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JOINT DISTRIBUTIONS OF OBSERVABLES AND MEASURES WITH INFINITE VALUES

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1. INTRODUCTION

Let us suppose that the set, L, of all experimentally verifiable propositions of physical system forms a quantum logic. According to Varadarajan[1], assume that the quantum logic L is an orthomodular orthocomplemented σ-lattice with the minimal and maximal elements 0 and 1, respectively, and with an orthocomplementation $\perp$: $a \mapsto a^{\perp}$, $a, a^{\perp} \in L$, which satisfies (i) $(a^{\perp})^{\perp} = a$, for any $a \in L$; (ii) if $a < b$, then $b^{\perp} < a^{\perp}$; (iii) $a \vee a^{\perp} = 1$, for any $a \in L$; (iv) if $a < b$, then $b = a \vee (a^{\perp} \wedge b)$. Here $<$ denotes a partial ordering on L, and $\wedge$ and $\vee$ denote the meet and the join.

Two elements a and b of L are said to be (i) orthogonal and write $a \perp b$ if $a < b^{\perp}$: (ii) compatible and write $a \leftrightarrow b$, if there are three mutually orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$, $b = b_1 \vee c$.

Physical quantities are identified with the observables of the quantum logic. An observable on L is a map x from the set, $B(R_1)$, of all Borel measurable subsets of the real line R_1, into L such that (i) $x(R_1) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$ if $E_i \cap E_j = \emptyset, i \neq j$. An observable is bounded if there is a compact subset $C \subset R_1$ such that $x(C) = 1$. Two observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for any $E, F \in B(R_1)$.

Physical states are identified with the states of the quantum logic, that is, a state is a map $m: L \to [0,1]$ with (i) $m(1) = 1$; (ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \perp a_j$, $i \neq j$.

The more general notion as a state is a measure or a signed measure. So, we say that a map $m: L \to R_1 \cup \{-\infty\} \cup \{+\infty\}$ is said to be a signed measure on L if (i) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \perp a_j$, $i \neq j$; (ii) $m(0) = 0$; (iii) from the values $\pm\infty$ it attains only one; for the sake of definiteness we consider $+\infty$ as the possible value. The positive signed measure is called a measure.

An element a is a carrier of a measure m if $m(b) = 0$ iff $b \perp a$. It is clear that if a carrier of a measure exists, then it is unique. The signed measure m is (i) finite if $|m(a)| < \infty$, for any $a \in L$; (ii) σ-finite if there is a sequence of mutually orthogonal elements $\{a_i\}_{i=1}^{\infty}$ with $\bigvee_{i=1}^{\infty} a_i = 1$ and $|m(a_i)| < \infty$ for any i. An observable x is σ-finite with respect to a signed measure m if there

is sequence $\{E_i\}_{i=1}^{\infty} \subset B(R_1)$ such that $E_i \cap E_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} E_i = R_1$ and $|m(x(E_i))| < \infty$, $i \geq 1$.

We say that a signed measure m is continuous from below (above) on an element $a \in L$ if, for any $a_1 < a_2 < \dots$ with $\bigvee_{i=1}^{\infty} a_i = a$ ($a_1 > a_2 > \dots$ with $\bigwedge_{i=1}^{\infty} a_i = a$ and at least for one n_0 $|m(a_{n_0})| < \infty$) we have $m(a) = \lim_i m(a_i)$. Similarly as in[2] we may prove that a finitely additive function on L with $m(0) = 0$ is a signed measure iff m is continuous from below on any element of L, or, equivalently, m is continuous from above on the minimal element 0.

2. JOINT DISTRIBUTIONS

For an observable x, an event $x(E)$ denotes that the measured value, ξ, of the corresponding physical quantity lies in a Borel subset $E \in B(R_1)$. If a quantum mechanical system is described by a measure m, the expression

$$\mu^{m}_{x_1 \dots x_n}(E_1 \times \dots \times E_n) = m\left(\bigwedge_{j=1}^{n} x_j(E_j)\right), \quad E_j \in B(R_1),\ j = 1, \dots, n, \qquad (2.1)$$

denotes the measure of the simultaneous measurement of the observables $x_1, \dots, x_n$ which give measured quantities lying in the Borel subsets $E_i \in B(R_1)$, $i = 1, \dots, n$.

According to Gudder[3], we say the observables $x_1, \dots, x_n$ have a joint distribution in a measure m if there is a measure $\mu^{m}_{x_1 \dots x_n}$ on the set $B(R_n)$ of all the Borel subsets of R_n such that (2.1) holds.

Gudder[3] introduced the notion of the joint distribution only for a state (it is named type I joint distribution, too). This type has been studied in[5-12]. Urbanik[4] defined another type of a joint distribution in a state (type II joint distribution) for the summable self-adjoint operators in a Hilbert space, and Gudder[3] generalized this notion for bounded observables on a sum logic.

For given observables $x_1, \dots, x_n$ the function $\mu^{m}_{x_1 \dots x_n}$, defined on all measurable rectangles of $B(R_n)$ via (2.1), may be extended to a measure on $B(R_n)$ for (i) any measure m; (ii) only some measures; (iii) no measure. According to this, we may say that the observables $x_1, \dots, x_n$ are (i) compatible; (ii) partially compatible; (iii) incompatible. This characterization was investigated in[5,6].

If m is a state, then the joint distribution, if it exists, is determined unambiguously on $B(R_n)$. For a measure m with $m(1) = \infty$, the uniqueness must be studied in more detail.

The notion of joint distribution in a measure may be generalized to any set $\{x_t: t\in T\}$ of observables in a natural way: we say that observables $\{x_t: t\in T\}$ have a joint distribution in a measure m if any finite subset of $\{x_t: t\in T\}$ has one. The generalization of this notion to σ-homomorphisms defined on a measurable space $(X, \mathcal{S})$ is straightforward (here $\mathcal{S}$ is a σ-algebra of subsets of X and a map $x: \mathcal{S}\to L$ is a σ-homomorphism if (i) $x(X)=1$; (ii) $x(E)\perp x(F)$ if $E\cap F=\emptyset$; (iii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$, $\{E_i\}\subset\mathcal{S}$.

S.P. Gudder in [7] posed the following problem:

VII. Joint distribution. Can a joint distribution be defined for noncompatible observables? The answer to that problem for states has been obtained in the papers [5,6,13,14].

In the present note we solve this problem for measures with $m(1)=\infty$. The solution will contain the answer for measures on a Hilbert space logic, too.

In the sequel we suppose that the observables $x_1,\dots,x_n$ are given and for the joint distribution $\mu^m_{x_1\dots x_n}$ of $x_1,\dots,x_n$ in a measure m we shall write simply μ.

LEMMA 2.1. Let observables $x_1,\dots,x_n$ be compatible. Then, for any measure m on L, there is a joint distribution. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

Proof. For compatible observables $x_1,\dots,x_n$, there is a unique σ-homomorphism $x: B(R_n)\to L$ such that $x(R_1\times\dots\times E_i\times\dots\times R_1)=x_i(E_i)$, $i=1,\dots,n$; see [1, Th.6.17]. Let us put $\mu(B)=m(x(B))$, $B\in B(R_n)$. Then μ is a well defined joint distribution.

The uniqueness of the joint distribution follows from the uniqueness of the extension of σ-finite measures defined on the set of all rectangles of $B(R_n)$, [2]. Q.E.D.

Define

$$a(E_1,\dots,E_n) = \bigvee_{i_1\dots i_n=0}^{1} \bigwedge_{j=1}^{n} x_j({}^{i_j}E_j),\quad E_1,\dots,E_n\in B(R_1), \tag{2.2}$$

where ${}^0E = E^c = R_1 - E$, ${}^1E = E$.

We put (if it exists)

$$a_0 = \bigwedge\{a(E_1,\dots,E_n): E_1,\dots,E_n\in B(R_1)\}. \tag{2.3}$$

In the paper [13] it is shown that the element a_0 exists, and, moreover, there is a sequence $\{a(E_1^k,\dots,E_n^k)\}_{k=1}^{\infty}$ such that

$$a_0 = \bigwedge_{k=1}^{\infty} a(E_1^k,\dots,E_n^k). \tag{2.4}$$

The element a_0 is called a commutator of $x_1,\dots,x_n$, and the main properties of the commutator are investigated in [12,13].

LEMMA 2.2. Let $x_1,\dots,x_n$ have a joint distribution in m. Then

$$\left.\begin{aligned}&\text{(i)}\quad m(a(E_1,\dots,E_n)) = m(1),\quad E_1,\dots,E_n\in B(R_1),\\ &\text{(ii)}\quad m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\wedge\bigwedge_{k=1}^{K} a(E_1^k,\dots,E_n^k)\Big) = m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big),\end{aligned}\right\} \tag{2.5}$$

for any $E_1,\dots,E_n, E_1^k,\dots,E_n^k\in B(R_1)$, $k=1,\dots,K$, where K may be an integer or ∞.

Proof. Part (i)

$$m(1) = m\Big(\bigwedge_{j=1}^{n} x_j(R_1)\Big) = \mu(R_1\times\dots\times R_1) = \mu((E_1\cup E_1^c)\times\dots\times(E_n\cup E_n^c)) = \sum_{i_1\dots i_n=0}^{1}\mu({}^{i_1}E_1\times\dots\times{}^{i_n}E_n) = m(a(E_1,\dots,E_n)).$$

Part (ii)

$$m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big) \geq m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\wedge\bigwedge_{k=1}^{K} a(E_1^k,\dots,E_n^k)\Big) = m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\wedge$$

$$\wedge\bigwedge_{k=1}^{K}\bigvee_{i_1\dots i_n=0}^{1}\bigwedge_{j=1}^{n} x_j({}^{i_j}E_j^k)\Big) \geq m\Big(\bigvee_{i_1\dots i_n=0}^{1}\bigwedge_{i=1}^{n} x_i(E_i)\wedge$$

$$\wedge\bigwedge_{k=1}^{K}\bigwedge_{j=1}^{n} x_j({}^{i_j}E_j^k)\Big) = m\Big(\bigvee_{i_1\dots i_n=0}^{1}\bigwedge_{j=1}^{n} x_j\big(E_j\cap\bigcap_{k=1}^{K}{}^{i_j}E_j^k\big)\Big) =$$

$$= \sum_{i_1\dots i_n=0}^{1}\mu\Big(\prod_{j=1}^{n}\big(E_j\cap\bigcap_{k=1}^{K}{}^{i_j}E_j^k\big)\Big) = \mu(E_1\times\dots\times E_n) = m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big).$$

Q.E.D.

COROLLARY 2.3. If $x_1,\dots,x_n$ have a joint distribution in m, then, for the commutator a_0, we have

$$m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\wedge a_0\Big) = m\Big(\bigwedge_{i=1}^{n} x_i(E_i)\Big), \tag{2.6}$$

for any $E_1,\dots,E_n\in B(R_1)$.

$$m(a_0) = m(1), \tag{2.7}$$

Proof. (2.6) follows from Lemma 2.2 and (2.4). For (2.7) it is sufficient to put $E_1=E_2=\dots=E_n=R_1$. Q.E.D.

LEMMA 2.4. Let $x_1,\dots,x_n$ have a joint distribution in a measure m. If there are $E\in B(R_1)$ and x_i such that $m(x_i(E))<\infty$, then

$$m(x_i(E)\wedge a_0^{\perp}) = 0. \tag{2.8}$$

Proof. From the results of the paper [13] follows that $a_0^{\perp}\leftrightarrow x_j(F)$ for any $F\in B(R_1)$ and any $j=1,\dots,n$. Hence $a_0^{\perp}\leftrightarrow x_i(E)$ and from

(2.6) we have

$$m(x_i(E)) = m(x_i(E) \wedge a_o) + m(x_i(E) \wedge a_o^{\perp}) = m(x_i(E)) + m(x_i(E) \wedge a_o^{\perp}),$$

consequently, (2.8) holds. Q.E.D.

LEMMA 2.5. Let $x_1, \dots, x_n$ have a joint distribution in a measure m. If at least one observable is σ-finite with respect to m, then

$$m(a_o^{\perp}) = 0. \qquad (2.9)$$

Proof. Let $\{E_n\}_{n=1}^{\infty} \subset B(R_1)$ be a sequence with $E_i \cap E_j = \emptyset, i \neq j$, $\bigcup_{n=1}^{\infty} E_n = R_1$, and, for some x_i, $|m(x_i(E_n))| < \infty$, $n \geq 1$. Since $a_0 \leftrightarrow x_i(E_n)$, for any n, then, due to [1, Lemma 6.10] $a_o^{\perp} \wedge \bigvee_{n=1}^{\infty} x_i(E_n) = \bigvee_{n=1}^{\infty} (a_o^{\perp} \wedge x_i(E_n))$. Check

$$m(a_o^{\perp}) = m(a_o^{\perp} \wedge 1) = m(a_o^{\perp} \wedge \bigvee_{n=1}^{\infty} x_i(E_n)) = \sum_{n=1}^{\infty} m(a_o^{\perp} \wedge x_i(E_n)) = 0,$$

when we use (2.8). Q.E.D.

THEOREM 2.6. Let $x_1, \dots, x_n$ be observables and let m be a measure. If (2.9) holds, then there is a joint distribution of $x_1, \dots, x_n$ in a measure m. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

If $x_1, \dots, x_n$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then (2.9) holds.

Proof. The first part of Theorem follows from the following. Let a_o be a commutator of $x_1, \dots, x_n$. Then, according to [13] $x_{io}(E) = x_i(E) \wedge a_o$, $E \in B(R_1)$, $i = 1, \dots, n$, defines an observable x_{io}, $i = 1, \dots, n$, in a quantum logic $L_{(0,a_o)} = \{b : b \in L, b < a_o\}$ (here the greatest element is a_o, an orthocomplementation "′" is defined via $b' = b^{\perp} \wedge a_o (b < a_o)$). Moreover, $x_{10}, \dots, x_{1n}$ are mutually compatible observables. Hence, due to Lemma 2.1, $x_{10}, \dots, x_{no}$ have a joint distribution in a measure $m_o = m|L_{(0,a_o)}$. From (2.9) we have

$$m(\bigwedge_{i=1}^{n} x_i(E_i)) = m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_o) + m(\bigwedge_{i=1}^{n} x_i(E_i) \wedge a_o^{\perp}) = m_o(\bigwedge_{i=1}^{n} x_{io}(E_i)),$$

which entails that $x_1, \dots, x_n$ have a joint distribution in m.

Repeating the same arguments as those in the proof of Lemma 2.1 we establish the uniqueness of a joint distribution.

The second part of the assertion of Theorem follows from Lemma 2.5. Q.E.D.

LEMMA 2.7. Let a_m be a carrier of a measure m. If $x_1, \dots, x_n$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then

$$a_m < a_o, \qquad (2.10)$$

and

$$a_m < a(E_1, \dots, E_n), \text{ for any } E_1, \dots, E_n \in B(R_1). \qquad (2.11)$$

If (2.10) holds, or, equivalently, (2.11) is true, then $x_1, \dots, x_n$ have a joint distribution in m. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

Proof. (2.10) and (2.11) follow from the definition of a carrier, Theorem 2.6 and (2.4). Q.E.D.

Note 1. The condition

$$m(a(E_1, \dots, E_n)^{\perp}) = 0, \quad \text{for any } E_1, \dots, E_n \in B(R_1), \qquad (2.12)$$

is necessary and sufficient condition for $x_1, \dots, x_n$ to have a joint distribution in a state (or finite measure) [5,6,13]. For a measure with $m(1) = \infty$ this condition is known only in special cases, see Lemma 2.7.

LEMMA 2.8. Let a logic L be σ-continuous, that is, for any $a_1 < a_2 < \dots$ and, any a, we have $(\bigvee_{i=1}^{\infty} a_i) \wedge a = \bigvee_{i=1}^{\infty} (a_i \wedge a)$. Let there hold for a measure m and observables $x_1, \dots, x_n$

$$m(\bigwedge_{j=1}^{n} x_j(E_1^j \cup E_2^j)) = \sum_{k_1 \dots k_n = 1}^{2} m(\bigwedge_{j=1}^{n} x_j(E_{k_j}^j)),$$
$$E_1^j \cap E_2^j = \emptyset, \; E_1^j, E_2^j \in B(R_1), \; j = 1, \dots, n. \qquad (2.13)$$

If at least one observable is σ-finite with respect to m, then there is a unique joint distribution of $x_1, \dots, x_n$ in m.

Proof. It is easy to verify that (2.13) implies that $\mu : E_1 \times \dots \times E_n \mapsto m(\bigwedge_{j=1}^{n} x_j(E_j))$, is a finitely additive function on the set $\mathcal{P}_n$ of all rectangles. The σ-continuity of a logic and the continuity of m from below entail that μ is a σ-additive and σ-finite function on $\mathcal{P}_n$. Therefore it may be extended to a measure on $B(R_n)$. Q.E.D.

The results of all the above assertions may be extended to the set of observables $\{x_t : t \in T\}$ such that there is at most countable subset $\mathcal{Q} \subset \bigcup\{R(x_t) : t \in T\}$, where $\mathcal{Q}$ generates the minimal sublogic of L containing the set $\bigcup\{R(x_t) : t \in T\}$ (here $R(x) := \{x(E) : E \in B(R_1)\}$. In particular, this is true for a sequence of observables. For given observables $\{x_t : t\in T\}$ we define the commutator, $a_0(T)$, of $\{x_t : t\in T\}$ (if it exists) via

$$a_0(T) = \wedge\{a_0(F) : F \text{ is a finite subset of } T\}, \tag{2.14}$$

where $a_0(F)$ is the commutator of observables $x_{t_1}, \dots, x_{t_s}$ and $F = \{t_1, \dots, t_s\}$.

From [13] it follows that $a_0(T)$ exists, and, moreover, there is a sequence of finite subsets $F_n \subset T$ such that

$$a_0(T) = \bigwedge_{n=1}^{\infty} a_0(F_n). \tag{2.15}$$

THEOREM 2.9. Let $\{x_t : t\in T\}$ be a system of observables for which there is at most countable subset $\mathcal{A} \subset \cup\{R(x_t) : t\in T\}$, where $\mathcal{A}$ generates the minimal logic containing all $R(x_t)$, $t\in T$. If $\{x_t : t\in T\}$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then

$$m(a_0(T)^{\perp}) = 0. \tag{2.16}$$

If (2.16) holds, then there is a joint distribution of $\{x_t : t\in T\}$. If at least one observable is σ-finite with respect to m, then there is a unique σ-finite measure μ on $\prod_{t\in T} B(R_1)$ such that

$$\mu\Big(\bigcap_{j=1}^{n} \pi_{t_j}^{-1}(E_j)\Big) = m\Big(\bigwedge_{j=1}^{n} x_{t_j}(E_j)\Big), \quad E_1, \dots, E_n \in B(R_1), \tag{2.17}$$

where π_t is the projection from R_1^T onto R_1.

Proof. It is clear that if $F_1 \subset F_2 \subset T$, then $a_0(F_2) < a_0(F_1)$. Let x_{t_0} be σ-finite with respect to m. Then (2.15) implies

$$a_0(T) = \bigwedge_{n=1}^{\infty} a_0(F_n) > \bigwedge_{n=1}^{\infty} a_0(F_n \cup \{t_0\}) > \bigwedge_{n=1}^{\infty} a_0\Big(\bigcup_{i=1}^{n} (F_i \cup \{t_0\})\Big) > a_0(T).$$

Theorem 2.6 entails $m(a_0(B_n)^{\perp}) = 0$, $n \geq 1$, where $B_n = \bigcup_{i=1}^{n} F_i \cup \{t_0\}$. The continuity of m from below gives (2.16).

Conversely, let (2.16) hold. Then, for any finite subset $F\subset T$, we have $m(a_0(F)^{\perp}) = 0$. Now we claim to show that there is a unique μ on $\prod_{t\in T} B(R_1)$ for which (2.17) holds. Let x_{t_0} be σ-finite with respect to m, and let for some $E \in B(R_1)$ have $0 < m(x_{t_0}(E)) < \infty$. Define a system of functions, $\{\mu_F^E : F$ is a finite subset of $T\}$, on $\prod_{t\in T} B(R_1)$ via

$$\mu_F^E\Big(\bigcap_{j=1}^{n} \pi_{t_j}^{-1}(E_j)\Big) = m\Big(x_{t_0}(E) \wedge \bigwedge_{j=1}^{n} x_{t_j}(E_j)\Big), \tag{2.18}$$

where $E_1, \dots, E_n \in B(R_1)$, $F = \{t_1, \dots, t_n\}$. The system $\{\mu_F^E : F$ is a finite subset of $T\}$ fulfills the conditions of Kolmogorov's

consistence theorem [23], hence there a unique measure μ^E on $\prod_{t\in T} B(R_1)$ with (2.18). Define $\mu(B) = \sum_{i=1}^{\infty} \mu^{E_i}(B)$, where $B \in \prod_{t\in T} B(R_1)$ and $\{E_i\}_{i=1}^{\infty}$ is a measurable partition of R_1 with $0 < m(x_{t_0}(E_i)) < \infty$, $i \geq 1$. The function μ is well defined and it is σ-additive and σ-finite. It is easy to check that (2.17) is fulfilled. The uniqueness of μ follows from the extension theorem for σ-finite measure on the set of all cylindrical sets. Q.E.D.

Analogically we may prove Lemma 2.7 for the case described in Theorem 2.9; it suffices to change a_0 to $a_0(T)$.

The proofs of the following two lemmas are simple and they are omitted.

LEMMA 2.10. Let at least one observable x_i, $i=1,\dots,n$, be σ-finite with respect to m. Then $x_1,\dots,x_n$ have a joint distribution in m iff $f_1 \circ x_1, \dots, f_n \circ x_n$ have it for all the Borel measurable real-values functions, where $f\circ x(E) := x(f^{-1}(E))$, $E \in B(R_1)$. In this case there holds

$$\mu^m_{f_1\circ x_1 \dots f_n \circ x_n}(E_1 \times \dots \times E_n) = \mu^m_{x_1 \dots x_n}(f_1^{-1}(E_1) \times \dots \times f_n^{-1}(E_n)).$$

LEMMA 2.11. Let M be a collection of measures on L and let a measure m_0 be a superposition of M, i.e., $m(a) = 0$, for any $m \in M$, implies $m_0(a) = 0$. Let, for any $m\in M$ and m_0, there be at least one observable which is σ-finite with respect to them. If $x_1, \dots, x_n$ have a joint distribution in any $m \in M$, then they have a joint distribution in m_0.

3. HILBERT SPACE LOGIC

One of the most important examples of quantum logics is the set, $L(H)$, of all closed subspaces of a Hilbert space of H over the real or complex field C. This is a case of the great importance in quantum mechanics. In this case observables may be identified with self-adjoint operators (not necessarily bounded), according to the spectral theorem.

The famous Gleason theorem [15] asserts that a state m on a separable Hilbert space H, dim $H \geq 3$, is induced by a positive von Neumann operator T via the formula

$$m(P) = tr(TP), \quad P \in L(H). \tag{3.1}$$

Here we identify the subspace P with its orthoprojector T^P onto P. We recall that a bounded operator T on H is said to be an operator with finite trace if $tr(T) := \sum_{a\in I} (Tx_a, x_a)$ is absolutely convergent series, independent of the used orthonormal basis $\{x_a : a \in I\}$.

The Gleason theorem has been generalized in [16,17] for all bounded signed measures on L(H) for a separable Hilbert space whose dimension is at least 3. Eilers and Horst [18] proved Gleason's theorem for finite measures on L(H) for non-separable Hilbert space, and Drisch [19] extended (3.1) for bounded signed measures on a logic L(H) of a non-separable Hilbert space whose dimension is a non-real measurable cardinal.

For the measures on L(H) with $m(H)=\infty$ we need the following notions. A bilinear form is a function $t: D(t)\times D(t)\to C$, where $D(t)$ is a linear submanifold of H (named the domain of t) such that t is linear in the first argument and antilinear in the second one. If $t(x,y)=t(y,x)$, for all $x,y\in D(t)$, then t is said to be symmetric; if for a symmetric bilinear form t we have $t(x,x)\geq 0$, then t is said to be positive. Let t be a symmetric bilinear form and $B\geq 0$ be a self-adjoint operator. Then $t\circ B$ denotes a symmetric bilinear form defined via $t\circ B(x,y)=t(B^{1/2}x,B^{1/2}y)$, when the corresponding assumptions on the domains of t and $B^{1/2}$ are satisfied. Symmetric bilinear form is said to be a bilinear form with finite trace if (i) $D(t)=H$; (ii) $t(x,y)=(Tx,y)$, for all $x,y\in H$, where T is an operator with finite trace. We put $\mathrm{tr}\, t=\mathrm{tr}(T)$, and we write $t\in \mathrm{tr}(H)$, where $\mathrm{tr}(H)$ is the set of all bounded operators with finite trace.

Lugovaja and Sherstnev [20] proved that, for any σ-finite measure m on L(H) of an infinite-dimensional separable Hilbert space there is a unique symmetric positive bilinear form t with a dense domain such that

$$m(P)=\begin{cases}\mathrm{tr}\, t\circ P & \text{if } t\circ P\in \mathrm{tr}(H),\\ \infty & \text{otherwise.}\end{cases}\qquad (3.2)$$

In the paper [21] this result has been extended to σ-finite f-bounded measures on L(H) of a Hilbert space whose dimension is a non-real measurable cardinal.

The joint distribution of observables on L(H) in a state has been studied in [3,5]. It was proved that $x_1,\dots,x_n$ have a joint distribution in a state m induced by $T\in \mathrm{tr}(H)$ via (3.1) iff

$$x_{i_1}(E_{i_1})\dots x_{i_n}(E_{i_n})T=x_1(E_1)\dots x_n(E_n)T,\qquad (3.3)$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and all $E_1,\dots,E_n\in B(R_1)$.

In the following we shall study the existence of a joint distribution for a measure m on L(H) with $m(H)=\infty$, and the condition analogous to (3.3) will be proved. First of all we begin with a finite-dimensional Hilbert space.

LEMMA 3.1. (Lugovaja-Sherstnev [20]). Let $\dim H=3$ and let m be a measure on L(H) with $m(H)=\infty$. If there are a one-dimensional Q and a two-dimensional P with $m(Q)<\infty$, $m(P)<\infty$, then $Q<P$.

Denote

$$P_m=\vee\{P: m(P)<\infty\}.\qquad (3.4)$$

The following lemma has been proved in [21].

LEMMA 3.2. Let $3\leq \dim H\leq\infty$ and let m be a measure with $m(H)=\infty$. If there is a two-dimensional Q_0 with $m(Q_0)<\infty$, then $m(Q)<\infty$ iff $Q<P_m$.

LEMMA 3.3. Let $4\leq \dim H<\infty$ and let m be a measure with $m(H)=\infty$. Let there be a three-dimensional Q_0 with $m(Q_0)<\infty$. If $m(M)=m(N)=0$, then $m(M\vee N)=0$.

Proof. Due to Lemma 3.2, $m(Q)<\infty$ iff $Q<P_m$. Hence $m(M\vee N)<\infty$. Applying the Gleason theorem to $m_0:=m|L_{(0,P_m)}=m|L(P_m)$ we see that $m(M\vee N)=0$. Q.E.D.

LEMMA 3.4. Let the conditions of Lemma 3.3 are fulfilled. Then any measure m on L(H) has a carrier.

Proof. Let us denote $\mathfrak{M}=\{P: m(P)=0\}$. It is clear that (i) $\mathfrak{M}\neq\emptyset$; (ii) if $Q<P$, $P\in\mathfrak{M}$, then $Q\in\mathfrak{M}$; (iii) if $P\perp Q$ and $P,Q\in\mathfrak{M}$, then $P\vee Q\in\mathfrak{M}$; (iv) if P_x and $P_y\in\mathfrak{M}$, then $P_x\vee P_y\in\mathfrak{M}$, where P_x denotes the one-dimensional subspace generated by a non-zero vector $x\in H$. Let us put $P_m^0=\vee\{P: m(P)=0\}$. Then from Lemma 3.3 and (i)-(iv) we have that $m(P_m^0)=0$. Define $A_m=P_m^{0\perp}$. Then A_m is a carrier of a measure m. Q.E.D.

We recall that a subset $\mathfrak{M}\subset L(H)$ with (i)-(iv), from the last proof, is said to be called an ideal.

THEOREM 3.5. Let the conditions of Lemma 3.3 are fulfilled. If, for $x_1,\dots,x_n$ we have

$$x_{i_1}(E_{i_1})\dots x_{i_n}(E_{i_n})A_m=x_1(E_1)\dots x_n(E_n)A_m,\qquad (3.5)$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and any $E_1,\dots,E_n\in B(R_1)$, where A_m is a carrier of a measure m, then $x_1,\dots,x_n$ have a joint distribution in m. Moreover, the condition (3.5) is equivalent to

$$A_{x_{i_1}}\dots A_{x_{i_n}}A_m=A_{x_1}\dots A_{x_n}A_m,\qquad (3.6)$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$, where A_x is an Hermitean operator corresponding to an observable x.

Proof. It is known [22] that (3.5) implies $(x_1(E_1)\wedge\dots\wedge x_n(E_n))A_m=x_1(E_1)\dots x_n(E_n)A_m$. Hence

$$a(E_1,\dots,E_n)A_m=\sum_{i_1\dots i_n=0}^{1}x_1(^{i_1}E_1)\dots x_n(^{i_n}E_n)A_m=IA_m=A_m,$$

where I is the identical operator on H. Therefore $a(E_1,\dots,E_n) > A_m$, for all $E_1,\dots,E_n$, consequently, $A_0 > A_m$, where A_0 is the commutator of $x_1,\dots,x_n$ and $m(A_0) = 0$. Repeating the first part of the proof of Theorem 2.6 we finish our proof. Q.E.D.

We see that measures with $m(H) = \infty$ on finite-dimensional Hilbert space are in some sense "pathological". More useful information we may obtain in an infinite-dimensional separable Hilbert space.

LEMMA 3.6. Any σ-finite measure on $L(H)$ of an infinite-dimensional separable Hilbert space has a carrier.

Proof. If $m(H) < \infty$, then the assertion follows immediately from Gleason's theorem.

Let now $m(H) = \infty$. Define $\mathfrak{M} = \{P : m(P) = 0\}$. We claim to show that $\mathfrak{M}$ is an ideal of $L(H)$. For that it is necessary to show that if $P_x, P_y \in \mathfrak{M}$, then $P_x \vee P_y \in \mathfrak{M}$. We may limit ourselves with $P_x \not\perp P_y$, $P_x \neq P_y$. The σ-finiteness of m entails that there is at least one three-dimensional P such that $m(P) < \infty$ and $Px \neq 0$, $Py \neq 0$. Then there is $z \in P$ such that $z \perp x$ and $z \perp y$. Applying the Lugovaja-Sherstnev lemma to a three-dimensional space $P = P_z \vee P_x \vee P_y$ we have that $m(P_x \vee P_y) < \infty$, and, consequently, $M(P) < \infty$, too. Using the Gleason theorem for a finite measure $m_0 = m \backslash L(P)$ we have $m(P_x \vee P_y) = 0$.

Now we show that if $P_{y_1},\dots,P_{y_n} \in \mathfrak{M}$, then $P = P_{y_1} \vee \dots \vee P_{y_n} \in \mathfrak{M}$. Lemma 3.2 implies that $m(P) < \infty$ and Lemma 3.3 entails that $m(P) = 0$.

Define the submanifold D generated by an ideal $\mathfrak{M}$ via $D = \{x : P_x \in \mathfrak{M}\} \cup \{0\}$ and let M be a subspace of H generated by D. Then $M = \vee\{P : m(P) = 0, \dim P < \infty\}$. The separability of a Hilbert space implies that there is a sequence of finite-dimensional subspaces of H, $\{P_n\}_{n=1}^{\infty}$, with $m(P_n) = 0$, such that $M = \bigvee_{n=1}^{\infty} P_n$. P_n may be chosen such that $P_1 < P_2 < \dots$. The continuity of m from below entails $m(M) = 0$. The element $A_m = M^{\perp}$ is a carrier of a measure m.

Q.E.D.

Note 2. The author does not know whether Lemma 3.6 holds for a non-separable Hilber space whose dimension is a non-real measurable cardinal. For that it is necessary and sufficient to show that $m(M) < \infty$. For more details, see the proof of Lemma 3.9.

The following elementary Lemma has been proved in /5/.

LEMMA 3.7. Let $M_1,\dots,M_n \in L(H)$, where H is an arbitrary Hilber space. Let $(i_1,\dots,i_n)$ be any permutation of $(1,\dots,n)$. If $0 \neq f \in {}^{i_1}M_1 \wedge \dots \wedge {}^{i_n}M_n$, where ${}^0M = M$, ${}^1M = M$, then

$$M_{j_1} \dots M_{j_n} f = M_1 \dots M_n f, \tag{3.7}$$

for any permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$.

THEOREM 3.8. Let H be an infinite-dimensional separable Hilber space. If $x_1,\dots,x_n$ have a joint distribution in m and at least one observable is σ-finite with respect to m, then (3.5) holds (for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$). If $x_1,\dots,x_n$ are bounded observables, then (3.6) holds.

If m is σ-finite and for $x_1,\dots,x_n$ there holds (3.5), then $x_1,\dots,x_n$ have a joint distribution in m. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

Proof. Since at least one observable is σ-finite with respect to m, we see that m is σ-finite measure, consequently, the carrier of m exists. Due to Lemma 2.7 $A_m < A_0 < a(E_1,\dots,E_n)$, where A_0 is the commutator of $x_1,\dots,x_n$ defined by (2.4). Therefore if $f \in A_m$, then $f \in a(E_1,\dots,E_n)$ and f is a finite linear combination of vectors from $x_1({}^{j_1}E_1) \wedge \dots \wedge x_n({}^{j_n}E_n)$ for any permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$. Due to Lemma 3.7,

$$x_{i_1}(E_{i_1}) \dots x_{i_n}(E_{i_n}) f = x_1(E_1) \dots x_n(E_n) f,$$

for any permutation of $(i_1,\dots,i_n)$ of $(1,\dots,n)$, and, consequently, (3.5) holds.

For bounded observables, (3.6) is a consequence of the spectral theorem for Hermitean operators.

The second part of the proof is analogous to the proof of Theorem 3.5. Q.E.D.

In the following the previous Theorem will be extended to a non-separable Hilbert space. We recall that a cardinal I is said to be non-real measurable if there is no positive measure ν, $\nu \neq 0$, on the power set of I with $\nu(\{a\}) = 0$, for each $a \in I$.

LEMMA 3.9. Let H be a Hilbert space whose dimension is a non-real measurable cardinal. Let m be a measure on $L(H)$ with $m(H) = \infty$. Let us put $A^{\perp} = \vee\{P : m(P) = 0\}$. If at least one observable is σ-finite with respect to m and $x_1,\dots,x_n$ have a joint distribution in m, then

$$x_{i_1}(E_{i_1}) \dots x_{i_n}(E_{i_n}) A = x_1(E_1) \dots x_n(E_n) A, \tag{3.8}$$

for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$ and all $E_1,\dots,E_n \in B(R_1)$.

If $m(A^{\perp}) < \infty$, m is σ-finite and (3.8) holds, then $x_1,\dots,x_n$ have a joint distribution in m. If at least one observable is σ-finite with respect to m, then the joint distribution is unique.

Proof. The first part of the theorem is similar to that in Theorem 3.8.

In the second part we show that $m(A^{\perp}) < \infty$ implies $m(A^{\perp}) = 0$, that is, A will be a carrier of m. The generalized Gleason theorem for a non-separable Hilbert space /21/ entails that there is a unique operator $T \in tr(H)$ such that $m(P) = tr(TP)$ whenever $P < A^{\perp}$. The operator T has a form $T = \sum_i \lambda_i f_i \otimes \bar{f}_i$, where $f_i \perp f_j, i \neq j$, $\|f_i\| = 1, f_i \in H, \lambda_i > 0$, for any i, $f \otimes \bar{f} : x \to (x,f)f, x \in H$. Hence $m(P) = 0$ iff $P \perp f_i$ for any i (here $P \perp f_i$ denotes that $x \perp f_i$, for all $x \in P$). Hence $A^{\perp} \perp f$, for any i, so that, $m(A^{\perp}) = 0$. For the rest of the proof we appeal Lemma 2.7. Q.E.D.

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Совместные распределения наблюдаемых и меры с бесконечными значениями

В рамках подхода квантовых логик к аксиоматизации квантовой механики изучаются совместные распределения наблюдаемых в мерах, принимающих бесконечные значения. Предложены необходимые и достаточные условия для существования совместных распределений. Модель квантовой логики пространства Гильберта изучается более подробно.

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Joint Distribution of Observables and Measures with Infinite Values

In the frame of the quantum logic approach to axiomatization of quantum mechanics we study the joint distribution of observables in measures attaining infinite values. The necessary and sufficient conditions to existence of the joint distribution are given. The Hilbert space quantum logic model is investigated in more detail.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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