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дубна

E5-85-91

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STATES ON ALGEBRAS OF UNBOUNDED OPERATORS

Submitted to Proceedings of XXI Karpacz
Winter School of Theoretical Physics, 1985

1985

1. Introduction

The aim of this talk is to summarize some results concerning states (or more general, linear functionals) on topological $*$-algebras of unbounded operators as well as to indicate some lines of investigations.

It is well-known that in the algebraic approach to quantum field theory and quantum statistics the basic objects are observables and states. They are given by a $*$-algebra with unit $\mathfrak{A}$ and positive, normalized linear (continuous) functionals ω on $\mathfrak{A}$. Via the GNS-construction any state gives a $*$-representation of $\mathfrak{A}$ as a $*$-algebra $\mathcal{A}$ of operators in an appropriate Hilbert space $\mathcal{H}$.

There are two essentially different cases which can arise:

1. $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the bounded operators on $\mathcal{H}$;
2. $\mathcal{A}$ is an algebra of unbounded operators acting on a dense invariant domain $\mathcal{D} \subset \mathcal{H}$.

The first case appears, for example, if one starts with a C^*-algebra $\mathfrak{A}$, and this is reflected in the C^*-approach to quantum theory (Haag-Araki-axiomatic /5/,/8/, quantum statistics /7/).

From the very beginning of the algebraic approach to quantum field theory it was clear that non-normable topological $*$-algebras and their representations as algebras of unbounded operators are very natural objects (e.g., the test function algebra and its representations via Wightman functionals /6/,/35/).

More recently the same fact proved true also for quantum statistics (cf. /15/ and the references there).

Thus, the last two decades have seen a fruitful development of the theory of topological algebras of unbounded operators and related objects such as states, ideals, groups of automorphisms and applications to quantum physics.

It should be remarked that despite the fact that it is easier to work in Hilbert space the (abstract) topological $*$-algebra $\mathfrak{A}$ and its sta-

tes are the more relevant objects because a representation is in general accompanied with a loss of information.
The talk is organized as follows. The second section contains the notions, notations and results about operator algebras needed later.
In section 3 we treat normal states and show how the results are related to the ideal theory in algebras of unbounded operators. Section 4 is devoted to singular states. There are mentioned a representation theorem and a decomposition result. In section 5 we collect some facts concerning perturbation of states.

2. Preliminaries

Let us start by repeating some notions and results in the bounded case. For a separable Hilbert space $\mathcal{H}$ we denote by $\mathfrak{S}_\infty(\mathcal{H})$, $\mathfrak{S}_1(\mathcal{H})$ the two-sided $*$-ideals of compact and nuclear operators respectively. Let $\mathcal{A}$ be a $*$-subalgebra with unit I of $\mathcal{B}(\mathcal{H})$. A linear functional ω on $\mathcal{A}$ is said to be positive if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. A positive linear functional ω is a state if $\omega(I) = 1$. One of the basic facts in the C^*-theory is that any positive functional is automatically continuous. The following classes of functionals are of current interest:

vector functionals:	$\omega(A) \equiv \omega_{\varphi,\psi}(A) = \langle A\varphi,\psi\rangle$, $\varphi,\psi \in \mathcal{H}$
vector states:	$\omega(A) \equiv \omega_\varphi(A) = \langle A\varphi,\varphi\rangle$, $\varphi \in \mathcal{H}, \|\varphi\| = 1$
normal functionals:	$\omega(A) \equiv \omega_T(A) = \mathrm{Tr}\, TA$, $T \in \mathfrak{S}_1(\mathcal{H})$
normal states:	$\omega(A) \equiv \omega_T(A)$, $T \geq 0$, $\mathrm{Tr}\, T = 1$
singular functionals:	continuous functionals ω on $\mathcal{B}(\mathcal{H})$ such that $\omega(A) = 0$ for all $A \in \mathfrak{S}_\infty(\mathcal{H})$.

The following facts are well-known (cf. /23/):

i) $\mathfrak{S}_\infty(\mathcal{H})' \cong \mathfrak{S}_1(\mathcal{H})$, $\mathfrak{S}_1(\mathcal{H})' \cong \mathcal{B}(\mathcal{H})$.
The spaces are isomorphic as Banach spaces, and the isomorphisms arise from the bilinear map $(A,T) \longrightarrow \mathrm{Tr}\, AT$ (with obvious interpretation).

ii) A (positive) linear functional on $\mathcal{B}(\mathcal{H})$ is normal if and only if it is ultraweakly continuous.

iii) Any $\omega \in \mathcal{B}(\mathcal{H})'$ can be uniquely decomposed into the sum of a normal and a singular functional.

To treat the unbounded case some definitions are needed /13/. For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathcal{H}$ the set of linear operators $\mathcal{L}^+(\mathfrak{D}) = \{A: A\mathfrak{D} \subset \mathfrak{D}\ ,\ A^*\mathfrak{D} \subset \mathfrak{D}\}$ is a $*$-algebra

with respect to the usual operations and the involution $A \longrightarrow A^+ = A^* | \mathfrak{D}$. An Op*-algebra $\mathcal{A}(\mathfrak{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathfrak{D})$ containing the identity operator I.
The graph-topology $t_\mathcal{A}$ on $\mathfrak{D}$ is given by the set of seminorms

$$\mathfrak{D} \ni \varphi \longrightarrow \|A\varphi\| \quad \text{for all } A \in \mathcal{A}(\mathfrak{D}).$$

For $\mathcal{A}(\mathfrak{D}) = \mathcal{L}^+(\mathfrak{D})$ we use the symbol t instead of $t_{\mathcal{L}^+}$.
An Op*-algebra $\mathcal{A}(\mathfrak{D})$ is said to be

closed: if $\mathfrak{D} = \bigcap_{A \in \mathcal{A}} \mathfrak{D}(\bar{A})$ or equivalently , if $\mathfrak{D}[t]$ is complete;

selfadjoint: if $\mathfrak{D} = \mathfrak{D}_* \equiv \bigcap_{A \in \mathcal{A}} \mathfrak{D}(A^*)$.

In Op*-algebras there can be defined a lot of topologies (cf. for example /3/,/4/,/10/,/13/,/14/,/15/, /24/,/25/,/34/). We mention only those used in this talk:

the uniform topology $\tau_\mathfrak{D}$, given by the family of seminorms

$$A \longrightarrow \|A\|_\mathfrak{M} = \sup_{\varphi,\psi \in \mathfrak{M}} |\langle A\varphi,\psi\rangle| \quad \text{for all } t_\mathcal{A}\text{-bounded sets } \mathfrak{M} .$$

The same topology is also given by the seminorms $\sup_{\varphi \in \mathfrak{M}} |\langle A\varphi,\varphi\rangle|$ and we will freely use the same symbol $\|\cdot\|_\mathfrak{M}$ for one of both the seminorms.

The topology $\tau^c_\mathfrak{D}$ is given by the family of seminorms

$$A \longrightarrow \|A\|_\mathfrak{M} , \quad \mathfrak{M} \text{ runs over all relatively } t_\mathcal{A}\text{-compact subsets of } \mathfrak{D} .$$

The uniform topology introduced in /13/ is closely related to the order structure in Op*-algebras as it was pointed out in /24/.
In the unbounded case one has to distinguish between two notions of positivity of a linear functional. A linear functional ω on $\mathcal{A}(\mathfrak{D})$ is said to be

positive: if $\omega(A^+A) \geq 0$ for all $A \in \mathcal{A}(\mathfrak{D})$;

strongly positive: if $\omega(A) \geq 0$ for all $A \geq 0$, i.e., $\langle A\varphi,\varphi\rangle \geq 0$, for all $\varphi \in \mathfrak{D}$.

The problem of the continuity of strongly positive functionals was investigated in /24/,/26/.
Next we introduce some ideals in $\mathcal{L}^+(\mathfrak{D})$ /29/-/31/,/34/,/17/. It turns out that the ideals are very useful to investigate several problems in algebras of unbounded operators. A task of this is given in the next sections. Moreover, these ideals can be used to transform some problems from the unbounded to the bounded case .

To simplify the representation we restrict ourselves for the rest of the talk to the case that $\mathfrak{D}[t]$ is an (F)-space and $\mathcal{L}^+(\mathfrak{D})$ is selfadjoint. Put

$$\mathfrak{B}(\mathfrak{D}) = \{T: T\mathcal{H}\subset\mathfrak{D},\ T^*\mathcal{H}\subset\mathfrak{D}\} = \{T: AT,\ AT^* \text{ are bounded for all } A\in\mathcal{L}^+(\mathfrak{D})\}.$$

$$\mathcal{J}_\infty(\mathfrak{D}) = \{T: T\in\mathcal{J}_\infty(\mathcal{H}),\ T\mathcal{H}\subset\mathfrak{D},\ T^*\mathcal{H}\subset\mathfrak{D}\} = \{T: \overline{AT},\ \overline{AT^*}\in\mathcal{J}_\infty(\mathcal{H}) \text{ for all } A\in\mathcal{L}^+(\mathfrak{D})\}$$

$$\mathcal{J}_1(\mathfrak{D}) = \{T: \overline{AT},\ \overline{AT^*}\in\mathcal{J}_1(\mathcal{H}) \text{ for all } A\in\mathcal{L}^+(\mathfrak{D})\}.$$

These sets are two-sided $*$-ideals in $\mathcal{L}^+(\mathfrak{D})$. With the help of $\mathfrak{B}(\mathfrak{D})$ one can describe the bounded sets of $\mathfrak{D}[t]$, /14/,/11/,/12/:

Proposition 2.1

Let $\mathcal{K}$ be the unit ball in $\mathcal{H}$. Then the family

$$\{B\mathcal{K} : B = B^*\in\mathfrak{B}(\mathfrak{D})\}$$

is a fundamental system of t-bounded sets.

For applications it is useful to have the following

Corollary 2.2

The family

$$\{B\mathcal{K} : B = B^*\in\mathfrak{B}(\mathfrak{D}),\ \mathcal{R}(B) \text{ t-dense in } \mathfrak{D},\ B^{-1} \text{ exists}\}$$

is a fundamental system of t-bounded sets. $\mathcal{R}(B)$ denotes the range of B.

This corollary can be proved using some standard considerations about bounded sets in (F)-spaces and the proof of Proposition 2.1 given in /12/.

The ideal $\mathcal{J}_\infty(\mathfrak{D})$ gives a description of the relatively t-compact sets in $\mathfrak{D}$.

Proposition 2.3

The family

$$\{C\mathcal{K} : C = C^*\in\mathcal{J}_\infty(\mathfrak{D}),\ \mathcal{R}(C) \text{ t-dense in } \mathfrak{D},\ C^{-1} \text{ exists}\}$$

is a fundamental system of relatively t-compact sets in $\mathfrak{D}$.

The proof combines Corollary 2.2 with the following fact (unfortunately, the author could not find this result in the literature).

Lemma 2.4

Let $\mathcal{H}$ be a separable Hilbert space, then the family $\{C\mathcal{K},\ C = C^*\in\mathcal{J}_\infty(\mathcal{H}),\ \mathcal{R}(C)\ \|\ \|\text{-dense in } \mathcal{H},\ C^{-1} \text{ exists}\}$ is a fundamental system of relatively $\|\ \|$-compact sets in $\mathcal{H}$.

An easy but important consequence of Propositions 2.1 and 2.3 is summarized in the following Corollary (cf. /14/,/15/ for the case $\mathfrak{D} = \mathfrak{D}^\infty(N) \equiv \bigcap_{k\geq 0}\mathfrak{D}(N^k),\ N = N^*$).

Corollary 2.5

The topologies $\tau_\mathfrak{D}$ and $\tau^c_\mathfrak{D}$ can be given by the following seminorms:

$\tau_\mathfrak{D}$: $A \longrightarrow \|BAB\|$ for all $B\in\mathfrak{B}(\mathfrak{D})$;

$\tau^c_\mathfrak{D}$: $A \longrightarrow \|CAC\|$ for all $C\in\mathcal{J}_\infty(\mathfrak{D})$.

Moreover the operators B and C can be chosen as in Corollary 2.2, Proposition 2.3 resp..

In the next sections we need some factorization results for ideals in $\mathcal{L}^+(\mathfrak{D})$. A basic result in this direction was obtained in /11/:

Let $\mathfrak{J}(\mathcal{H})$ be a metrizable and complete ideal in $\mathfrak{B}(\mathcal{H})$, $\mathfrak{J}(\mathfrak{D})$ the corresponding ideal in $\mathcal{L}^+(\mathfrak{D})$ (see the definition of $\mathcal{J}_1(\mathfrak{D})$). Then $\mathfrak{J}(\mathfrak{D})$ factorizes as follows:

$$\mathfrak{J}(\mathfrak{D}) = \mathfrak{B}_\ell\cdot\mathfrak{J}(\mathcal{H})\cdot\mathfrak{B}_\ell^*,$$

where $\mathfrak{B}_\ell = \{T: T\mathcal{H}\subset\mathfrak{D}\}$.

This can be a little bit generalized, and we formulate one variant as a lemma.

Lemma 2.6

i) $\mathfrak{J}(\mathfrak{D})$ can be factorized as

$$\mathfrak{J}(\mathfrak{D}) = \mathfrak{B}(\mathfrak{D})\cdot\mathfrak{J}(\mathcal{H})\cdot\mathfrak{B}(\mathfrak{D}).$$

ii) $\mathcal{J}_1(\mathfrak{D})$ (and many other ideals) factorizes as

$$\mathcal{J}_1(\mathfrak{D}) = \mathfrak{B}(\mathfrak{D})\cdot\mathcal{J}_\infty(\mathcal{H})\cdot\mathcal{J}_1(\mathcal{H})\cdot\mathcal{J}_\infty(\mathcal{H})\cdot\mathfrak{B}(\mathfrak{D}).$$

Clearly, ii) follows from i) and well-known properties of nuclear operators (or about series). The property i) is a consequence of the following observation. $T\in\mathfrak{B}_\ell$ implies $TT^*\in\mathfrak{B}_\ell$ hence $TT^*\in\mathfrak{B}(\mathfrak{D})$. But this means $(TT^*)^{1/2}\in\mathfrak{B}(\mathfrak{D})$ /17/ or /30/. Then one uses the polar decomposition in the form $T = (TT^*)^{1/2}U$.

3. Normal states

For simplicity we will consider only normal functionals on $\mathcal{L}^+(\mathfrak{D})$. They can be defined with the help of the ideal $\mathcal{J}_1(\mathfrak{D})$. Namely, a linear functional ω is said to be normal, if

$$\omega(A) = \operatorname{Tr} TA \quad \text{for some } T\in\mathcal{J}_1(\mathfrak{D}) \text{ and all } A\in\mathcal{L}^+(\mathfrak{D}).$$

The normality of functionals, especially of states, was investigated by several authors /3/,/4/,/9/,/10/,/12/,/17/,/25/,/26/,/28/,/34/,/37/. We mention here only the most significant results.
In /34/ and independently in /4/ the connection between normality and continuity with respect to ultrastrong and ultraweak topologies in algebras of unbounded operators was established. Among other things the analogous result to ii) in section 2 was proved.
The results of Schmüdgen /25/,/26/ clarify the connection between normality of strongly positive functionals, continuity with respect to the topology $\tau^c_{\mathcal{D}}$ and the topological structure of $\mathcal{D}[t]$. Many former results can be derived from these as special cases. Let us quote some typical results from /25/,/26/:

i) Every strongly positive linear functional on $\mathcal{L}^+(\mathcal{D})$ is normal if and only if $\mathcal{D}[t]$ is a Montel space (i.e., any bounded set is relatively compact).

ii) A linear functional on $\mathcal{L}^+(\mathcal{D})$ is normal if and only if it is $\tau^c_{\mathcal{D}}$-continuous.

iii) If $\mathcal{D}[t]$ is a Montel space then any strongly positive linear functional on $\mathcal{L}^+(\mathcal{D})$ is $\tau^c_{\mathcal{D}}$-continuous.

The aim of iii) was to use ii) for one direction of i) instead of the long proof in /28/.
Results on normality of functionals in the context of the rigged Hilbert space $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}'$ are contained in /12/.

Now we will give an impression how these problems are connected with ideal theory. For example let us give an <u>alternative proof of ii)</u>

1. Let ω be $\tau^c_{\mathcal{D}}$-continuous on $\mathcal{L}^+(\mathcal{D})$, i.e.

$$|\omega(A)| \leq \|A\|_{\mathcal{M}} = \|CAC\|, \quad C \in \mathcal{S}_\infty(\mathcal{D}), \quad \mathcal{R}(C) \text{ dense in } \mathcal{D}[t].$$

Consider the functional $\hat{\omega}$ on $\mathcal{S} = \{X = CYC,\ Y \in \mathcal{L}^+(\mathcal{D})\} \subset$ $\subset \mathcal{S}_\infty(\mathcal{H})$ defined by

$$\hat{\omega}(X) = \hat{\omega}(CYC) = \omega(Y).$$

Then $|\hat{\omega}(X)| \leq \|X\|$, i.e., $\hat{\omega}$ is $\|\ \|$-continuous. $\mathcal{R}(C)$ $\|\ \|$-dense in $\mathcal{H}$ implies that $\mathcal{S}$ is dense in $\mathcal{S}_\infty(\mathcal{H})$ with respect to the operator norm. Thus, $\hat{\omega}$ has a continuous extension to $\mathcal{S}_\infty(\mathcal{H})$ also denoted by $\hat{\omega}$. But this means that $\hat{\omega}$ is normal,

$$\hat{\omega}(X) = \operatorname{Tr} SX, \quad S \in \mathcal{S}_1(\mathcal{H}),\ X \in \mathcal{S}_\infty(\mathcal{H}).$$

For $X = CYC$ we obtain

$$\hat{\omega}(X) = \omega(Y) = \operatorname{Tr} SCYC = \operatorname{Tr} CSCY = \operatorname{Tr} TY$$

for all $Y \in \mathcal{L}^+(\mathcal{D})$. It is easy to see that $T = CSC \in \mathcal{S}_1(\mathcal{D})$.
Therefore ω is normal on $\mathcal{L}^+(\mathcal{D})$.

2. Let ω be normal, i.e., $\omega(A) = \operatorname{Tr} TA$, $T \in \mathcal{S}_1(\mathcal{D})$. Lemma 2.6 ii) gives the factorization $T = CDREF$ with $C, F \in \mathcal{B}(\mathcal{D})$, $R \in \mathcal{S}_1(\mathcal{H})$, $D, E \in \mathcal{S}_\infty(\mathcal{H})$. Therefore,

$$|\omega(A)| = |\operatorname{Tr} TA| = |\operatorname{Tr} REFACD| \leq \|REFACD\|_1 \leq \|R\|_1 \cdot \|EFACD\|.$$

Furthermore $\|EFACD\| = \sup_{\varphi,\psi \in \mathcal{K}} |\langle EFACD\varphi, \psi\rangle| \leq \sup_{\varphi,\psi \in \mathcal{K}} |\langle ACD\varphi, F^*E^*\psi\rangle|$.
The sets $D\mathcal{K}$ and $E^*\mathcal{K}$ are relatively $\|\ \|$-compact. Since the operators from $\mathcal{B}(\mathcal{D})$ are continuous from $\mathcal{H}[\|\ \|]$ to $\mathcal{D}[t]$, the sets $CD\mathcal{K}$ and $F^*E^*\mathcal{K}$ are relatively t-compact and $\mathcal{M} = CD\mathcal{K} \cup$ $\cup F^*E^*\mathcal{K}$ is relatively t-compact, too. Thus, the estimation above gives

$$|\omega(A)| \leq \|R\|_1 \cdot \|A\|_{\mathcal{M}}$$

i.e., ω is $\tau^c_{\mathcal{D}}$-continuous.
Let us remark that this result gives also a characterization of the normal functionals on $\mathcal{B}(\mathcal{H})$.

4. Singular states

To extend the notion of singularity of a linear functional to the unbounded case one needs in $\mathcal{L}^+(\mathcal{D})$ an appropriate notion of "compact operators". By "appropriate" we mean that this set endowed with the topology $\tau_{\mathcal{D}}$ should have some of the good properties of $\mathcal{S}_\infty(\mathcal{H})$: It turns out that the $\tau_{\mathcal{D}}$-closure (in $\mathcal{L}^+(\mathcal{D})$) of the minimal ideal $\mathcal{F}(\mathcal{D})$ of the finite dimensional operators in $\mathcal{L}^+(\mathcal{D})$ is the best candidate for this. We use the notion $\mathfrak{C}(\mathcal{D}) = \overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$. $\mathfrak{C}(\mathcal{D})$ is the only $\tau_{\mathcal{D}}$-closed two-sided *-ideal in $\mathcal{L}^+(\mathcal{D})$, it consists of all operators mapping t-bounded sets in relatively t-compact sets, weakly convergent sequences in t-convergent sequences and so on (cf./12/, /21/,/29/,/31/,/34/).

<u>Definition 4.1</u>

A $\tau_{\mathcal{D}}$-continuous linear functional ω on $\mathcal{L}^+(\mathcal{D})$ is said to be singular if $\omega(A) = 0$ for all $A \in \mathfrak{C}(\mathcal{D})$.

In contrast to the C^*-case where positive functionals are automatically continuous we have to include a continuity condition in the definition to obtain reasonable results.
Our aim is to give here a description of all singular states on $\mathcal{L}^+(\mathcal{D})$ and an example of a decomposition result. Proofs and further properties of singular functionals can be found in /38/.
First let us recall some facts from the bounded case. It has often been mentioned that $\mathcal{B}(\mathcal{H})$, $\mathcal{S}_1(\mathcal{H})$, $\mathcal{S}_\infty(\mathcal{H})$ are the non-commutative analogs of l^∞, l^1 and c_0 (the zero-sequences).

The complex homomorphisms of l^{∞} are given by the elements of $\beta\mathbb{N}$ (the Stone-Čech-compactification of $\mathbb{N}$) which can be identified with ultrafilters on $\mathbb{N}$. There are two types of ultrafilters: fixed (consisting of all subsets of $\mathbb{N}$ containing a fixed element of $\mathbb{N}$) and free ultrafilters giving the element of $\beta\mathbb{N}\setminus\mathbb{N}$. The formula

$$\omega_{\mathfrak{U}}((x_n)) = \lim_{\mathfrak{U}} x_n \quad , (x_n)\in l^{\infty} \ , \ \mathfrak{U} \text{ - ultrafilter}$$

gives the complex homomorphisms of l^{∞}.

If $\mathfrak{U}$ is fixed at $k\in\mathbb{N}$, then

$$\omega_{\mathfrak{U}} = \omega_k \quad \text{with} \quad \omega_k((x_n)) = x_k \ .$$

If $\mathfrak{U}$ is free, then

$$\omega_{\mathfrak{U}}((x_n)) = 0 \text{ for all } (x_n)\in c_0 \ ,$$

i.e., the free ultrafilters give rise to singular states.

This is a guide to construct singular states on $\mathcal{B}(\mathcal{H})$.

Let (ψ_n) be a sequence of unit vectors in $\mathcal{H}$ weakly converging to zero. Then $\omega_{\mathfrak{U}}$ defined by

$$(1) \qquad \omega_{\mathfrak{U}}(A) = \lim_{\mathfrak{U}} \langle A\psi_n, \psi_n\rangle$$

gives a state on $\mathcal{B}(\mathcal{H})$ which is singular if $\mathfrak{U}$ is free. Moreover, Wils /36/ showed that every singular state has this form, namely, there is a fixed sequence of unit vectors (ψ_n) weakly converging to zero such that any singular state on $\mathcal{B}(\mathcal{H})$ is given by (1) with an appropriate $\mathfrak{U}$.

We call (ψ_n) a Wils-sequence. Let us remark that this sequence is not uniquely determined. For example, any sequence $(V\psi_n)$, V a unitary operator, also gives this result.

A little bit more is known. While the only normal pure states on $\mathcal{B}(\mathcal{H})$ are the vector states, pure singular states are obtained if one takes in (1) an orthonormal basis (φ_n). But the question of whether the converse statement is also true seems to be open up to now.

We will turn now to the unbounded case. Again, the idea is useful to transform the problem to the bounded case by means of the ideal $\mathcal{B}(\mathcal{D})$.

Let (ψ_n) be a Wils-sequence, $B\in\mathcal{B}(\mathcal{D})$, then

$$(2) \qquad \omega_{B,\mathfrak{U}}: \qquad \omega_{B,\mathfrak{U}}(A) = \lim_{\mathfrak{U}} \langle AB\psi_n, B\psi_n\rangle \quad , A\in\mathcal{L}^+(\mathcal{D})$$

gives a positive singular functional on $\mathcal{L}^+(\mathcal{D})$ (even a strongly positive one). To see this remark that the $\tau_{\mathcal{D}}$-continuity follows from

$$|\omega_{B,\mathfrak{U}}(A)| = |\lim_{\mathfrak{U}} \langle AB\psi_n, B\psi_n| \le \sup_{\varphi,\psi\in\mathcal{M}} |\langle A\varphi,\psi\rangle|$$

where $\mathcal{M} = \{B\psi_n\}$.

The singularity follows from the fact that $S\in\mathcal{C}(\mathcal{D})$ implies $BSB^{*}\in$ $\in \mathcal{J}_{\infty}(\mathcal{H})$ for all $B\in\mathcal{B}(\mathcal{D})$ /21/.

To get states one has to form

$$\tilde{\omega}_{B,\mathfrak{U}} = \omega_{B,\mathfrak{U}} / \omega_{B,\mathfrak{U}}(I).$$

Remark that the Cauchy-Schwarz-inequality

$$|\omega_{B,\mathfrak{U}}(A)|^2 \le \omega_{B,\mathfrak{U}}(I)\cdot\omega_{B,\mathfrak{U}}(A^+A)$$

implies $\omega_{B,\mathfrak{U}}\equiv 0$ if $\omega_{B,\mathfrak{U}}(I) = 0$. Hence, either $\omega_{B,\mathfrak{U}}\equiv 0$ or $\tilde{\omega}_{B,\mathfrak{U}}$ is well defined.

One can prove that (2) gives all positive singular functionals on $\mathcal{L}^+(\mathcal{D})$. More exactly,

Proposition 4.2

The positive singular functionals on $\mathcal{L}^+(\mathcal{D})$ are given by

$$\omega_{B,\mathfrak{U}}: \qquad \omega_{B,\mathfrak{U}}(A) = \lim_{\mathfrak{U}} \langle AB\psi_n, B\psi_n\rangle \quad , A\in\mathcal{L}^+(\mathcal{D}),$$

where $\mathfrak{U}$ is a free ultrafilter, (ψ_n) a Wils-sequence, B an appropriate operator from $\mathcal{B}(\mathcal{D})$ which can be taken to be selfadjoint and to have dense range in $\mathcal{D}[t]$.

Remark 4.3

This Proposition reflects also the fact that on $\mathcal{L}^+(\mathcal{D})$ with $\mathcal{D}[t]$ a Montel space there do not exist singular functionals. Indeed, $\mathcal{D}[t]$ is Montel if and only if $\mathcal{B}(\mathcal{D}) = \mathcal{J}_{\infty}(\mathcal{D})$. Hence $\mathcal{L}^+(\mathcal{D}) =$ $= \mathcal{C}(\mathcal{D})$ and any singular functional is zero on $\mathcal{L}^+(\mathcal{D})$.

The following Proposition is a variant of a decomposition result.

Proposition 4.4

Any $\tau_{\mathcal{D}}$-continuous linear functional ω on $\mathcal{L}^+(\mathcal{D})$ can be uniquely decomposed into the sum of a normal and a singular one.

Remark 4.5

The author does not know reasonable and well-founded arguments concerning the physical significance of singular states.

But if one has in mind that equilibrium states are closely connected with normal states (Gibbs-states are normal states and KMS-states are in some cases locally Gibbs-states), then one could be lead to the impression that singular states are in some sense (in which?) related to non-equilibrium.

5. Perturbation of states

While in the bounded case there exist several perturbation results (cf. e.g. /7/ and the references there) in the unbounded case only little is known. I will sketch here one direction of investigations. In connection with Gibbs-states one-parameter semigroups of the kind

$\{e^{-tH}\}_{t\geq 0}$, $H = H^*$ play an important role. To associate Gibbs-states with $\{e^{-tH}\}$ one must suppose that e^{-tH} is in $\mathcal{J}_1(\mathcal{H})$ (bounded case) or in $\mathcal{J}_1(\mathcal{D})$ (unbounded case) for the t under consideration. Then one is interested in the question for which $V = V^*$ $e^{-t(H+V)}$ is in $\mathcal{J}_1(\mathcal{D})$ and whether $e^{-t(H+V)}$ depends continuously on V (in a sense to be defined). To give an example of an exact result let us fix some notions (/18/,/22/,/32/,/33/).

Suppose $\mathcal{D} = \mathcal{D}^\infty(H) = \bigcap_{n\geq 0} \mathcal{D}(H^n)$, $H = H^* \geq I$. Put

$$OP_r(H) = \{A \in \mathcal{L}^+(\mathcal{D}) : \|H^s A \varphi\| \leq c_s \|H^{s+r}\varphi\| \text{ for all } \varphi \in \mathcal{D}, s \in \mathbb{R}\}$$

The operators $A \in OP_r(H)$ could be called "operators of order r" (more exactly: "operators of order $\leq$ r with respect to H"). It turns out that the Op^*-algebra $OP_0(H)$ is suitable to describe perturbations. First we quote here a generalization of the Trotter formula in the context of $\mathcal{L}^+(\mathcal{D})$. It is important that one can replace the strong operator convergence by the convergence with respect to t.

Proposition 5.1 /33/

Let $V = V^* \in OP_0(H)$. Then

i) $e^{-t(H+V)} \in \mathcal{L}^+(\mathcal{D})$;

ii) the Trotter formula with respect to the t-convergence is valid:

$$e^{-s(H+V)}\varphi = \operatorname*{t-lim}_{n\to\infty} \left(e^{-(s/n)H} e^{-(s/n)V}\right)^n \varphi \quad \text{for all } \varphi \in \mathcal{D}$$

The next proposition deals with the continuous dependence on V.

Proposition 5.2 /18/,/33/

Suppose that H^{-n} is nuclear for some $n \in \mathbb{N}$ and $V = V^* \in OP_0(H)$. Then

i) $e^{-t(H+V)} \in \mathcal{J}_1(\mathcal{D})$ for all $t > 0$;

ii) the map: $OP_0(H) \ni V \longrightarrow e^{-t(H+V)} \in \mathcal{J}_1(\mathcal{D})$ is continuous if one defines the topologies by means of the following sets of seminorms:

$OP_0(H)$: $V \longrightarrow \|H^k V H^k \varphi\|$ for all $\varphi \in \mathcal{D}$, $k = 0,1,2,\ldots$

$\mathcal{J}_1(\mathcal{D})$: $T \longrightarrow \|H^k T H^k\|$ for all $k = 0,1,2,\ldots$

(this is the so-called β^*-topology, cf. /16/).

Remark 5.3

The assumption $V \in OP_0(H)$ is rather restrictive for applications. So it would be very desirable to generalize Proposition 5.2 (and also Proposition 5.1). Some steps were done in /19/,/20/. There the matrix representation of V with respect to the eigenvectors of H was used.

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Received by Publishing Department
on February 11, 1985.

Тиммерман В. E5-85-91

Состояния на алгебрах неограниченных операторов

Дан обзор некоторых основных результатов по нормальным состояниям на алгебрах неограниченных операторов. Показано, как эти результаты связаны с теорией идеалов. Приведены несколько известных фактов относительно возмущения нормальных состояний, а также некоторые новые результаты по сингулярным состояниям.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Timmermann W. E5-85-91

States on Algebras of Unbounded Operators

There are reviewed some of the fundamental results on normal states on algebras of unbounded operators. It is indicated how these results are related with ideal theory. Few known facts concerning perturbation of normal states are included. There are contained some new results on singular states.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985