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DIRECT EVALUATION OF THE IWASAWA AND TRIANGLE DECOMPOSITIONS FOR THE REAL FORMS OF LIE ALGEBRAS gl(n+1,C)

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1. Introduction

The Iwasawa[1] and triangle [2] decompositions play the crucial role in the construction of infinite-dimensional unitary irreducible representations of noncompact semisimple Lie groups by the induced representation method[3]. Explicit forms of this decompositions have been constructed by many authors for particular examples of real semisimple Lie algebras[4]. The usual indirect method for evaluation of the Iwasawa decomposition of a noncompact semisimple Lie algebra involves search for solutions to certain simultaneous eigenvalue type equations; for details see the book by Hermann[5]. The direct method has been formulated by Cornwell[6]. It is based on the canonical form of Lie algebra and the fact that for any two Cartan subalgebras, there exists an inner automorphism which maps one into the other. This inner automorphism, however must be guessed for each particular case.

The present paper gives the Iwasawa decomposition for real forms of Lie algebras gl(n+1,ℂ) in a much more simply way. For any real form of the algebra gl(n+1,ℂ), we construct the above mentioned Cartan automorphism; explicit form of this automorphism then specifies the decompositions directly. Organisation of the paper is the following. In Section 2, we describe in general terms the construction of the decompositions in the case when the Cartan automorphism which gives real forms is defined explicitly. For the algebras gl(n+1,ℂ), we give explicit forms of the Cartan automorphism in Section 3. In the last section, these automorphisms will be used for construction of the Iwasawa and triangle decompositions.

2. The Iwasawa and triangle decompositions

In this section, we have given a brief survey of the Iwasawa and triangle decompositions. We followed Chapter 5 of Helgasson[7] and Chapter 3,§3 of Zhelobenko and Stern[2].

Let $\tilde{g}$ be a semisimple Lie algebra. We denote by $\tilde{h}$ a Cartan subalgebre $\tilde{g}$ and by Δ the root system of $\tilde{g}$ with respect $\tilde{h}$. Any semisimple Lie algebra has a Cartan-Weyl basis $\{ H_1, \dots, H_n\ E_\alpha\ ;\ \alpha \in \Delta \}$ for which:

$$\{ H_1, \dots, H_n \} \quad \text{is a basis in } \tilde{h} \tag{1a}$$

$$[h, E_\alpha] = \alpha(h) E_\alpha \tag{1b}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad (N_{\alpha\beta} = -N_{-\alpha,-\beta}) \text{ for } \alpha+\beta \in \Delta \tag{1c}$$

$$B(E_\alpha, E_{-\alpha}) = 1, \tag{1d}$$

where $B(\cdot,\cdot)$ is a Killing form on $\tilde{g}$.

Using this Cartan-Weyl basis of $\tilde{g}$ we can define an antilinear mapping on $\tilde{g}$ by

$$\psi(H_i) = -H_i \qquad \text{for } i=1,2,\dots,n \tag{2a}$$

$$\psi(E_\alpha) = -E_{-\alpha} \qquad \text{for all } \alpha \in \Delta . \tag{2b}$$

This mapping is the involutive antilinear automorphism on $\tilde{g}$ and the real form corresponding to

$$g_\psi = \{ X \in \tilde{g};\ \psi(X) = X \} \tag{3}$$

is compact.

A linear automorphism is called a Cartan automorphism if

$$\theta^2 = 1 \tag{4a}$$

$$\theta \cdot \psi = \psi \cdot \theta \tag{4b}$$

$$\theta(\tilde{h}) = \tilde{h} . \tag{4c}$$

If θ is a Cartan automorphism on $\tilde{g}$, then θ defines the real forms of $\tilde{g}$ as a set for

$$g_\theta = \{ X \in \tilde{g};\ \theta \cdot \psi(X) = X \}. \tag{5}$$

With the help of the Cartan automorphism θ, we can construct very simply a Cartan decomposition of $\tilde{g}$

$$g_\theta = g_\theta^t \oplus g_\theta^p, \qquad \text{where} \tag{6a}$$

$$g_\theta^t = \{ Y \in g_\theta\ ;\ \theta(Y) = Y \} \tag{6b}$$

$$g_\theta^p = \{ Y \in g_\theta\ ;\ \theta(Y) = -Y \}. \tag{6c}$$

The subalgebra g_θ^t is a maximal compact subalgebra in the algebra g_θ and the subspace g_θ^p is a complementary subspace to g_θ^t with respect to Killing form $B(\cdot,\cdot)$.

Semilarly we put

$$h_\theta = \{ X \in \tilde{h},\ \theta\psi(X) = X \} \tag{7a}$$

and then we have

$$h_\theta = h_\theta^t \oplus h_\theta^p \quad \text{where} \tag{7b}$$

$$h_\theta^t = \{ Y \in h_\theta,\ \theta(Y) = Y \} \tag{7c}$$

$$h_\theta^p = \{ Y \in h_\theta,\ \theta(Y) = -Y \}. \tag{7d}$$

The h_θ^p is a maximal commutative subalgebra in g_θ^p.

Now, let $\{H_1, \dots, H_q\}$ be a basis in h_θ^p and further let $\{iH_{q+1}, \dots, iH_n\}$ be a basis in h_θ^t. Then $\{H_1, \dots, H_n\}$ is a basis in $\tilde{h}$. By Δ_+ we shall denote a system of positive roots with respect this basis.

For any $\alpha \in \Delta$ we define

$$\alpha^\theta(H) = \alpha(\theta(H)) \qquad \text{for any } H \in \tilde{h}. \tag{8}$$

After performing the following easy calculation

$$[H, \theta(E_\alpha)] = \theta[\theta(H), E_\alpha] = \alpha(\theta(H))\theta(E_\alpha) \tag{9}$$

we become that α^θ is also the element of Δ.

Using this fact we may define

$$P_\theta^+ = \{ \alpha;\ \alpha \in \Delta_+,\ \alpha \neq \alpha^\theta \} \tag{10a}$$

$$\tilde{n}_\theta^+ = \sum_{\alpha \in P_\theta^+} g^\alpha , \quad n_\theta^+ = \tilde{n}_\theta^+ \cap g_\theta \tag{10b}$$

$$\tilde{n}_\theta^- = \sum_{\alpha \in P_\theta^+} g^{-\alpha} , \quad n_\theta^- = \tilde{n}_\theta^- \cap g_\theta \tag{10c}$$

where $g^\alpha = \mathbb{C}\{E_\alpha\}$

$$g_\theta^o = \{ Y \in g_\theta\ ;\ [Y, h_\theta^p] = 0 \} \tag{10d}$$

and we get the required Iwasawa and triangle decompositions of the real form g_θ at last

$$g_\theta = g_\theta^t \oplus h_\theta^p \oplus n_\theta^+ \qquad \text{(Iwasawa decomposition)} \qquad (11a)$$

$$g_\theta = n_\theta^+ \oplus g_\theta^o \oplus n_\theta^- \qquad \text{(triangle decomposition).} \qquad (11b)$$

3. The Cartan automorphisms of Lie algebras gl(n+1,ℂ)

The algebra gl(n+1,ℂ) is the $(n+1)^2$ - dimensional complex Lie algebra with the standard basis $\{E_{ij};\ i,j=1,2,\ldots,n+1\}$ the elements of which obey

$$[E_{ij}, E_{kl}] = \delta_{kl} E_{il} - \delta_{il} E_{kj}. \qquad (12)$$

This algebra is a direct sum of its one dimensional centrum (generated by the element $E = \sum_{i=1}^{n+1} E_{ii}$) and the simple subalgebra sl(n+1,ℂ) $\sim A_n$ whose generators are E_{ij}, $i \neq j$, $A_i = E_{ii} - \frac{1}{n} E$, $i = 1,2,\ldots,n$.

The standard Cartan subalgebra $\check{h}$ in $\tilde{g}$ = gl(n+1,ℂ) is generated by the "diagonal" elements E_{ii}, its dimension, i.e. rank of $\tilde{g}$ equals n+1. We choose the following Cartan-Weyl basis (for details see the book by Zhe-Xian Wan/7/ p.221)

$$H_i = E_{ii} \qquad (13a)$$

$$E_{\lambda_i - \lambda_k} = \frac{1}{\sqrt{2(n+1)}} E_{ik} \qquad i,k=1,2,\ldots,n+1. \qquad (13b)$$

The relations (11) imply that (13b) are the root vectors corresponding to the root $\lambda_i - \lambda_k$ because for $H(\lambda_1,\ldots,\lambda_{n+1}) = \sum_{i=1}^{n+1} \lambda_i E_{ii}$ we get

$$[H(\lambda_1,\ldots,\lambda_{n+1}), E_{ik}] = (\lambda_i - \lambda_k) E_{ik}. \qquad (14)$$

For the root system Δ we get

$$\Delta = \{\lambda_i - \lambda_k;\ 1 \le i,k \le n+1,\ i \neq k\}. \qquad (15)$$

The equality

$$-(\lambda_i - \lambda_k) = (\lambda_k - \lambda_i) \qquad (16)$$

implies that in this case the mapping (2a-b) equals

$$\psi(E_{ik}) = -E_{ki}. \qquad (17)$$

We define a linear mapping θ' on $\tilde{g}$ in the basis $\{E_{ij};\ i,j=1,2,\ldots,n+1\}$

$$\theta'(E_{ij}) = -E_{ji}, \qquad (18)$$

further we define for any $q=1,\ldots,\left[\frac{n+1}{2}\right]$ linear mappings θ_q on $\tilde{g}$ in this way:

$$\theta_q(E_{st}) = E_{q+s,q+t}, \qquad \theta_q(E_{q+s,q+t}) = E_{st}, \qquad (19a)$$

$$\theta_q(E_{s,q+t}) = E_{q+s,t}, \qquad \theta_q(E_{q+s,t}) = E_{s,q+t}, \qquad (19b)$$

$$\theta_q(E_{\alpha,s}) = -E_{\alpha,q+s}, \qquad \theta_q(E_{\alpha,q+s}) = -E_{\alpha,s}, \qquad (19c)$$

$$\theta_q(E_{\alpha,\beta}) = E_{\beta,\alpha} \qquad (19d)$$

where $s,t=1,2,\ldots,q$ $\qquad \alpha,\beta = 2q+1,\ldots,n+1$

and for (n+1) even we put

$$\theta''(E_{st}) = -E_{q+t,q+s} \qquad \theta''(E_{q+t,q+s}) = -E_{st} \qquad (20a)$$

$$\theta''(E_{s,q+t}) = E_{t,q+s} \qquad (20b)$$

$$\theta''(E_{q+s,t}) = E_{q+t,s} \quad \text{where } s,t=1,2,\ldots,q=\frac{n+1}{2}. \qquad (20c)$$

Theorem: The linear mappings θ', θ_q, θ'' are Cartan automorphisms on $\tilde{g}$.

Proof: One can by direct calculation verify that the conditions (4a-c) are fulfilled.

4. Explicit forms of the decompositions

Using the method described in Section 2 we shall construct the explicit forms of the Iwasawa and triangle decompositions for the real forms of the Lie algebra gl(n+1,ℂ) which are defined with the help of the automorphisms θ', θ_q and θ''. By the construction we will use following equalities

$$\psi(X + \psi(X)) = X + \psi(X) \qquad (21a)$$

$$\psi(iX - i\psi(X)) = iX - i\psi(X) \qquad \text{for any } X \in \tilde{g} \qquad (21b)$$

$$\theta(Y + \theta(Y)) = Y + \theta(Y) \qquad (21c)$$

$$\theta(Y - \theta(Y)) = -(Y - \theta(Y)) \qquad \text{for any } Y \in g_\theta \qquad (21d)$$

which are the direct consequence of the definitions (2a-b) and (4a-c). The calculations of the decompositions are simple and we carry only final results.

I. The case θ'.

For a subalgebra $g^t_{\theta'}$ and a subspace $g^p_{\theta'}$ we have

$$g^t_{\theta'} = R\{E_{ij} - E_{ji};\ i,j = 1,2,\dots,n+1\} \tag{22a}$$

$$g^p_{\theta'} = R\{E_{ij} + E_{ji};\ i,j = 1,2,\dots,n+1\}. \tag{22b}$$

In this case a subalgebra $h^t = 0$ and the set $\{E_{11},\dots,E_{n+1,n+1}\}$ forms a basis in $h^p_{\theta'}$. For $P^+_{\theta'}$, with respect to this basis we get

$$P^+_{\theta'} = \{\lambda_i - \lambda_k \quad ;\ i < k,\ i,k = 1,2,\dots,n+1\} \tag{23}$$

and this imply consequently that

$$n^+_{\theta'} = R\{E_{ij};\ i < j,\ i,j = 1,2,\dots,n+1\} \tag{24a}$$

$$n^-_{\theta'} = R\{E_{ij};\ i > j,\ i,j = 1,2,\dots,n+1\} \tag{24b}$$

and

$$g^o_{\theta'} = h^p_{\theta'}\ . \tag{24c}$$

II. The case θ_q, $q=1,2,\dots,\left[\frac{n+1}{2}\right]$.

For a subalgebra $g^t_{\theta_q}$ and a subspace $g^p_{\theta_q}$ we get

$$\begin{aligned} g^t_{\theta_q} = R\{&(E_{st}-E_{q+t,q+s}-E_{ts}+E_{q+s,q+t}),\ i(E_{st}+E_{q+t,q+s}+E_{ts}+E_{q+s,q+t}),\\ &(E_{s,q+t}-E_{t,q+s}+E_{q+s,t}-E_{q+t,s}),\ i(E_{s,q+t}+E_{t,q+s}+E_{q+s,t}+E_{q+t,s}),\\ &(E_{s,\alpha}+E_{\alpha,q+s}-E_{q+s,\alpha}-E_{\alpha,s}),\ i(E_{s,\alpha}-E_{\alpha,q+s}-E_{q+s,\alpha}+E_{\alpha,s}),\\ &(E_{\alpha,\beta}-E_{\beta,\alpha}),\ i(E_{\alpha,\beta}+E_{\alpha,\beta})\}, \end{aligned} \tag{25a}$$

$$\begin{aligned} g^p_{\theta_q} = R\{&(E_{st}-E_{q+t,q+s}+E_{ts}-E_{q+s,q+t}),\ i(E_{st}+E_{q+t,q+s}-E_{ts}-E_{q+s,q+t}),\\ &(E_{s,q+t}-E_{t,q+s}-E_{q+s,t}+E_{q+t,s}),\ i(E_{s,q+t}+E_{t,q+s}-E_{q+s,t}-E_{q+t,s}),\\ &(E_{s,\alpha}+E_{\alpha,q+s}+E_{q+s,\alpha}+E_{\alpha,s}),\ i(E_{s,\alpha}-E_{\alpha,q+s}+E_{q+s,\alpha}-E_{\alpha,s})\}. \end{aligned} \tag{25b}$$

We put

$$\{(E_{11}-E_{q+1,q+1}),\dots,(E_{qq}-E_{2q,2q})\} \tag{26a}$$

a basis in a subalgebra $h^p_{\theta_q}$ and

$$\{i(E_{11}+E_{q+1,q+1}),\dots,i(E_{qq}+E_{2q,2q}),iE_{2q+1,2q+1},\dots,iE_{n+1,n+1}\} \tag{26b}$$

in a subalgebra $h^t_{\theta_q}$.

For a set of the roots $P^+_{\theta_q}$ we get

$$\begin{aligned} P^+_{\theta_q} = \{&\lambda_s-\lambda_t,\ \lambda_{q+t}-\lambda_{q+s},\ s<t,\ \lambda_s-\lambda_{q+t},\ \lambda_s-\lambda_\alpha,\\ &\lambda_\alpha-\lambda_{q+s}\quad s,t = 1,2,\dots,q,\ \ \alpha = 2q+1,\dots,n+1\} \end{aligned} \tag{27}$$

and further

$$\begin{aligned} n^+_{\theta_q} = R\{&(E_{st}-E_{q+t,q+s}),\ i(E_{st}+E_{q+t,q+s});\ s<t,\\ &(E_{s,q+t}-E_{t,q+s}),\ i(E_{s,q+t}+E_{t,q+s}),\\ &(E_{s,\alpha}+E_{\alpha,q+s}),\ i(E_{s,\alpha}-E_{\alpha,q+s})\}, \end{aligned} \tag{28a}$$

$$\begin{aligned} n^-_{\theta_q} = R\{&(E_{st}-E_{q+t,q+s}),\ i(E_{st}+E_{q+t,q+s}),\ s>t\\ &(E_{q+s,t}-E_{q+t,s}),\ i(E_{q+s,t}+E_{q+t,s}),\\ &(E_{\alpha,s}+E_{q+s,\alpha}),\ i(E_{\alpha,s}-E_{q+s,\alpha})\}, \end{aligned} \tag{28b}$$

$$\begin{aligned} g^o_{\theta_q} = R\{&(E_{ss}-E_{q+s,q+s}),\ i(E_{ss}+E_{q+s,q+s}),\\ &(E_{\alpha,\beta}-E_{\beta,\alpha}),\ i(E_{\alpha,\beta}+E_{\beta,\alpha})\} \end{aligned} \tag{28c}$$

where $s,t = 1,2,\dots,q \qquad \alpha,\beta = 2q+1,\dots,n+1$.

III. The case θ''.

In the case n+1 even we have the automorphism θ'' yet.

For a subalgebra $g^t_{\theta''}$ and a subspace $g^p_{\theta''}$ we get

$$\begin{aligned} g^t_{\theta''} = R\{&(E_{st}+E_{q+s,q+t}-E_{ts}-E_{q+t,q+s}),\ i(E_{st}-E_{q+s,q+t}+E_{ts}-E_{q+t,q+s}),\\ &(E_{s,q+t}-E_{q+s,t}+E_{t,q+s}-E_{q+t,s}),\ i(E_{s,q+t}+E_{q+s,t}+E_{t,q+s}+E_{q+t,s})\}, \end{aligned} \tag{29a}$$

$$\begin{aligned} g^p_{\theta''} = R\{&(E_{st}+E_{q+s,q+t}+E_{ts}+E_{q+t,q+s}),\ i(E_{st}-E_{q+s,q+t}-E_{ts}+E_{q+t,q+s}),\\ &(E_{s,q+t}-E_{q+s,t}-E_{t,q+s}+E_{q+t,s}),\ i(E_{s,q+t}+E_{q+s,t}-E_{t,q+s}-E_{q+t,s})\}. \end{aligned} \tag{29b}$$

The subalgebra $h^p_{\theta''}$ has a basis

$$\{(E_{11}+E_{q+1,q+1}),\ldots,(E_{qq}+E_{2q,2q})\} \tag{30a}$$

and the subalgebra $h^{t}_{\theta''}$ has a basis

$$\{i(E_{11}-E_{q+1,q+1}),\ldots,i(E_{qq}-E_{2q,2q})\}. \tag{30b}$$

For a set of the roots $P^{+}_{\theta''}$ we have

$$P^{+}_{\theta''}=\{\lambda_s-\lambda_t,\lambda_{q+s}-\lambda_{q+t},\lambda_s-\lambda_{q+t},\lambda_{q+s}-\lambda_t,\quad s<t\} \tag{31}$$

and further

$$n^{+}_{\theta''}=R\{(E_{st}+E_{q+s,q+t}),\ i(E_{st}-E_{q+s,q+t});\ s<t \tag{32a}$$
$$(E_{s,q+t}-E_{q+s,t}),\ i(E_{s,q+t}+E_{q+s,t});\ s<t\},$$

$$n^{-}_{\theta''}=R\{(E_{st}+E_{q+s,q+t}),\ i(E_{st}-E_{q+s,q+t});\ s>t \tag{32b}$$
$$(E_{s,q+t}-E_{q+s,t}),\ i(E_{s,q+t}+E_{q+s,t});\ s>t\}$$

$$g^{0}_{\theta''}=R\{(E_{ss}+E_{q+s,q+s}),\ i(E_{ss}-E_{q+s,q+s}) \tag{32c}$$
$$(E_{s,q+s}-E_{q+s,s}),\ i(E_{s,q+s}+E_{q+s,s})\}$$

where $s,t=1,2,\ldots,q$.

Dimensions of the subalgebras g^{p}_{θ} and h^{p}_{θ} are characteristic values of the given real form g_{θ}. For the automorphisms θ', θ_q, θ'' we get from (22b), (25b), (26a), (29b) and (30a)

θ	$\dim g^{p}_{\theta}$	$\dim h^{p}_{\theta}$
θ'	$\frac{n(n+1)}{2}$	$n+1$
θ''	$q(2q-1)$	$q=\frac{n+1}{2}$
θ_q	$2q(n+1-q)$	q

A comparison of these results with a list of the real forms $gl(n+1,C)$ in book/2/ p.85 implies a following theorem.

<u>Theorem:</u> The algebra $g_{\theta'}$ is isomorphic $gl(n+1,R)$, further the algebras g_{θ_q} are isomorphic $u(q,n+1-q)$ for any $q=1,2,\ldots,[\frac{n+1}{2}]$, and the algebra $g_{\theta''}$ is isomorphic $u^{*}(n+1)$.

<u>A conclusion</u>

In the present paper we give the explicit form of the Iwasawa and triangle decompositions for the real forms of Lie algebras $gl(n+1,C)$. However, we can use this method for the real forms of complex Lie algebras B_n, C_n, D_n. Some positive indication has been obtained already; the results will be presented in a forthcoming series of papers.

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