_n$.

(I) $t^1 \geq t^2 \geq \ldots \geq t^n$ and $t^{1\prime} \geq t^{2\prime} \geq \ldots \geq t^{n\prime}$

and

(II) $q' \overset{c}{\succsim} q$.

Then: there exists a stochastic matrix T, with $q' = Tq$ and $c = Tc$ that can always be chosen as a finite product of matrices of the type:

$$T=\frac{1}{c^k+c^\ell}\begin{array}{cc}
\begin{array}{ccccc} \quad & k & \quad & \ell & \quad \end{array} & \\
\left(\begin{array}{ccccc}
1\;\ddots\;1 & & & & \\
 & x\cdot c^\ell+c^k & & c^k(1-x) & \\
 & & 1\;\ddots\;1\;\ddots\;1 & & \\
 & c^\ell(1-x) & & x\cdot c^k+c^\ell & \\
 & & & & 1\;\ddots\;1
\end{array}\right) & \begin{array}{c} \\ k \\ \\ \ell \\ \\ \end{array}
\end{array}$$

---

* In more mathematical texts (in connection with "majorization") this sign is used in the opposite sense!

with $x \in [0,1]$ and $1 \leq k < \ell \leq n$, see[6]. It is just this generalized version that opens up the possibility of answering the question for the sets of accessibility in the case of different heat capacities. One has to take it - instead of the theorem of Hardy, Littlewood and Polya - as origin for the above-mentioned construction procedure, and what remains to be done is straightforward because it runs by analogy with[5].

4. REMARKS

We want to stress again that $\mathcal{Z}(q^*, c)$ is formed by all of these processes. For a single one, e.g., for a special equation (+) one may only say that no state out of $\mathcal{S}_n \setminus \mathcal{Z}(q^*, c)$ can belong to its trajectory. But even this is a valuable piece of information, especially, in connection with the following remark.

Although it seems that solutions of master equations are typical continuous c-concave processes, there are examples of quite different non-linear evolution equations which also show this property, as was worked out by Alberti and Crell[7].

Finally we want to mention that our considerations imply a stronger relation than " $\overset{c}{\succsim}$ ".

Thus we define:

Let $q^*, q, c \in \mathcal{S}_n$, $c \notin \partial\mathcal{S}_n$.

Then $q \overset{c}{\ggg} q^* \overset{def}{\iff} q \in \mathcal{Z}(q^*, c)$.

This relation should be a useful tool for investigating continuous dissipative motion.

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Цылка Х. E5-86-173
Какие состояния являются достижимыми с помощью c-процессов?

Рассматривается частная проблема динамических систем, так называемая "достижимость состояний". Под этим термином мы понимаем следующее: являются ли состояния допустимыми или нет с какого-то фиксированного начального состояния с помощью определенного процесса. В этой работе изучаются множества состояний, достижимых с помощью непрерывных c-процессов. Оказывается, что они представляют собой невыпуклые полиэдры.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Zylka Ch. E5-86-173
Which States are Accessible by Continuous c-Concave Processes?

We consider a special question concerning dynamical systems called "accessibility of states". By this term we mean the problem whether or not states are attainable from a fixed initial state by a certain process. In this note the sets of states which are attainable by continuous c-processes are investigated. They turn out to be nonconvex polyhedra.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986