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# ON ACCURACY OF COMPUTATION OF THE LORENZ ATTRACTORS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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## I. Introduction

More than twenty years passed from the moment when Lorenz [1] has discovered strange attractor of his system with the help of the numerical computation by computers. For that time the Lorenz attractors were obtained for many other systems of nonlinear ordinary differential equations [2-4] by the numerical computation also. The possibility of using the Lorenz attractors for turbulence investigation leads to intensive numerical and theoretical research of the attractors [2-8]. Besides, the numerical solutions of systems of ordinary differential equations are initial informations for further investigations. However the Lorenz attractors are insteady solutions of systems of ordinary differential equations and the Lorenz attractors calculated by their difference schemes can be strongly differed from their original Lorenz attractors. For this reason the elaboration of the highly exact method for the numerical computation of the Lorenz attractors of systems of ordinary differential equations is the subject of this work.

It is shown that most of systems of ordinary differential equations having the Lorenz attractors belongs to the class of bilinear dynamical systems, the general form of which is the following

$$\dot{x} = Ax + B(x,x), \tag{1.1}$$

where $x \in R^n$, $A \in R^{n\times n}$ and $B(x,x)$ is a vector-column of n quadratic forms

$$B_r(x,x) = \sum_{i=1}^{n} \sum_{j=i}^{n} B_r^{ij} x_i x_j$$

$$(B_r^{ij} \in R \ ; \quad r = 1,\dots,n).$$

In general the right part of systems of ordinary differential equations having Lorenz attractors is an analytical function and they can be represented in the form

$$\dot{x}_r = \sum_{K=1}^{M} \sum_{\substack{\alpha_1 = 1,\dots,n \\ \dots\dots \\ \alpha_K = (\alpha_{K-1}),\dots,n \\ S_1+\dots+S_K = m}} a_r^{\alpha_1,\dots,\alpha_K} x_{\alpha_1}\dots x_{\alpha_K} \qquad (1.2)$$

$(a_r^{\alpha_1,\dots,\alpha_K} \in R; \quad r = 1,\dots,n)$.

For any initial condition the Cauchy problem solution of system (1.1) or (1.2) can be expanded into power series. It follows from analyticity of their right part. In this work the method for the numerical computation of the Lorenz attractors of the systems with any accuracy degree is elaborated on the basis of expansion of the Cauchy problem solutions into power series. In particular we shall show that, the numerical solutions in Lorenz's discovery /1/ being Lorenz attractors of his difference scheme are not those of Lorenz system of differential equations. This testifies that the original Lorenz attractors of systems of ordinary differential equations can be obtained only with the help of grounded calculation methods.

## 2. Computation algorithms

For any point $x \in R^n$ at the moment $t \in R$ the Cauchy problem solution $\varphi(t) = x$ of system (1.1) or (1.2) can be represented by the vector of power series

$$\varphi(t+\tau) = \sum_{m=0}^{\infty} C_m(t)\tau^m ,$$

where $C_m(t)$ is an n-dimensional vector of Taylor coefficients $C_m(t) = (C_m^1(t),\dots,C_m^r(t),\dots,C_m^n(t))^T$. By induction on $m$ it is easy to show that the vector of Taylor coefficients satisfies the following recurrent equation /9,10/

$$C_{m+1}(t) = \frac{1}{m+1}\left(AC_m(t) + \beta(C_K(t), C_{m-K}(t))\right),$$
$$C_0(t) = \varphi(t) = x, \qquad (2.1)$$

$$\beta_r(C_K(t), C_{m-K}(t)) = \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{K=0}^{m} B_r^{ij} C_K^i(t) C_{m-K}^j(t); \quad r = 1,\dots,n.$$

Choosing an integration step $h$ and a length $L$ of the approximating polynomial we have the following system of difference equations



$$\varphi(t+h) = \sum_{m=0}^{L} C_m(t)h^m, \quad t = 0, h, 2h, \dots \qquad (2.2)$$

Then systems of recurrent equations (2.1) and (2.2) put together the algorithms for the numerical computation of the Lorentz attractors of system (1.1) with any accuracy degree with the help of corresponding choice of the length $L$.

The simple method of error check of computation of the Lorentz attractors is gradual increase of the length $L$. For example, we wish to calculate $N$ points of the attractor, the relative error of computation of every point is not greater than $\delta > 0$. Then we must choose the length $L$ such that for any other length $L' > L$ $|(\varphi'(t) - \varphi(t))/\varphi'(t)| < \delta$ for all $t = 1, 2, \dots N$, where $\varphi'(t)$ and $\varphi(t)$ are solutions calculated by $L'$-approximation and $L$-approximation respectively. The method needs a lot of times of calculation with various length of the approximating polynomial.

The other method, which allows one to investigate the error of computation with every length $L$, is studying the behaviour of the deviations $y(t)$ from the Cauchy problem solution by applying a small perturbation $y(0)$ at t=0. If $\varphi(0) = x$ then the initial perturbations $\theta_m(0)$ $(m = 0, 1, \dots, L)$ of the coefficients $C_m(0)$ are determined as $\theta_m(0) = C_m^*(0) - C_m(0)$, where $C_m^*(0)$ and $C_m(0)$ are the coefficients of the power series at points $x + y(0)$ and $x$ respectively. We represent system (2.1) in the form

$$X(t+1) = F(X(t)), \qquad (2.3)$$

where $X(t)$ is $(L+1)\times n$ dimensional vector

$$X(t) = \begin{pmatrix} C_0(t) \\ C_1(t) \\ \vdots \\ C_m(t) \\ \vdots \\ C_L(t) \end{pmatrix}, \quad F(X(t)) = \begin{pmatrix} F_0(X(t)) \\ F_1(X(t)) \\ \vdots \\ F_m(X(t)) \\ \vdots \\ F_L(X(t)) \end{pmatrix},$$

$$F_0(X(t)) = \sum_{m=0}^{L} C_m(t)h^m$$

$$F_1(X(t)) = AF_0(X(t)) + \beta(F(X_0(t)))$$

$$\dots\dots\dots\dots\dots\dots$$

$$F_m(X(t)) = \frac{1}{m}\left(AF_{m-1}(X(t)) + \beta\left(F_K(X(t)), F_{m-1-K}(X(t))\right)\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_L(X(t)) = \frac{1}{L}\left(AF_{L-1}(X(t)) + \beta\left(F_K(X(t)), F_{L-1-K}(X(t))\right)\right)$$

The behaviour of the deviations $\theta_m(t)$ from the Taylor coefficients $C_m(t)$ is defined as

$$\theta(t) = F^t(X(0)+\theta(0)) - F^t(X(0)).$$

Because of small amplitude of $\theta(0)$ we oan use linear approximation. Then from (2.1), (2.2) and (2.3) the deviations $y(t)$ from the Lorenz attractor $\varphi(t)$ are determined by the following system of equations

$$\theta(t+1) = G(t,x)\,\theta(t), \tag{2.4a}$$

$$y(t) = D\theta(t), \tag{2.4b}$$

where $G(t,x) = \left(\frac{\partial F(X)}{\partial X}\right)_{|X=X(t)}$ is the Jaoobi matrix with the coefficients

$$G_{m\mu}(t) = h^{\mu-1} d_{m-1}(t), \quad (\mu, m = 1,\dots,L+1) \tag{2.4c}$$

$d_m(t)$ are (nx)=-matrices defined by the following reccurent equation

$$d^{\gamma r}_{m+1}(t) = \frac{1}{m+1}\left(\sum_{\alpha=1}^{n} A_{r\alpha} d^{\gamma\alpha}_m(t) + \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{K=0}^{m} B^{ij}_r d^{\gamma i}_K(t)\, C^j_{m-K}(t) + d^{\gamma j}_{m-K}(t)\, C^i_K(t)\right)\Big) \tag{2.4d}$$

$$(\gamma, r = 1,\dots,n)$$

$d_0(t) = I$ (identity matrix)

$$D = (I_{n\times n}, 0). \tag{2.4e}$$

So system of equations (2.4) describes the behaviour of the deviations $y(t)$ from the Lorenz attraotor by applying a small perturbation $y(0)$ and can be used for its error oheok in computation process.

Generalizing the results for system (1.2) we obtain the analogous computation algorithms. The Taylor coefficients $C_m(t)$ of the



Cauchy problem solution $\varphi(t) = x$ of system (1.2) are determined by the following recurrent equation /10/.

$$C^r_{m+1}(t) = \frac{1}{m+1}\sum_{K=1}^{M}\sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = (\alpha K-1),\dots,n \\ S1+\dots+SK=m}} a_r^{\alpha 1,\dots,\alpha K}\, C^{\alpha 1}_{S1}(t)\dots C^{\alpha K}_{SK}(t)$$

The matrices of coefficients $d_m(t)$ in system (2.4) are defined by the algorithm

$$d^{\gamma r}_{m+1}(t) = \frac{1}{m+1}\sum_{K=1}^{M}\sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = (\alpha K-1),\dots,n \\ S1+\dots+SK=m}} a_r^{\alpha 1,\dots,\alpha K}\sum_{\sigma=1}^{K} d^{\gamma\alpha\sigma}_{S\sigma}(t)\left(C^{\alpha\sigma}_{S\sigma}(t)\right)^{-1} C^{\alpha 1}_{S1}(t)\dots C^{\alpha K}_{SK}(t)$$

3. Lorenz attractors

The computation algorithms given in the previous paragraph permit to calculate the Lorenz attractors of systems of ordinary differential equations with any accuracy degree due to choice of the length $L$ of the approximating polynomial (2.2). For their illustration now we represent numerical solution of Lorenz system

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{3.1}$$

by the algorithms for the same data, with whioh the strange attractors have been discovered by Lorenz, that is $\sigma = 10$, $r = 28$, $b = 8/3$, $y(0) = 1$, $z(0) = 0$, $x(0) = 0$ and $h = 0{,}01$.
We wish to computate the first six thousands of points of the Lorenz attractor with the relative error smaller than $10^{-6}$. The computation has been realized by computer EC 1061.

The Taylor coefficients $C_m = (C^x_m, C^y_m, C^z_m)^T$ of the solution of system (3.1) are determined by the recurrent equations

$$c^x_{m+1}(t) = \frac{1}{m+1}\left(-\sigma c^x_m(t) + \sigma c^y_m(t)\right)$$

$$c^y_{m+1}(t) = \frac{1}{m+1}\left(r c^x_m(t) - c^y_m(t) - \sum_{k=0}^{m} c^x_k(t) c^z_{m-k}(t)\right)$$

$$c^z_{m+1}(t) = \frac{1}{m+1}\left(-b c^z_m(t) + \sum_{k=0}^{m} c^x_k(t) c^y_{m-k}(t)\right).$$

The coefficients $d_m^{\gamma r}(t)$ of system (2.4) are defined by the recurrent equations

$$d^{xx}_{m+1}(t) = \frac{1}{m+1}\left(-\sigma d^{xx}_m(t) + \sigma d^{xy}_m(t)\right)$$

$$d^{xy}_{m+1}(t) = \frac{1}{m+1}\left(r d^{xx}_m(t) - d^{xy}_m(t) - \sum_{k=0}^{m}\left(d^{xx}_k(t) c^z_{m-k}(t) + c^x_k(t) d^{xz}_{m-k}(t)\right)\right)$$

$$d^{xz}_{m+1}(t) = \frac{1}{m+1}\left(-b d^{xz}_m(t) + \sum_{k=0}^{m}\left(d^{xx}_k(t) c^y_{m-k}(t) + c^x_k(t) d^{xy}_{m-k}(t)\right)\right)$$

$$d^{yx}_{m+1}(t) = \frac{1}{m+1}\left(-\sigma d^{yx}_m(t) + \sigma d^{yy}_m(t)\right)$$

$$d^{yy}_{m+1}(t) = \frac{1}{m+1}\left(r d^{yx}_m(t) - d^{yy}_m(t) - \sum_{k=0}^{m}\left(d^{yx}_k(t) c^z_{m-k}(t) + c^x_k(t) d^{yz}_{m-k}(t)\right)\right)$$

$$d^{yz}_{m+1}(t) = \frac{1}{m+1}\left(-b d^{yz}_m(t) + \sum_{k=0}^{m}\left(d^{yx}_k(t) c^y_{m-k}(t) + c^x_k(t) d^{yy}_{m-k}(t)\right)\right)$$

$$d^{zx}_{m+1}(t) = \frac{1}{m+1}\left(-\sigma d^{zz}_m(t) + \sigma d^{zy}_m(t)\right)$$

$$d^{zy}_{m+1}(t) = \frac{1}{m+1}\left(r d^{xz}_m(t) - d^{zy}_m(t) - \sum_{k=0}^{m}\left(d^{zx}_k(t) c^z_{m-k}(t) + c^x_k(t) d^{zz}_{m-k}(t)\right)\right)$$

$$d^{zz}_{m+1}(t) = \frac{1}{m+1}\left(-b d^{zz}_m(t) + \sum_{k=0}^{m}\left(d^{xz}_k(t) c^y_{m-k}(t) + c^x_k(t) d^{zy}_{m-k}(t)\right)\right)$$

The computation results have shown that for the length of the approximating polynomial $L = 20$ the relative error of the first six thousands iterations is smaller than $10^{-6}$. The comparison of our exact computation results with the Lorenz computation results /1/ is represented in table 1. In it we see that the error of the first thousands interactions of the Lorenz computation is greater than 0,1 and smaller than 1. Then it rapidly increases. The error of coordinate $y$ at point $t = 16,43$ (iteration 1643) exceeds its absolute value. Thus the Lorenz attractor discovered in /1/ is that of the



system of difference equations of the double approximation of system (3.1). But it is not the Lorenz attractor of the Lorenz system (3.1).

In table 2 the values of coordinates x,y and z from numerical computation with L = 10; 15; 20 are represented for comparison. With L=10 the error of the first 2600 iterations is smaller $10^{-6}$, then it quickly increases and exceeds the absolute value of coordinates (interaction 4000). With L=15 the error of the first 3700 iterations is smaller than $10^{-6}$, then it increases fast and exceeds the absolute value of coordinates (interation 5000).

Numerical computation has shown that for every $L \geq 2$ the discrete dynamical system, which is the system of difference equations of the L-approximation of system (3.1), has the Lorenz attractors. Although the Lorenz attractors corresponding to different value L coincide only in the first interactions their characteristics of the relative maximum of coordinate z little differ one from another, because the number of iterations from the relative maximum to the next is not large (less than 100 iterations ). In table 3 we represent coordinates x,y,z obtained from numerical computation with $L \geq 20$ at iterations such that z gets relative maximum for the first six thousands iterations. (their error is smaller than $10^{-6}$ ).

4. Conclusion

Discrete dynamical systems being difference schemes with different degrees of approximation of a system of ordinary differential equations can have the Lorenz attractors. However out of instability of the corresponding Cauchy problem solution every of ones can approximate the Lorenz attractors of the system of ordinary differential equations only in some initial interval of time. The elaborated algorithms in the work permit one to calculate their Lorenz attractors in arbitrary given initial interval of time with any accuracy degree.

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Table I

The comparison of the Lorenz computation results with the exact computation results of Lorenz system

| N | Lorenz solution | | | Exact solution | | | Error of Lorenz solution | | |
|---|---|---|---|---|---|---|---|---|---|
| | $x$ | $y$ | $z$ | $x^*$ | $y^*$ | $z^*$ | $\lvert x^*-x\rvert$ | $\lvert y^*-y\rvert$ | $\lvert z^*-z\rvert$ |
| 0000 | 000.0 | 000.I | 000.0 | 000.0 | 000.I | 000.0 | 0.0 | 0.0 | 0.0 |
| 0045 | 0I7.4 | 005.5 | 048.3 | 0I7.4 | 005.6 | 048.3 | 0.0 | 0.I | 0.0 |
| 0I07 | -009.I | -008.3 | 028.7 | -009.I | -008.3 | 028.7 | 0.0 | 0.0 | 0.0 |
| 0I68 | -009.2 | -008.4 | 028.8 | -009.2 | -008.4 | 028.8 | 0.0 | 0.0 | 0.0 |
| 0230 | -009.2 | -008.4 | 028.9 | -009.2 | -008.4 | 029.0 | 0.0 | 0.0 | 0.I |
| 0292 | -009.2 | -008.3 | 029.0 | -009.3 | -008.4 | 029.I | 0.I | 0.I | 0.I |
| 0354 | -009.3 | -008.3 | 029.2 | -009.3 | -008.4 | 029.2 | 0.0 | 0.I | 0.0 |
| 04I6 | -009.3 | -008.3 | 029.3 | -009.4 | -008.4 | 029.4 | 0.I | 0.I | 0.I |
| 0478 | -009.4 | -008.2 | 029.5 | -009.5 | -008.4 | 029.5 | 0.I | 0.2 | 0.0 |
| 0540 | -009.4 | -008.2 | 029.6 | -009.5 | -008.4 | 029.7 | 0.I | 0.2 | 0.I |
| 0602 | -009.5 | -008.2 | 029.8 | -009.6 | -008.4 | 029.9 | 0.I | 0.2 | 0.I |
| 0664 | -009.6 | -008.3 | 030.0 | -009.7 | -008.5 | 030.0 | 0.I | 0.2 | 0.0 |
| 0726 | -009.7 | -008.3 | 030.2 | -009.9 | -008.6 | 030.2 | 0.2 | 0.3 | 0.0 |
| 0789 | -009.7 | -008.I | 030.4 | -009.6 | -008.5 | 030.5 | 0.I | 0.4 | 0.I |
| 085I | -009.9 | -008.3 | 030.7 | -0I0.I | -008.7 | 030.7 | 0.2 | 0.4 | 0.0 |
| 09I4 | -0I0.0 | -008.I | 030.9 | -0I0.2 | -008.6 | 03I.0 | 0.2 | 0.5 | 0.I |
| 0977 | -0I0.0 | -008.0 | 03I.2 | -0I0.3 | -008.6 | 03I.3 | 0.3 | 0.6 | 0.I |
| I040 | -0I0.2 | -008.0 | 03I.5 | -0I0.5 | -008.7 | 03I.6 | 0.3 | 0.7 | 0.I |
| II03 | -0I0.4 | -008.I | 03I.9 | -0I0.8 | -009.0 | 03I.9 | 0.4 | 0.9 | 0.0 |
| II67 | -0I0.5 | -007.9 | 032.3 | -0I0.9 | -008.9 | 032.3 | 0.4 | I.0 | 0.I |
| I23I | -0I0.7 | -007.9 | 032.8 | -0II.2 | -009.I | 032.7 | 0.5 | I.2 | 0.I |
| I295 | -0II.I | -008.2 | 033.3 | -0II.6 | -009.7 | 033.I | 0.5 | I.5 | 0.2 |
| I36I | -0II.I | -007.7 | 033.9 | -0II.9 | -009.5 | 033.8 | 0.8 | I.8 | 0.I |
| I427 | -0II.6 | -007.9 | 034.7 | -0I2.4 | -0I0.2 | 034.4 | 0.8 | 2.3 | 0.3 |
| I495 | -0I2.0 | -007.7 | 035.7 | -0I3.0 | -0I0.8 | 035.2 | I.0 | 3.I | 0.5 |
| I566 | -0I2.5 | -007.2 | 037.I | -0I3.9 | -0II.7 | 036.4 | I.4 | 4.5 | 0.7 |
| I643 | -0I3.9 | -007.7 | 039.6 | -0I5.5 | -0I6.6 | 035.7 | I.6 | 8.9 | 3.9 |
| I722 | 0I4.0 | 007.5 | 040.I | 0I4.6 | 0I0.9 | 038.6 | 0.6 | 3,4 | I.5 |
| I798 | -0I3.5 | -007.2 | 039.I | -0I3.4 | -020.9 | 023.8 | 0.I | I3.7 | I5.3 |
| I882 | 0I4.6 | 007.4 | 04I.3 | 0I0.6 | 004.I | 036.0 | 4.0 | 3.I | 5.3 |
| I952 | -0I2.7 | -007.8 | 037.0 | 0I4.8 | 0II.8 | 038.2 | 27.5 | I9.6 | I.2 |
| 2029 | -0I3.5 | -007.0 | 039.0 | -0I3.5 | -02I.0 | 024.I | 0.0 | I4.0 | I7.9 |
| 2II0 | 0I4.6 | 008.3 | 040.8 | 0I2.2 | 006.9 | 036.8 | 2.4 | I.4 | 4.0 |
| 2I83 | -0I2.8 | -007.0 | 037.9 | 0I4.3 | 0I0.2 | 038.6 | 27.I | I7.2 | 0.7 |
| 2268 | -0I4.4 | -006.6 | 04I.5 | -0I6.6 | -I8.I | 036.9 | 2.2 | II.5 | 4.6 |
| 2337 | 0I2.6 | 007.9 | 036.8 | 0I3.2 | 0I2.0 | 034.6 | 0.6 | 4.I | 2.2 |



| N | $x$ | $y$ | $z$ | $x^*$ | $y^*$ | $z^*$ | $\lvert x^*-x\rvert$ | $\lvert y^*-y\rvert$ | $\lvert z^*-z\rvert$ |
|---|---|---|---|---|---|---|---|---|---|
| 24I2 | 0I3.7 | 008.I | 038.9 | 0I2.9 | 008.0 | 037.4 | 0.8 | 0.I | I.5 |
| 250I | -0I5.3 | -008.0 | 042.3 | 006.I | -00I.9 | 033.I | 2I.4 | 6.I | 9.2 |
| 2569 | 0II.9 | 007.6 | 035.7 | -0I2.I | -004.6 | 039.4 | 24.0 | I2.2 | 2.7 |
| 2639 | 0I2.9 | 008.2 | 037.I | 008.8 | 0I6.I | 0I3.4 | 4.I | 7.9 | 23.7 |

**Table 2**

The comparison of the **strange** attractors of Lorenz system calculated with different degrees of approximation

| N | L = 10 | | L = 15 | | L ≥ 20 |
|---|---|---|---|---|---|
| | $x$ | $\lvert x^*-x\rvert$ | $x$ | $\lvert x^*-x\rvert$ | $x^*$ |
| 2604 | 0.4566 | | 0.45666 | | 0.4566 |
| 3760 | $0.3I7IIxI0^{I}$ | $0.7xI0^{-2}$ | $0.3I78IxI0^{I}$ | | $0.3I782xI0^{I}$ |
| 3800 | $0.96803xI0^{I}$ | $0.939xI0^{-I}$ | $0.97743xI0^{I}$ | $0.IxI0^{-3}$ | $0.97742xI0^{I}$ |
| 4000 | $0.48579xI0^{I}$ | $0.4489IxI0^{I}$ | 0.28708 | $0.873xI0^{-I}$ | 0.3688I |
| 4200 | $0.22967xI0^{I}$ | $0.73339xI0^{I}$ | $-0.50I65xI0^{I}$ | $0.I07xI0^{-I}$ | $-0.50272xI0^{I}$ |
| 4400 | $-0.II660xI0^{2}$ | $0.786I3xI0^{I}$ | $-0.37826xI0^{I}$ | $0.I727xI0^{-I}$ | $-0.37987xI0^{I}$ |
| 4600 | $-0.7I247xI0^{I}$ | $0.48593xI0^{I}$ | $-0.23I59xI0^{I}$ | $0.505xI0^{-I}$ | $-0.22654xI0^{I}$ |
| 4800 | $0.63338xI0^{I}$ | $0.54932xI0^{I}$ | $0.I0503xI0^{2}$ | $0.I324xI0^{I}$ | $0.II827xI0^{2}$ |
| 5000 | $-0.754IIxI0^{I}$ | $0.I0636xI0^{2}$ | $0.8II80xI0^{I}$ | $0.5023xI0^{I}$ | $0.30950xI0^{I}$ |
| 5200 | $0.9I458xI0^{I}$ | $0.I6II2xI0^{I}$ | 0.37068 | $0.70502xI0^{I}$ | $0.I0757xI0^{2}$ |
| N | $y$ | $\lvert y^*-y\rvert$ | $y$ | $\lvert y^*-y\rvert$ | $y^*$ |
| 2604 | 0.9435I | $0.IxI0^{-4}$ | 0.94352 | | 0.94352 |
| 3760 | $0.50377xI0^{I}$ | $0.35xI0^{-2}$ | $0.504I2xI0^{I}$ | | $0.504I2xI0^{I}$ |
| 3800 | $0.3I44IxI0^{I}$ | 0.2269 | $0.33444xI0^{I}$ | $0.3xI0^{-3}$ | $0.3344IxI0^{I}$ |
| 4000 | $0.72303xI0^{I}$ | $0.I422xI0^{2}$ | $-0.6896IxI0^{I}$ | $0.I48xI0^{-I}$ | $-0.69I09xI0^{I}$ |
| 4200 | $0.28980xI0^{I}$ | $0.77004xI0^{I}$ | $-0.48089xI0^{I}$ | $0.65xI0^{-2}$ | $-0.48024xI0^{I}$ |
| 4400 | $-0.774I4xI0^{I}$ | $0.35I32xI0^{I}$ | $-0.42I2IxI0^{I}$ | $0.I659xI0^{-I}$ | $-0.42278xI0^{I}$ |
| 4600 | $-0.I2754xI0^{2}$ | $0.I2828xI0^{2}$ | 0.85592 | $0.776IxI0^{-I}$ | 0.7783I |
| 4800 | $-0.45879xI0^{I}$ | $0.I97I6xI0^{2}$ | $0.I4067xI0^{2}$ | $0.I06IxI0^{I}$ | $0.I5I29xI0^{2}$ |
| 5000 | $-0.I9669xI0^{I}$ | $0.73947xI0^{I}$ | $0.I304IxI0^{2}$ | $0.7643xI0^{I}$ | $0.5398xI0^{I}$ |
| 5200 | $0.I5532xI0^{2}$ | $0.I6047xI0^{2}$ | $0.2405IxI0^{I}$ | $0.29207xI0^{I}$ | -0.5I564 |
| N | $z$ | $\lvert z^*-z\rvert$ | $z$ | $\lvert z^*-z\rvert$ | $z^*$ |
| 2604 | $0.I5290xI0^{2}$ | | $0.I5290xI0^{2}$ | | $0.I5290xI0^{2}$ |
| 3760 | $0.I7720xI0^{2}$ | 0.202 | $0.I7923xI0^{2}$ | $0.IxI0^{-2}$ | $0.I7922xI0^{2}$ |
| 3800 | $0.34994xI0^{2}$ | $0.IIxI0^{-I}$ | $0.35005xI0^{2}$ | | $0.35005xI0^{2}$ |
| 4000 | $0.I9443xI0^{2}$ | $0.I049IxI0^{2}$ | $0.29834xI0^{2}$ | 0.I | $0.29934xI0^{2}$ |

| N | z | \|z*-z\| | z | \|z*-z\| | z* |
|---|---|---|---|---|---|
| 4200 | 0.17731x10$^{2}$ | 0.5593x10$^{1}$ | 0.23283x10$^{2}$ | 0.41x10$^{-1}$ | 0.23324x10$^{2}$ |
| 4400 | 0.34985x10$^{2}$ | 0.14397x10$^{2}$ | 0.20565x10$^{2}$ | 0.23x10$^{-1}$ | 0.20588x10$^{2}$ |
| 4600 | 0.12986x10$^{2}$ | 0.12476x10$^{2}$ | 0.25685x10$^{2}$ | 0.21 | 0.25475x10$^{2}$ |
| 4800 | 0.35387x10$^{2}$ | 0.8159x10$^{1}$ | 0.24719x10$^{2}$ | 0.2509x10$^{1}$ | 0.27228x10$^{2}$ |
| 5000 | 0.32102x10$^{2}$ | 0.20034x10$^{2}$ | 0.17677x10$^{2}$ | 0.5609x10$^{1}$ | 0.12068x10$^{2}$ |
| 5200 | 0.17053x10$^{2}$ | 0.22472x10$^{2}$ | 0.22402x10$^{2}$ | 0.17123x10$^{2}$ | 0.39525x10$^{2}$ |

Table 3

Coordinates $x,y,z$ at iterations such that $z$ gets relative maximum for the first six thousands iteration ($L \geqslant 20$, error is smaller than $10^{-6}$ )

| N | x | y | z |
|---|---|---|---|
| 45 | 0.17483x10$^{2}$ | 0.56779x10$^{1}$ | 0.48308x10$^{2}$ |
| 107 | -0.91268x10$^{1}$ | -0.83380x10$^{1}$ | 0.28788x10$^{2}$ |
| 169 | -0.91632x10$^{1}$ | -0.83010x10$^{1}$ | 0.28893x10$^{2}$ |
| 230 | -0.92838x10$^{1}$ | -0.84447x10$^{1}$ | 0.29010x10$^{2}$ |
| 292 | -0.93275x10$^{1}$ | -0.84260x10$^{1}$ | 0.29136x10$^{2}$ |
| 354 | -0.93778x10$^{1}$ | -0.84130x10$^{1}$ | 0.29270x10$^{2}$ |
| 416 | -0.94363x10$^{1}$ | -0.84084x10$^{1}$ | 0.29413x10$^{2}$ |
| 478 | -0.95051x10$^{1}$ | -0.84157x10$^{1}$ | 0.29566x10$^{2}$ |
| 540 | -0.95865x10$^{1}$ | -0.84396x10$^{1}$ | 0.29729x10$^{2}$ |
| 603 | -0.95566x10$^{1}$ | -0.82192x10$^{1}$ | 0.29908x10$^{2}$ |
| 665 | -0.96678x10$^{1}$ | -0.82757x10$^{1}$ | 0.30101x10$^{2}$ |
| 727 | -0.98033x10$^{1}$ | -0.83717x10$^{1}$ | 0.30306x10$^{2}$ |
| 790 | -0.98135x10$^{1}$ | -0.81879x10$^{1}$ | 0.30531x10$^{2}$ |
| 852 | -0.10012x10$^{2}$ | -0.83812x10$^{1}$ | 0.30773x10$^{2}$ |
| 915 | -0.10084x10$^{2}$ | -0.82786x10$^{1}$ | 0.31044x10$^{2}$ |
| 978 | -0.10195x10$^{2}$ | -0.82373x10$^{1}$ | 0.31338x10$^{2}$ |
| 1041 | -0.10366x10$^{2}$ | -0.82927x10$^{1}$ | 0.31662x10$^{2}$ |
| 1105 | -0.10392x10$^{2}$ | -0.79939x10$^{1}$ | 0.32024x10$^{2}$ |
| 1168 | -0.10752x10$^{2}$ | -0.83777x10$^{1}$ | 0.32430x10$^{2}$ |
| 1233 | -0.10746x10$^{2}$ | -0.78824x10$^{1}$ | 0.32902x10$^{2}$ |
| 1297 | -0.11199x10$^{2}$ | -0.83408x10$^{1}$ | 0.33444x10$^{2}$ |
| 1363 | -0.11337x10$^{2}$ | -0.79556x10$^{1}$ | 0.34102x10$^{2}$ |
| 1470 | -0.11526x10$^{2}$ | -0.75318x10$^{1}$ | 0.34886x10$^{2}$ |
| 1498 | -0.12077x10$^{2}$ | -0.76447x10$^{1}$ | 0.35935x10$^{2}$ |
| 1569 | -0.12856x10$^{2}$ | -0.78009x10$^{1}$ | 0.37422x10$^{2}$ |
| 1648 | -0.14229x10$^{2}$ | -0.79720x10$^{1}$ | 0.40128x10$^{2}$ |
| 1724 | 0.13661x10$^{2}$ | 0.76726x10$^{1}$ | 0.39186x10$^{2}$ |
| 1809 | -0.14501x10$^{2}$ | -0.69094x10$^{1}$ | 0.41462x10$^{2}$ |
| 1879 | 0.12451x10$^{2}$ | 0.73607x10$^{1}$ | 0.36950x10$^{2}$ |
| 1955 | 0.13364x10$^{2}$ | 0.69381x10$^{1}$ | 0.39140x10$^{2}$ |
| 2040 | -0.14404x10$^{2}$ | -0.65720x10$^{1}$ | 0.41504x10$^{2}$ |
| 2109 | 0.12717x10$^{2}$ | 0.81314x10$^{1}$ | 0.36863x10$^{2}$ |
| 2185 | 0.13346x10$^{2}$ | 0.71103x10$^{1}$ | 0.38973x10$^{2}$ |
| 2273 | -0.14982x10$^{2}$ | -0.72714x10$^{1}$ | 0.42184x10$^{2}$ |
| 2341 | 0.12121x10$^{2}$ | 0.77181x10$^{1}$ | 0.35968x10$^{2}$ |
| 2412 | 0.12969x10$^{2}$ | 0.80347x10$^{1}$ | 0.37465x10$^{2}$ |
| 2492 | 0.14011x10$^{2}$ | 0.72255x10$^{1}$ | 0.40238x10$^{2}$ |
| 2567 | -0.13529x10$^{2}$ | -0.75900x10$^{1}$ | 0.38977x10$^{2}$ |
| 2655 | 0.15426x10$^{2}$ | 0.85605x10$^{1}$ | 0.42188x10$^{2}$ |
| 2723 | -0.12284x10$^{2}$ | -0.81997x10$^{1}$ | 0.35896x10$^{2}$ |
| 2794 | -0.13028x10$^{2}$ | -0.83273x10$^{1}$ | 0.37348x10$^{2}$ |
| 2873 | -0.14138x10$^{2}$ | -0.78857x10$^{1}$ | 0.40006x10$^{2}$ |
| 2950 | 0.13620x10$^{2}$ | 0.72483x10$^{1}$ | 0.39426x10$^{2}$ |
| 3031 | -0.14556x10$^{2}$ | -0.80541x10$^{1}$ | 0.40744x10$^{2}$ |
| 3104 | 0.13074x10$^{2}$ | 0.75434x10$^{1}$ | 0.38084x10$^{2}$ |
| 3191 | 0.15318x10$^{2}$ | 0.84966x10$^{1}$ | 0.42007x10$^{2}$ |
| 3260 | -0.12159x10$^{2}$ | -0.75910x10$^{1}$ | 0.36151x10$^{2}$ |
| 3332 | -0.12888x10$^{2}$ | -0.74708x10$^{1}$ | 0.37756x10$^{2}$ |
| 3414 | -0.14762x10$^{2}$ | -0.83302x10$^{1}$ | 0.40968x10$^{2}$ |
| 3486 | 0.12977x10$^{2}$ | 0.77778x10$^{1}$ | 0.37693x10$^{2}$ |
| 3568 | 0.14299x10$^{2}$ | 0.72707x10$^{1}$ | 0.40806x10$^{2}$ |
| 3640 | -0.13218x10$^{2}$ | -0.80329x10$^{1}$ | 0.37986x10$^{2}$ |
| 3726 | -0.14923x10$^{2}$ | -0.77649x10$^{1}$ | 0.41713x10$^{2}$ |
| 3795 | 0.12625x10$^{2}$ | 0.82048x10$^{1}$ | 0.36609x10$^{2}$ |
| 3869 | 0.13560x10$^{2}$ | 0.82615x10$^{1}$ | 0.38515x10$^{2}$ |
| 3986 | 0.16659x10$^{2}$ | 0.73035x10$^{1}$ | 0.45611x10$^{2}$ |
| 4049 | -0.10332x10$^{2}$ | -0.80755x10$^{1}$ | 0.31825x10$^{2}$ |
| 4112 | -0.10633x10$^{2}$ | -0.83436x10$^{1}$ | 0.32202x10$^{2}$ |
| 4176 | -0.10815x10$^{2}$ | -0.83072x10$^{1}$ | 0.32638x10$^{2}$ |
| 4241 | -0.10877x10$^{2}$ | -0.79304x10$^{1}$ | 0.33141x10$^{2}$ |
| 4306 | -0.11149x10$^{2}$ | -0.79235x10$^{1}$ | 0.33731x10$^{2}$ |
| 4372 | -0.11436x10$^{2}$ | -0.78187x10$^{1}$ | 0.34439x10$^{2}$ |
| 4439 | -0.11921x10$^{2}$ | -0.79612x10$^{1}$ | 0.35333x10$^{2}$ |
| 4508 | -0.12622x10$^{2}$ | -0.82978x10$^{1}$ | 0.36519x10$^{2}$ |
| 4582 | -0.13317x10$^{2}$ | -0.77866x10$^{1}$ | 0.38386x10$^{2}$ |

| N | x | y | z |
|---|---|---|---|
| 4679 | -0.16093xI0$^2$ | -0.85661xI0$^I$ | 0.43586xI0$^2$ |
| 4745 | 0.11208xI0$^2$ | 0.76551xI0$^I$ | 0.34115xI0$^2$ |
| 48II | 0.11791xI0$^2$ | 0.81075xI0$^I$ | 0.34928xI0$^2$ |
| 4880 | 0.11969xI0$^2$ | 0.73549xI0$^I$ | 0.35959xI0$^2$ |
| 495I | 0.12813xI0$^2$ | 0.76196xI0$^I$ | 0.37481xI0$^2$ |
| 503I | 0.13890xI0$^2$ | 0.68720xI0$^I$ | 0.40253xI0$^2$ |
| 5106 | -0.13269xI0$^2$ | -0.70104xI0$^I$ | 0.38894xI0$^2$ |
| 5196 | 0.14856xI0$^2$ | 0.65007xI0$^I$ | 0.42462xI0$^2$ |
| 5263 | -0.11911xI0$^2$ | -0.76939xI0$^I$ | 0.35552xI0$^2$ |
| 5333 | -0.12479xI0$^2$ | -0.75537xI0$^I$ | 0.36845xI0$^2$ |
| 5408 | -0.13572xI0$^2$ | -0.77370xI0$^I$ | 0.38951xI0$^2$ |
| 5497 | 0.15255xI0$^2$ | 0.78537xI0$^I$ | 0.42337xI0$^2$ |
| 5565 | -0.11979xI0$^2$ | -0.76235xI0$^I$ | 0.35754xI0$^2$ |
| 5635 | -0.12900xI0$^2$ | -0.82581xI0$^I$ | 0.37136xI0$^2$ |
| 5712 | -0.14109xI0$^2$ | -0.84576xI0$^I$ | 0.39507xI0$^2$ |
| 5793 | 0.14165xI0$^2$ | 0.73292xI0$^I$ | 0.40476xI0$^2$ |
| 5867 | -0.13227xI0$^2$ | -0.73492xI0$^I$ | 0.38546xI0$^2$ |

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Ву Суан Минь E5-86-132

О точности вычисления аттракторов Лоренца систем обыкновенных дифференциальных уравнений


Разработан метод высокой точности вычисления аттракторов Лоренца систем обыкновенных дифференциальных уравнений с аналитической правой частью. Он основан на разложении их решений задачи Коши в степенные ряды. Показано, что оригинальные аттракторы Лоренца систем обыкновенных дифференциальных уравнений могут быть получены только с помощью разностных схем высокой степени аппроксимации.



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Vu Xuan Minh E5-86-132

On Accuracy of Computation of the Lorenz Attractors of Systems of Ordinary Differential Equations


The highly exact method for the numerical computation of the Lorenz attractors of systems of ordinary differential equations is presented. This is based on expansion of their Cauchy problem solutions into power series. It is shown that original Lorenz attractors of systems of ordinary differential equations can be obtained only with the help of the highly exact method.



The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1986