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E5-86-145

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ON THE TRUNCATED HENON SYSTEM

1986

1. INTRODUCTION

In the last decade there is a rapidly increasing number of papers concerning numerical investigations of dynamical systems. One of the most extensively studied discrete systems is the so-called Henon system[6] given by the following pair of equations:

$$(H) \qquad \begin{aligned} x_{n+1} &= y_n + 1 - A x_n^2 \\ y_{n+1} &= B x_n . \end{aligned}$$

This is one of the simplest nonlinear maps of the plane into itself and it appears that it reflects a lot of important properties of dynamical systems.

If $B = 1$, then the map is area preserving. In the dissipative case, i.e., $B < 1$, the trajectories, not escaping to infinity can approach several kinds of attractors: fixed points, periodic orbits, different types of non-periodic attractors (e.g., strange, chaotic, stochastic attractors).

Up to now there is no generally accepted terminology (for possible definitions see [5,8,10]). We will use the term "strange attractor" for those non-periodic attractors which have sensitive dependence on initial conditions, i.e. the largest characteristic exponent (Lyapunov exponent) is positive. Such attractors are called "chaotic" in[5]. In the system (H) Henon found such a strange attractor for $A = 1.4$, $B = 0.3$. An exhaustive investigation of (H) was performed by Simó[9]. In the study of systems of type(H) one difficulty stems from the fact that there are divergent points. Thus, for some values of the parameter it can be even very difficult to find non-divergent points.For this and other reasons from the very beginning there were also considered systems constrained onto bounded regions (e.g., compact manifolds). One possibility is to truncate the system under consideration modulo m, m is natural number. For a two-dimensional systems this means, for example, that the system is constrained onto a cartain 2-torus. So, in[2] Chirikov and Izraelev studied the system

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$$\phi_{n+1} = [\phi_n + k(\psi_n^2 - \psi_n + 1/6) - \epsilon(\phi_n - \phi_0)]$$

$$\phi_{n+1} = [\psi_n + \phi_{n+1} - 1/2],$$

where the brackets [ ] mean truncation modulo 1. For $\epsilon = 0$, the mapping is area preserving and has a stochastic component for several values of k (the term "stochastic" is here used somewhat loosely for irregular motion which is mixing and has positive KS-entropy, for details see[2]). If $0 < \epsilon < 0.1$, the stochasticity is destroyed and only periodic orbits appear. Then stochasticity again appears at $\epsilon = 0.1$. Thus, there is a gap between stochasticity in the area preserving case ($\epsilon = 0$) and stochasticity in the dissipative case ($\epsilon > 0$). In[7] McLaughlin observed that this behavior found by Chirikov and Izraelev is not universal. For this he studied the truncated Henon system

$$x_{n+1} = [y_n + 1 - Ax_n^2] \quad [\ ] \text{ means truncation}$$

$$y_{n+1} = [Bx_n] \quad \text{modulo } 2.$$

The figures in[7] for A = 1.4, B = 0.6, 0.9, 0.9995, 1.0 respectively seem to show a continuous transition from stochasticity in the area preserving case (B = 1) to stochasticity in the dissipative case (B < 1). But the figures show that x and y vary in the interval $-2 \leq x, y \leq 2$ instead of in $0 \leq x, y \leq 2$ as it should be if one truncates modulo 2. The reason may be that there was used the standard modulo-function in the machine (denote it by MOD). This function acts as follows:
1.3 MOD 2 = 1.3; 2.4 MOD 2 = 0.4; −1.3 MOD 2 = −1.3; −2.4 MOD −2.4 MOD = − 0.4 and so on. In contrast to this, as it is well-known, the usual modulo-function (denote it by mod) acts in this way: 1.3 mod 2 = 1.3; 2.4 mod = 0.4; −1.3 mod 2 = 0.7; −2.4 mod 2 = 1.6 and so on.

Let us remark that in[2] there was used the mod-function. But in the literature the use of MOD can be found in several places (e.g.[1,4]). We will denote the corresponding truncated Henon systems by H(MOD m), H(mod m) respectively. Clearly, these systems are quite different.

In this paper we investigate some special properties of H(MOD m) and H(mod m) for m = 1,2. So in section 2 a mechanism of generation of periodic orbits and some kind of crisis for periodic orbits are described. Section 3 contains some remarks concerning the role of dissipation. All numerical calculations were performed on ESER 1061 with double precision.

2. THE ROLE OF TRUNCATION

It is well-known that in one-dimensional (discrete) systems there cannot coexist different attractors, while in multi-dimensional systems even infinitely many attractors can coexist. For the Henon system (H) in[3] and[9] there are given several examples for the coexistence of attractors. It is clear that in general a truncation will change the dynamics and hence also the structure of attractors. Let us give an example.

Table 1 A = 1.o752; B = 0.3

system	*kind of attractors*	*characteristic exponent*
(H)	*24-orbit* *strange attractor*	*−0.0208* *0.102*
H(MOD 2)	*same attractor and exponents as in (H)*	
H(mod 2)	*strange attractor*	*0.73*
H(MOD 1) *H(mod 1)*	*fixed point (0,0)*	*−0.602*

That one obtains the same results for (H) and H(MOD 2) depends on the fact that the 24-orbit and the strange attractor of (H) are contained in the square $-2 \leq x, y \leq 2$. The corresponding domains of attraction are quite different because in truncated systems one has no divergent points. We give an illustration in figures 1 and 2. They show the domains of attraction for the systems (H) and H(MOD 2) in the context of table 1. Figure 1 contains three types of points: divergent points located outside the two parabolae, convergent points inside the parabolae. The black points go to the 24-orbit, the "white" points (one must imagine a lattice) run to the strange attractor. Figure 2 contains only two types of points: again the black points go to 24-orbit, while the "white" points run to the strange attractor.

Now we consider the system H(mod 1) for A=1 and vary the parameter B. Then one gets the following interesting set of periodic attractors (period 1 means here the fixed point (0.0)):

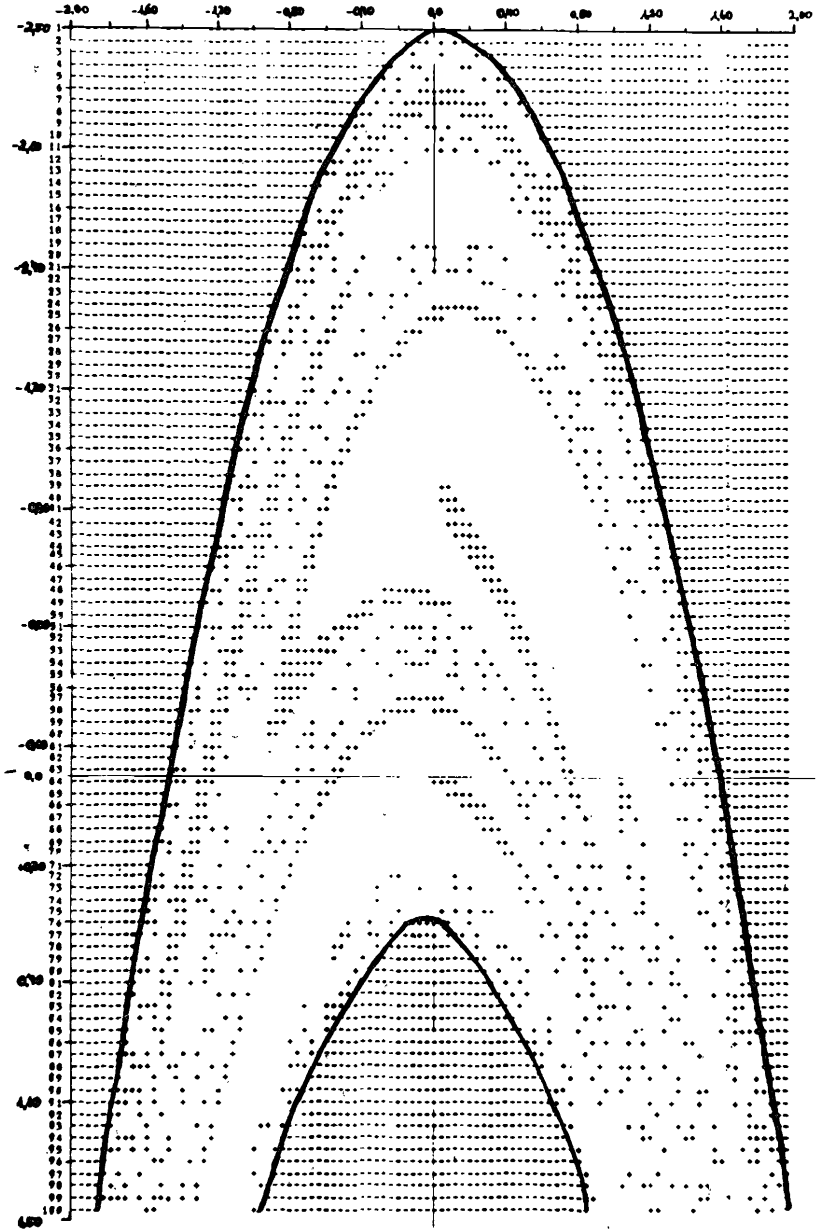

Fig. 1.

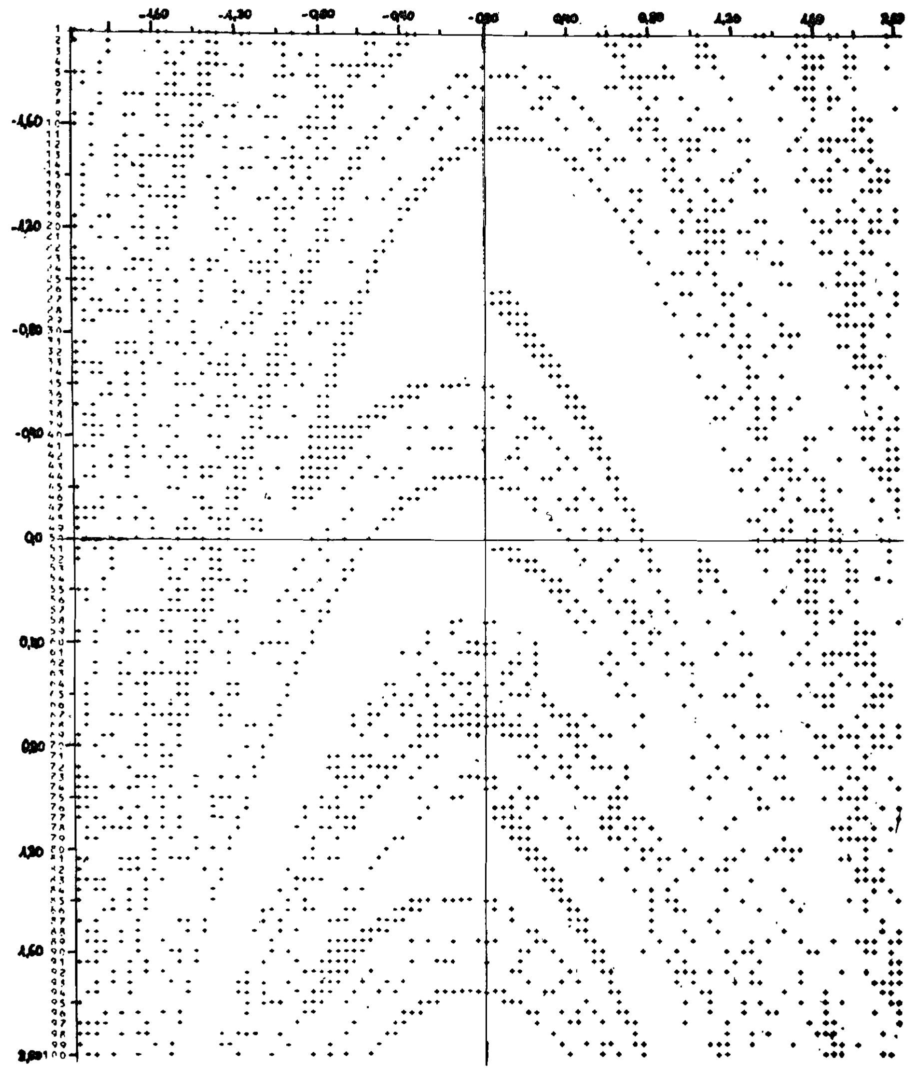

Fig. 2.

Table 2 H(mod 1), A = 1

value of B	periods of coexisting attractors
0.002 ⋮ 0.246	1 , 3
0.247 ⋮ 0.263	1 , 3 , 8
0.264 ⋮ 0.267	1 , 3 , 8 , 10
0.268 ⋮ 0.2694	1 , 3 , 8 , 10 , 12
0.2695 ⋮ 0.26954	1 , 3 , 8 , 10
0.26955 ⋮ 0.27017	1 , 3 , 8
0.27018 ⋮ 0.2736	1 , 3
0.2737 ⋮ 0.27546	1 , 6
0.27547	1

Let us add some values of the corresponding characteristic exponents.

Table 3 H(mod 1), A = 1

period	value of B	characteristic exponent
3	0.002	-1.839
	0.246	-0.053
	0.2736	-0.001
8	0.247	-0.222
	0.27017	-0.178
10	0.264	-0.285
	0.26954	-0.275
12	0.268	-0.341

These orbits are interesting for several reasons.
1. First of all one may wonder that there is no period doubling except the transition from B = o.2736 to 0.2737 (3 → 6). As it can be seen from the characteristic exponents the orbits do not succeed to bifurcate before disappearing (the exponent does not reach zero). They dissappear not by bifurcation but by quite another mechanism. To understand this one has to look how the orbits develop if B is increased. Take for example the 6-orbit.

Table 4 H(mod 1), A = 1, 6-orbit for different values of B

B=0.27540	B=0.27543	B=0.27546
(0.063246,0.275266)	(0.062713,0.275362)	(0.062184,0.275458)
⋮	⋮	⋮
(0.999513,0.062273)	(0.999753,0.062220)	(0.999991,0.062167)

We have explicitly given only two points of the 6-orbit for the different B-values. One sees that one point (the last in the table) wanders to the "boundary" **x** = 1. If it goes across this "boundary", the orbit disappears. The same machanism can be observed for the other orbits. i.e., the vanishing of the 8-, 10-, 12-,... orbit. But because it is more exactly to consider the system on the unit torus than on the square, the "boundary" **x** = 1 is only a fictive one. So the term "boundary crisis" for this mechanism would be misleading. What really happens is that in course of the evolution of the orbit under consideration, one point of the orbit wanders into the domain of attraction of the orbit with the next smaller period. Thus, one could call this phenomenon "crisis by capture".

2. The next interesting point is the question how the orbits of period 8, 10, 12,... arise. It seems that all these orbits are generated by the same mechanism. To see this, let us give some coordinates of orbits explicitly.

H(mod 1), A = 1, B = 0.268

8-orbit

(0.932547,0.071735) ↓	(0.010445,0.056024) ↓	(0.999672,0.014988) ↓
(0.202091,0.249923) ↓	(0.055925,0.002799)	(0.015644,0.267912) ↓
(0.209082,0.054160)		(0.267667,0.004193)

10-orbit

(0.929280,0.071816)	(0.013509,0.055121)	(0.000602,0.014724)
(0.208254,0.249047)	(0.054939,0.003620)	(0.014723,0.000161)
(0.205254,0.055812)		
(0.999945,0.003846)		
(0.004057,0.267985)		
(0.267969,0.001087)		

12-orbit

(0.928468,0.071823)	(0.014265,0.054893)	(0.000832,0.014657)
(0.209770,0.248829)	(0.054690,0.003823)	(0.014656,0.000223)
(0.204826,0.056218)		
(0.000008,0.003928)	(0.999987,0.001053)	
(0.003928,0.000002)	(0.001079,0.267996)	
	(0.267995,0.000289)	

For the 8-orbit the arrows indicate the order in which the points run through the orbit. For the other orbits there is the same scheme, i.e., the clusters should be read from left to right and within the clusters from above to below. There can be seen a surprising symmetry within the pairs of points and within the whole structure of the orbit (3-, 2-,... 2-. 3-cluster). We do not know how this "pair creation" can be explained analytically.

3. There is a natural question: are there further coexisting orbits, namely orbits with period 14, 16, 18,...? This is indeed the case. In the parameter interval $0.268 \leq B \leq 0.2694$ there are contained further intervals were orbits of period 14,16,18 appear. Up to this period we found the orbits numerically, but we guess that there may be even infinitely many. To find further orbits

that there may be even infinitely many. To find further orbits numerically is a difficult task for two reasons. First, the intervals of B where these orbits exist will be very small. Secondly, the domains of attraction also become very narrow. As an illustration of this last fact serves the following table. There was considered an equally spaced lattice of 2500 initial points, and it was counted how much of them run to the corresponding orbits:

Table 5 H(mod 1), A = 1, B = 0.268

period	*number of attracted points*	*rate (%)*
1	1385	55.4
3	625	25.0
8	356	14.2
10	114	4.6
12	20	0.8

Let us yet mention that for the system H(MOD 1) there exist the same orbits as indicated above. The investigation of the structure of the domains of attraction leads to the following fact which may be worthy to note. A lattice of 10 000 initial points (the same spacing as for H(mod 1) leads to almost exactly the same rates of points converging to the corresponding attractors. The rates are 53.6%,25.9%,15.1%,4.6%,0.8% respectively. Up to now we gave examples for values of the parameters A and B so that the behavior of H(mod 1) and H(MOD 1) was the same. But of course there are also values which lead to quite different behaviors. For A = 1.4, B = 0.3 the system H(mod 1) seems to have only the fixed point (0,0) , but this could not be proved analytically. On the other hand, in H(MOD 1) there coexist the fixed point and an 3-orbit which bifurcates by period doubling:

Table 6 H(MOD 1), A = 1.4

value of B	*period*	*characteristic exponent*
⋮		
0.316	3	-0.003695
0.317	6	-0.000148
⋮		
0.366		-0.003597
0.367		-0.004508
⋮	12	
0.368		-0.017801

Let us add a remark concerning the question how one can find such periodic orbits and follow their evolution. To follow the evolution of a given periodic orbit for different values of a parameter the best way is to take a point of the orbit corresponding to one value of the parameter as initial point for the next value. This must be combined with some control about the rate of increasing or decreasing the parameter. If nothing is known before, one can try to look for orbits (or generally for attractors) by systematic testing a lattice of initial points. This may be very time consuming. But in systems like H and its truncations one can make use of the fact that for B = 0 it changes into a one-dimensional system which can be investigated much easier. Then take the parameter value A near to the value in which we are interested for the original system and investigate this one-dimensional system. If one finds some orbit, it is very likely (because of continuity) that it will persist for small B. Now one can try to follow the evolution of this orbit as described above changing if necessary both parameters. Recently there was described a method for a systematic search of periodic orbits applicable also to higher dimensional systems (cf. /11/ and the references therein).

3. THE ROLE OF DISSIPATION

As already mentioned in the introduction McLaughlin studied the system H(MOD 2) for A = 1.4 and B varying from 1 to 0.6. There is only one stochastic component (likely a strange attractor) for every B under consideration and the transition from B = 1 to B < 1 is continuous. He claimed that this is a continuous transition from the area preserving to the dissipative case (cf.Introduction). But the trouble is that for B = 1 the map is not globally area preserving as can be seen from figures 3-5. Figure 3 shows the image of the square $-2 \leq x, y \leq 2$ under H(MOD 2). The regions indicated by the numbers "2" are covered twice. Figure 4 is the image of figure 3 (of course only the black region) under H(MOD 2). Here the numbers "2" also indicate the regions covered twice. Finally, figure 5 is obtained from figure 4 in the same way. Let us remark that while in figure 3 there are no white islands, the succeeding figures are more and more complicated and foliated.
In contrast to this, the system H(mod 2) is area preserving for B = 1 and has indeed such a continuous transition as described above. We will return to this point in another paper.
Let us add some comments on another remark contained in /7/. There is written: however, for the truncation used by Chirikov and Izraelev (i.e., mod 1), the origin becomes an attractor for B = 1 and all iterates eventually fall into its domain of at-

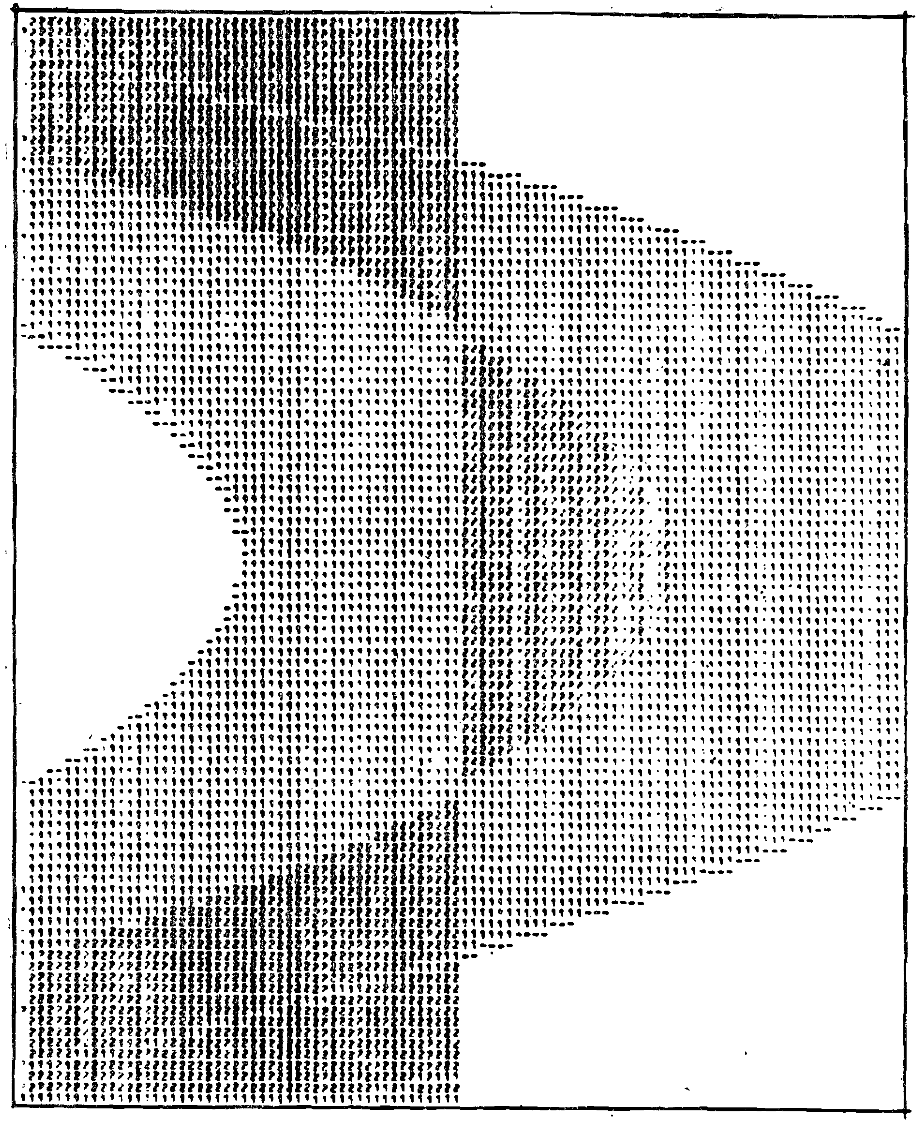

Fig.3.

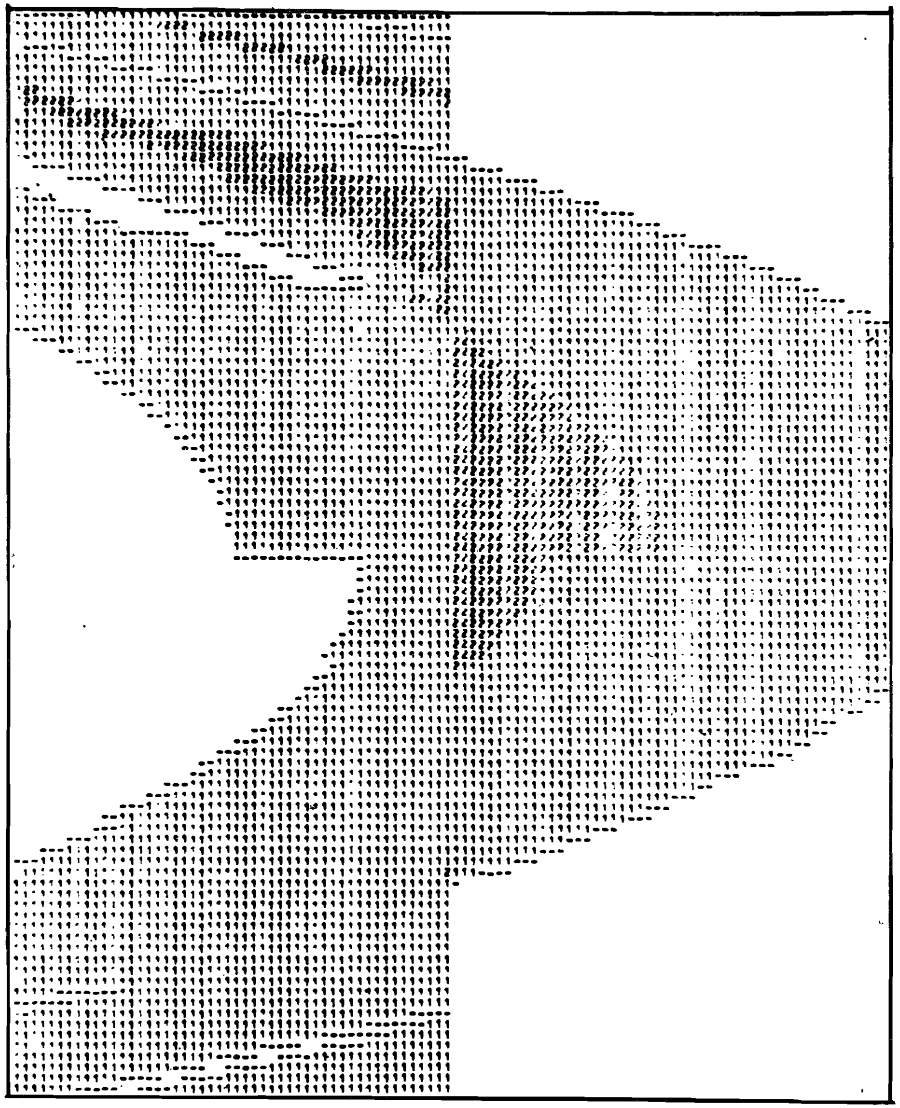

Fig.4.

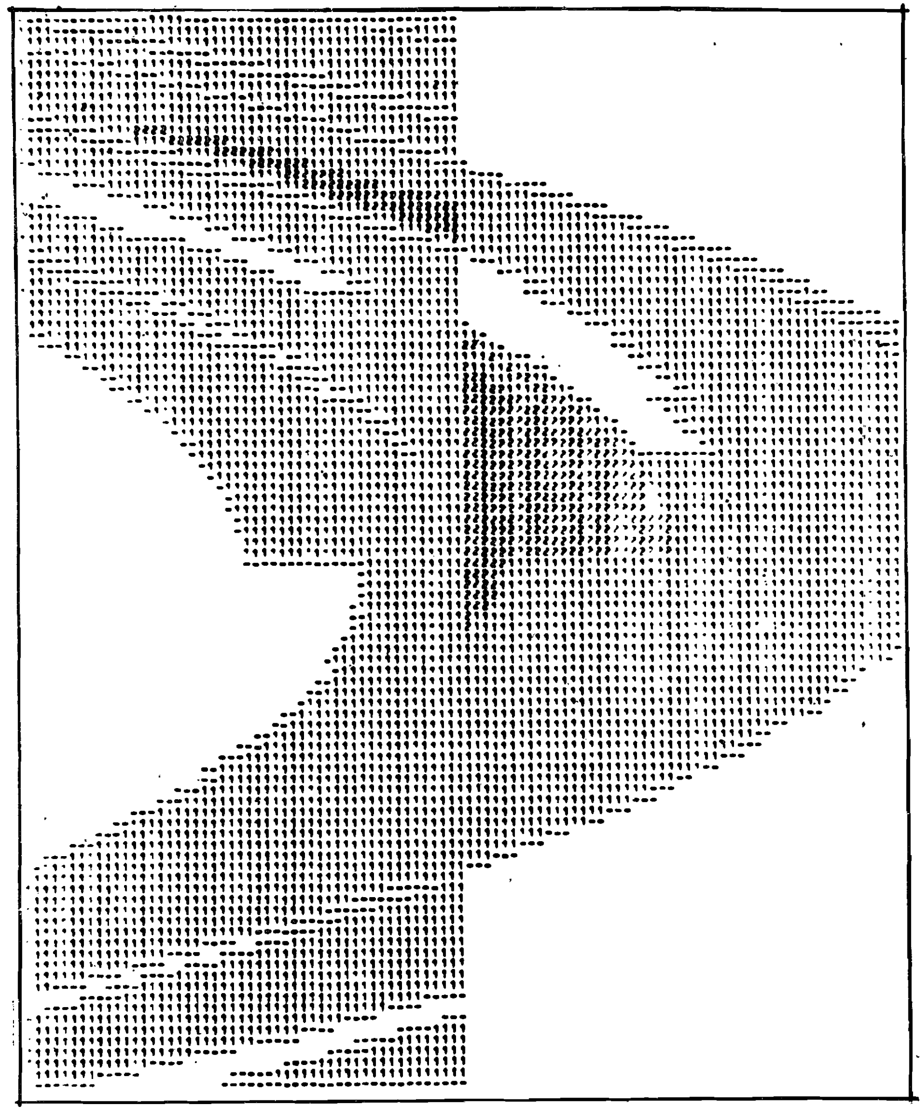

Fig.5.

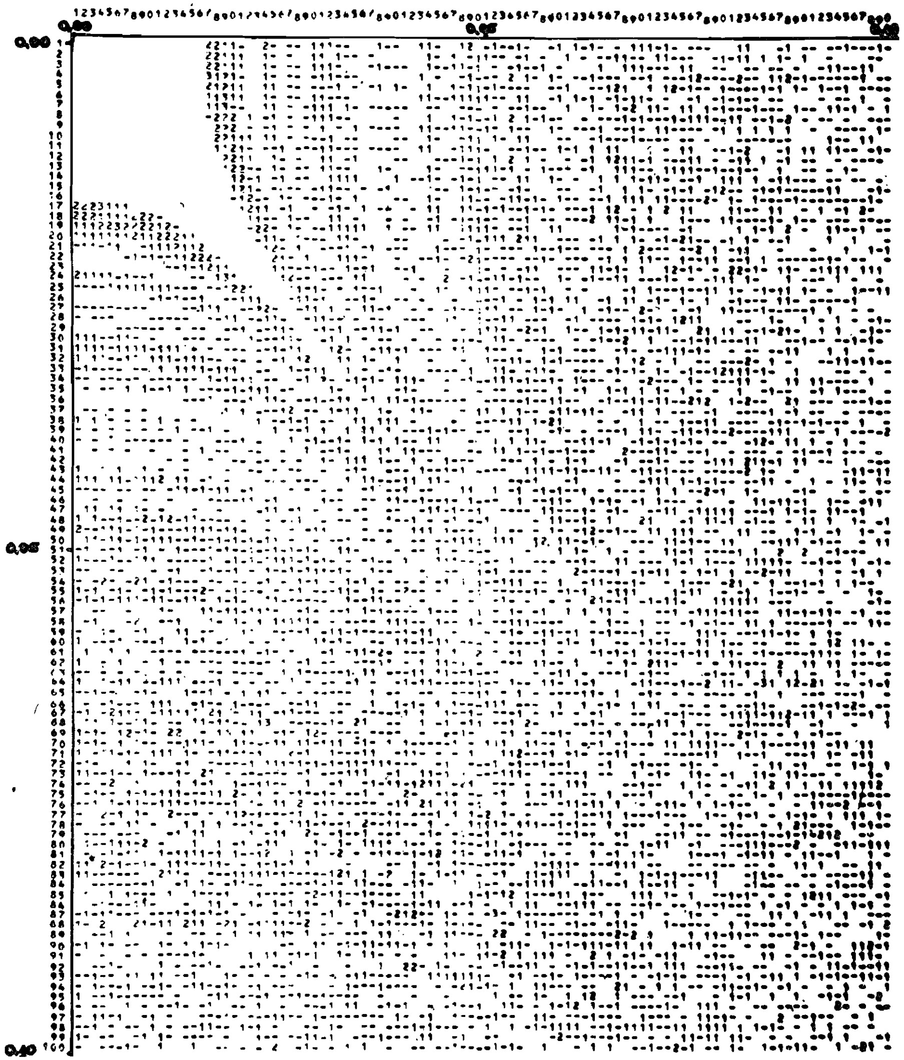

Fig.6

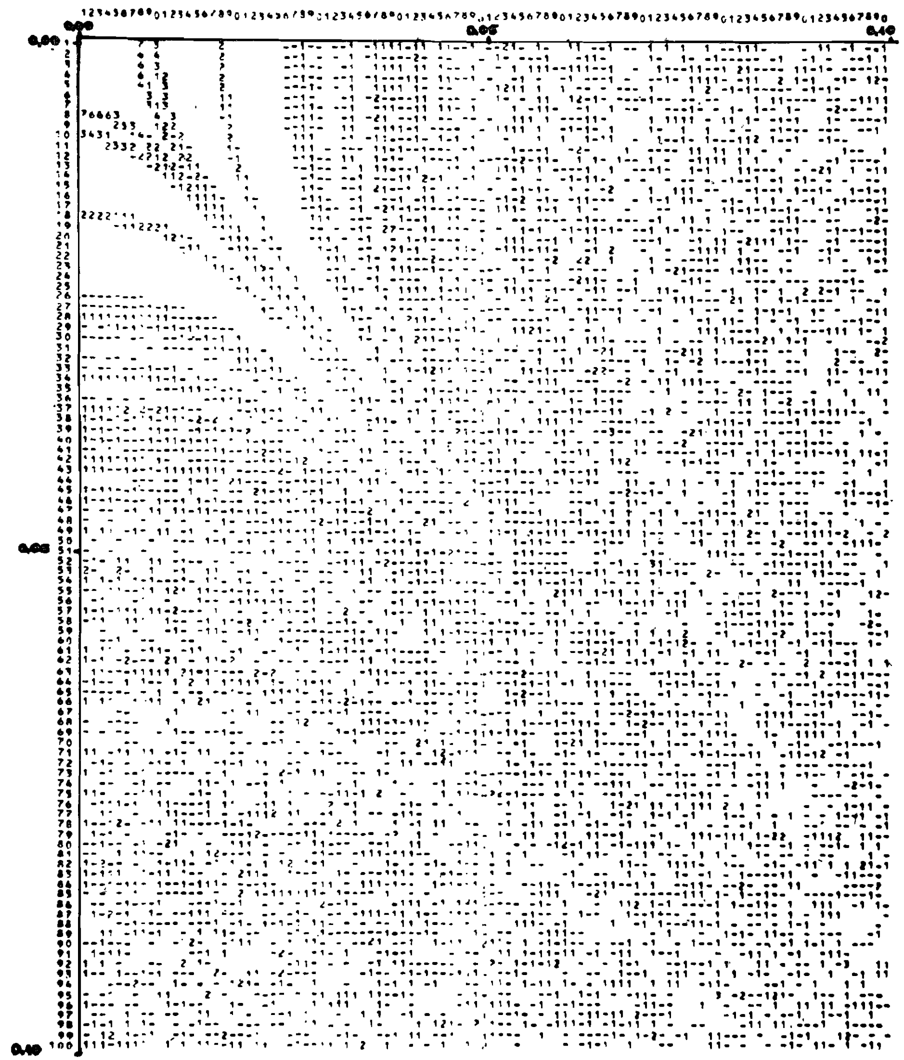

Fig.7

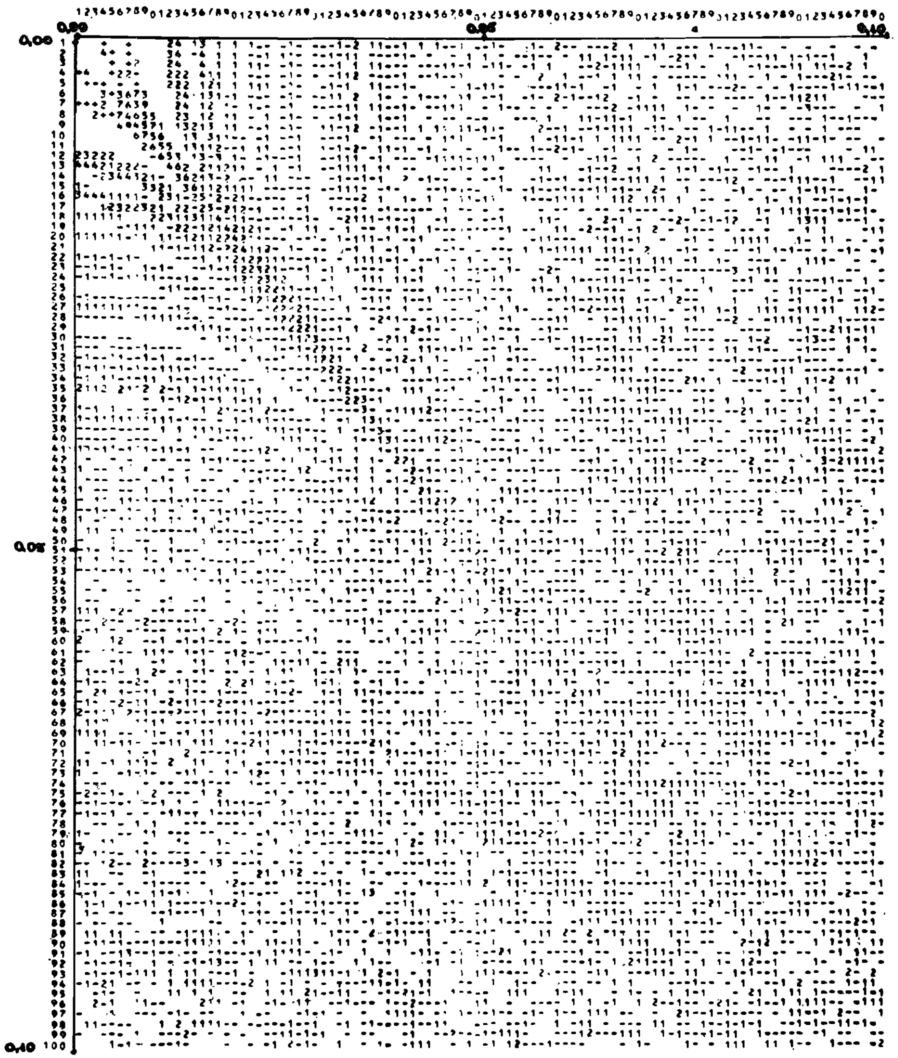

Fig. 8.

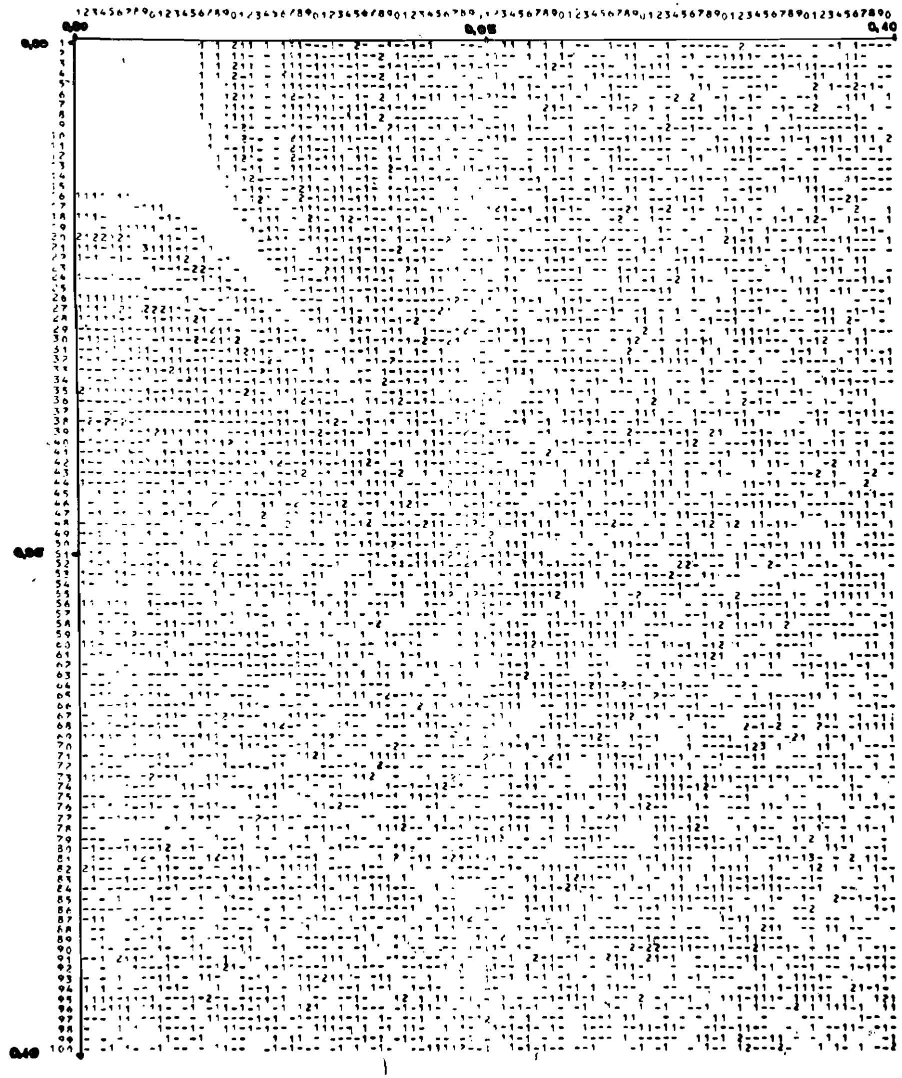

Fig. 9.

traction".We tried to verify this. It is clear, that for both systems, H(MOD 1) and H(mod 1) the origin is a fixed point independently of the values of A and B. For H(MOD 1) neither analytically nor numerically this cited fact could be established. There were tested several initial points (not too near to (0.0)) with sufficiently many iterations. But they did not reach the origin.

For H(mod 1) we observed an interesting effect near the origin. First remark that for $B = 1$ the origin cannot be attracting since the map is area preserving. But for $A = 1.4$, $B = 0.9995$ near the origin appears some kind of pulsation. This is illustrated by the following computer experiment. We have plotted 10 successive histogramms of 10^6 iterations of an initial point in any histogramm. For the first histogramm the initial point (0.5, 0.5) was used. In each of the following the last point of the preceeding 10^6 interations serves as initial point. Figures 6-9 show the 1st, 3rd, 5th, 6th histogramms respectively. In all the further histogramms the points never came so near to (0.0) as in the 5th histogramm. Moreover, the histogramms show only the part $0 \le x, y \le 0.1$, but the points are dispersed over the whole unit square. Thus, we could not identify any attractor in this case. It would be interesting to give an explanation of this phenomenon.

ACKNOWLEDGEMENT

The authors are grateful to B.V.Chirikov and F.M.Izraelev for valuable discussions on dynamical systems.

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Received by Publishing Department
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Изучается поведение простых дискретных динамических систем, полученных разными обрезаниями модели Хенона. С помощью численных экспериментов найден новый тип исчезновения периодических орбит. Также обсуждается переход от системы, для которой площадь сохраняется, к диссипативной системе.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1986

Timmermann B., Timmermann W. E5-86-145
On the Truncated Henon System

The behavior of simple discrete dynamical systems, given by several truncations of the Henon map, is studied. With the help of numerical experiments a new kind of disappearing of periodic orbits is described. The transition from the area preserving case to the dissipative regime is also discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986