объединенный
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исследований
дубна

E5-85-727

F.Loffler*, W.Timmermann

ON THE STRUCTURE OF THE STATE SPACE OF MAXIMAL Op*-ALGEBRAS

Submitted to "Publications of the Research Institute for Mathematical Sciences"

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* Sektion Mathematik, Karl-Marx-Universität DDR-7010 Leipzig

1985

1. Introduction

In this paper we continue the investigation of the structure of the state space of $\mathcal{L}^+(\mathcal{D})$ begun in /15/.
The paper is organized as follows. Section 2 contains the necessary definitions, notations and auxiliary results. In section 3 we prove the main results about the pure state space and the vector state space of $\mathcal{L}^+(\mathcal{D})$. These two sets of states coincide. Moreover there is given a representation theorem for the pure (and vector) state space analogous to the bounded case.
Moreover section 3 contains several results concerning sequences of states. Only some of them correspond to the bounded case.
In section 4 we investigate the state space of more general Op^*-algebras $\mathcal{A}(\mathcal{D})$. Among other things it is proved that the state space is the w^*-closed convex hull of the vector states. If $\mathcal{A}(\mathcal{D})$ is selfadjoint and topologically irreducible (cf.Def.4.3) then the state space is the w^*-closed convex hull of pure states.

2. Preliminaries

For a dense linear manifold $\mathcal{D}$ in a separable Hilbert space $\mathcal{H}$ the set $\mathcal{L}^+(\mathcal{D}) = \{A: A\mathcal{D} \subset \mathcal{D},\ A^*\mathcal{D} \subset \mathcal{D}\}$ is a $*$-algebra with respect to the usual operations and the involution $A \longrightarrow A^+ = A^* | \mathcal{D}$. An Op^*-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing the identity operator I. The <u>graph topology</u> $t_{\mathcal{A}}$ on $\mathcal{D}$ induced by $\mathcal{A}(\mathcal{D})$ is given by the family of seminorms $\varphi \longrightarrow \|A\varphi\|$ for all $A \in \mathcal{A}(\mathcal{D})$. Denote $t_{\mathcal{L}^+(\mathcal{D})}$ simply by t. This topologization of $\mathcal{D}$ gives rise to a canonical rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}'[t']$ and a canonical dual pair $(\mathcal{D}, \mathcal{D}')$. Here t' is the strong topology in $\mathcal{D}'$. Let $\sigma = \sigma(\mathcal{D}, \mathcal{D}')$ be the weak topology in $\mathcal{D}$. Remember that a sequence $(\varphi_n) \subset \mathcal{D}$ is σ-convergent to zero ($\varphi_n \xrightarrow{\sigma} 0$) if and only if (φ_n) is t-bounded and $\langle \psi, \varphi_n \rangle \longrightarrow 0$ for all $\psi \in \mathcal{D}$ hence for all $\psi \in \mathcal{H}$.
An Op^*-algebra $\mathcal{A}(\mathcal{D})$ is called
<u>closed</u> if $\mathcal{D} = \bigcap_{A \in \mathcal{A}} \mathcal{D}(\bar{A})$ or equivalently if $\mathcal{D}[t]$ is complete;
<u>selfadjoint</u> if $\mathcal{D} = \mathcal{D}_* = \bigcap_{A \in \mathcal{A}} \mathcal{D}(A^*)$.

In Op^*-algebras there can be defined a lot of topologies (cf.e.g. /10/-/12/,/17/,/18/). We mention only those used here:
the <u>uniform topology</u> $\tau_{\mathcal{D}}$ given by the family of seminorms

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi \in \mathcal{M}} |\langle \varphi, A\psi \rangle|,$$

where $\mathcal{M}$ runs over all $t_{\mathcal{A}}$-bounded subsets of $\mathcal{D}$;
<u>the topology $\tau^c_{\mathcal{D}}$</u> given by the family of seminorms $\|\cdot\|_{\mathcal{N}}$ where $\mathcal{N}$ now runs over all relatively $t_{\mathcal{A}}$-compact subsets of $\mathcal{D}$. Later on we will frequently use the fact that $\tau_{\mathcal{D}}$ and $\tau^c_{\mathcal{D}}$ are not only defined on $\mathcal{L}^+(\mathcal{D})$ but also on $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, hence on $\mathcal{B}(\mathcal{H})$, too /12/,/8/.
The most important domains $\mathcal{D}$ are those where t is a Frechét-topology, i.e. $\mathcal{D}[t]$ is complete and t can be given by $\{\|A_n \cdot\| : n \in \mathbb{N}\}$ with $A_n = A_n^+$, $I = A_0 \leq A_1 \leq \dots$. A special type of such domains is of the form $\mathcal{D} = \mathcal{D}^\infty(T) \equiv \bigcap_{n \geq 0} \mathcal{D}(T^n)$, $T = T^* \geq I$.
In what follows we allways assume that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint and $\mathcal{D}[t]$ is an (F)-space. Some of the results are valid in more general situations. To simplify notations we denote a bounded operator $A \in \mathcal{L}^+(\mathcal{D})$ and its closure $\bar{A} \in \mathcal{B}(\mathcal{H})$ by the same letter A. The following sets are two-sided $*$-ideals in $\mathcal{L}^+(\mathcal{D})$ and play an important role in the description of $\tau_{\mathcal{D}}, \tau^c_{\mathcal{D}}$ (/8/,/11/,/12/,/15/,/19/,/20/):

$$\mathcal{B}(\mathcal{D}) = \{T: T\mathcal{H} \subset \mathcal{D},\ T^*\mathcal{H} \subset \mathcal{D}\} = \{T: AT,\ AT^* \text{ bounded for all } A \in \mathcal{L}^+(\mathcal{D})\}$$

$$\mathcal{J}_\infty(\mathcal{D}) = \{T: T \in \mathcal{J}_\infty(\mathcal{H}) \cap \mathcal{B}(\mathcal{D})\} = \{T: AT,\ AT^* \in \mathcal{J}_\infty(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathcal{D})\}.$$

Here $\mathcal{J}_\infty(\mathcal{H})$ denotes the $*$-ideal of compact operators on $\mathcal{H}$.
Now we collect some properties needed later on.

<u>Proposition 2.1</u>

Let $\mathcal{K}$ be the unit ball in $\mathcal{H}$.
i) The family $\{B\mathcal{K} : B \in \mathcal{B}(\mathcal{D}),\ B \geq 0,\ B^{-1} \text{ exists},\ \mathcal{R}(B) \text{ t-dense in } \mathcal{D}\}$ is a fundamental system of t-bounded sets. Hence $\tau_{\mathcal{D}}$ can be given by the seminorms $A \longrightarrow \|BAB\|$, B as above.
ii) Let $B = \int_0 \lambda\, dE_\lambda \in \mathcal{B}(\mathcal{D})$ and $P_a = \int_a dE_\lambda$, $A_a = P_a A P_a$ for $0 < a < b$, $A \in \mathcal{L}^+(\mathcal{D})$. Then
$P_a, A_a \in \mathcal{B}(\mathcal{D})$ and $\lim_{a \to 0} \|B^\alpha (A - A_a) B^\alpha\| = 0$ for all $\alpha > 0$.
iii) $\mathcal{B}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ hence in $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{H})$.

The ideal $\mathcal{J}_\infty(\mathcal{D})$ gives a description of the relatively t-compact sets in $\mathcal{D}$ /20/. For completeness we include a proof. It is based on the following fact which seems to be not included in standard books on Hilbert spaces (the authors are grateful to K.D.Kürsten for suggesting a short proof):

Let $\mathcal{H}$ be a separable Hilbert space, then the family

(1) $\{C\mathcal{K} : C \geq 0,\ C \in \mathfrak{S}_\infty(\mathcal{H}),\ \mathcal{R}(C) \text{ dense in } \mathcal{H}\}$ is a fundamental system of relatively $\| \ \|$-compact sets in $\mathcal{H}$.

Proposition 2.2

The family $\{S\mathcal{K} : S \geq 0,\ S \in \mathfrak{S}_\infty(\mathcal{D}),\ \mathcal{R}(S) \text{ t-dense in } \mathcal{D},\ S^{-1} \text{ exists}\}$ is a fundamental system of relatively t-compact sets in $\mathcal{D}$.

Proof:

It is easy to see that any set of the family is relatively t-compact. To prove the other direction we first of all remark that the proof of (1) (or some other considerations) gives immediately the following estimation: if $\mathcal{N} \subset \mathcal{H}$ is relatively $\| \ \|$-compact and $\|\varphi\| \leq a$ for all $\varphi \in \mathcal{N}$, then C can be chosen to satisfy $a \leq \|C\| \leq 2a$.

Now let $\mathcal{M} \subset \mathcal{D}$ be relatively t-compact, hence t-bounded. By Proposition 2.1 there is a $B \geq 0$, $B \in \mathcal{B}(\mathcal{D})$ with $B\mathcal{K} \supset \mathcal{M}$. Without restriction of generality we assume that $\|B\| \leq 1$. Put $B = \int_0^1 \lambda \, dE_\lambda$,

$P_n = \int_{J_n} dE_\lambda$, $J_n = (2^{-2n}, 2^{-2(n-1)}]$, $n = 1,2,\dots$, $\mathcal{H}_n = P_n\mathcal{H} \subset \mathcal{D}$.

Then $B_n = P_nBP_n = BP_n \in \mathcal{L}^+(\mathcal{D})$ (even $\in \mathcal{B}(\mathcal{D})$), $P_n \in \mathcal{B}(\mathcal{D})$, and $I = \sum_n \oplus P_n$.

Moreover $\mathcal{M}$-relatively t-compact implies that $\mathcal{M}_n = P_n\mathcal{M} \subset \mathcal{H}_n$ is relatively $\| \ \|$-compact and from $B\mathcal{K} \supset \mathcal{M}$ it follows that $B_n\mathcal{K} = B_n\mathcal{K}_n \supset \mathcal{M}_n$, where $\mathcal{K}_n$ is the unit ball in $\mathcal{H}_n$.

Consequently for all $\varphi \in \mathcal{M}_n$ it is $\|\varphi\| \leq \|B_n\| \leq 2^{-2n+2}$. Applying (1) to $\mathcal{M}_n$, $\mathcal{H}_n$ we get a compact $S_n' \in \mathcal{B}(\mathcal{H}_n)$ with $S_n' \geq 0$, $S_n'\mathcal{K}_n \supset \mathcal{M}_n$ and

(2) $$2^{-2n+2} \leq \|S_n'\| \leq 2 \cdot 2^{-2n+2}.$$

Clearly S_n' can be considered as an element of $\mathcal{B}(\mathcal{H})$. Put $S_n = 2^{3/2} \cdot \|S_n'\|^{1/2} \cdot S_n'$. Then $\|S_n\| = \|S_n'\|^{1/2} \cdot 2^{3/2}$ and $S_n(\|S_n'\|^{1/2} \cdot 2^{-3/2})\mathcal{K} = S_n'\mathcal{K}_n \supset \mathcal{M}_n$.

The operator $S = \sum \oplus S_n$ has the following properties:

i) $S = S^*$ is compact because $S_n = S_n^*$ are compact and the series $\sum \oplus S_n$ is $\| \ \|$-convergent as can be seen from

$$\|S\| \leq \sum \|S_n\| = 2^{3/2} \sum \|S_n'\|^{1/2} \leq 2^3 \sum_n 1/2^n < \infty .$$

ii) $S\mathcal{K} \supset \mathcal{M}$. Indeed let $\varphi \in \mathcal{M}$, $\varphi = \sum \varphi_n$, $\varphi_n \in \mathcal{M}_n$. Then there exist $\psi_n \in (\|S_n'\|^{1/2} \cdot 2^{-3/2})\mathcal{K}$ with $S_n\psi_n = \varphi_n$. Because $\|\psi_n\| \leq 2^{-n}$ it follows that $\psi = \sum \psi_n$ has norm $\|\psi\| \leq \sum_n 2^{-n} = 1$, i.e. $\psi \in \mathcal{K}$ and $S\psi = \varphi \in \mathcal{M}$.

iii) $S \in \mathfrak{S}_\infty(\mathcal{D})$. Because $S \in \mathfrak{S}_\infty(\mathcal{H})$ it is enough to show that $S \in \mathcal{B}(\mathcal{D})$. Let $A \in \mathcal{L}^+(\mathcal{D})$ be arbitrary, then

$$\|AS\varphi\| = \|AB^{1/2} B^{-1/2} S\varphi\| \leq \|AB^{1/2}\| \cdot \|B^{-1/2} S\varphi\| .$$

The first term is bounded because $B^{1/2} \in \mathcal{B}(\mathcal{D})$, for the second factor remark that

$$\|B^{-1/2}S\varphi\|^2 = \sum \|B_n^{-1/2} S_n\varphi_n\|^2 \leq \sum \|B_n^{-1/2} S_n\|^2 \cdot \|\varphi_n\|^2 \leq C\|\varphi\|^2,$$

where $C = \sup_n \|B_n^{-1/2} S_n\|^2 = \sup_n 2^{2n} \|S_n'\| \cdot 2^{-2n+2} \leq 16$.

Moreover S^{-1} exists. So S is the desired operator. The assertion about $\mathcal{R}(S)$ can be proved as in the $\mathcal{B}(\mathcal{D})$-case using if necessary a larger t-relatively compact set $\mathcal{N} \supset \mathcal{M}$.

q.e.d.

Next we consider linear functionals on closed Op^*-algebras $\mathcal{A}(\mathcal{D})$. We restrict ourselves to $\tau_{\mathcal{D}}$-continuous functionals.

Let $E(\mathcal{A}) = \{\omega \in \mathcal{A}(\mathcal{D})[\tau_{\mathcal{D}}]' : \omega \geq 0,\ \omega(I) = 1\}$ be the <u>state space</u> of $\mathcal{A}(\mathcal{D})$. Here $\omega \geq 0$ means $\omega(A) \geq 0$ for all $A \in \mathcal{A}(\mathcal{D})$ with $\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}$ (strongly positive functionals).

Further let $\sigma_0(\mathcal{A}) = \sigma(\mathcal{A}', \mathcal{A})$ be the w^*-topology in $\mathcal{A}(\mathcal{D})[\tau_{\mathcal{D}}]'$. We need the following subsets of $E(\mathcal{A})$:

<u>vector states:</u> $V_0(\mathcal{A}) = \{\omega \in E(\mathcal{A}) : \omega(A) = \langle \varphi, A\varphi \rangle \text{ for some } \varphi \in \mathcal{D},\ \|\varphi\| = 1\}$,

<u>vector state space</u>: $V(\mathcal{A}) = \sigma_0(\mathcal{A})$-closure of $V_0(\mathcal{A})$;

<u>pure states</u>: $P_0(\mathcal{A}) = \{\omega \in E(\mathcal{A}) : \omega_1 \in \mathcal{A}',\ \omega_1 \geq 0,\ \omega_1 \leq \omega \text{ implies } \omega_1 = \lambda\omega,\ \lambda \in [0,1]\}$;

<u>pure state space</u> $P(\mathcal{A}) = \sigma_0(\mathcal{A})$-closure of $P_0(\mathcal{A})$.

Clearly, $\omega \in P_0(\mathcal{A})$ if and only if ω cannot be represented as a non-trivial convex combination of $\omega_1, \omega_2 \in E(\mathcal{A})$. The corresponding subsets of $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]'$ are denoted simply by E, V_0, V, P_0, P respectively, and $\sigma_0(\mathcal{L}^+(\mathcal{D}))$ we denote by σ_0. To define normal and singular functionals on $\mathcal{L}^+(\mathcal{D})$ we introduce the following two-sided $*$-ideals in $\mathcal{L}^+(\mathcal{D})$ /13/,/9/,/14/,/17/:

$$\mathfrak{S}_1(\mathcal{D}) = \{T \in \mathcal{L}^+(\mathcal{D}) : AT,\ AT^* \in \mathfrak{S}_1(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathcal{D})\}$$

$\mathcal{F}(\mathcal{D}) = \{F \in \mathcal{L}^+(\mathcal{D}) : \dim F\mathcal{D} < \infty\}$

$\mathcal{C}(\mathcal{D}) = \tau_{\mathcal{D}}$-closure of $\mathcal{F}(\mathcal{D})$.

Here $\mathcal{J}_1(\mathcal{H})$ denotes the *-ideal of nuclear operators on $\mathcal{H}$.

Definition 2.3

A linear functional ω on $\mathcal{L}^+(\mathcal{D})$ is said to be
normal if $\omega(A) = \mathrm{Tr}\, AT$ for some $T \in \mathcal{J}_1(\mathcal{D})$ and all $A \in \mathcal{L}^+(\mathcal{D})$;
singular if ω is $\tau_{\mathcal{D}}$-continuous and $\omega(C) = 0$ for all $C \in \mathcal{C}(\mathcal{D})$.

Let us remark that while normal functionals are automatically $\tau_{\mathcal{D}}$-continuous (even $\tau^c_{\mathcal{D}}$-continuous, /17/) we have included the $\tau_{\mathcal{D}}$-continuity in the definition of singular functionals. Moreover the notion of singularity used here is a direct generalization of that from the bounded case and has nothing to do with that used by Inoue, cf. e.g./6/.
In /15/ we described a procedure relating $\tau_{\mathcal{D}}$-continuous functionals on $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{H})$ based on the $\tau_{\mathcal{D}}$-density of $\mathcal{B}(\mathcal{D})$ (see Proposition 2.1.iii)). Let $\omega \in \mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]'$ ($\in \mathcal{B}(\mathcal{H})[\tau_{\mathcal{D}}]'$), by restriction to $\mathcal{B}(\mathcal{D})$ and extension to $\mathcal{B}(\mathcal{H})$ (to $\mathcal{L}^+(\mathcal{D})$) one gets a unique $\tau_{\mathcal{D}}$-continuous functional on $\mathcal{B}(\mathcal{H})$ (on $\mathcal{L}^+(\mathcal{D})$) which we denote by $\tilde{\omega}$ ($\hat{\omega}$). Some properties of this procedure were collected in /15/.

3. The vector state space and the pure state space of $\mathcal{L}^+(\mathcal{D})$

This section is devoted to the description of P and V. In /5/ Glimm proved among other things the following result about $V(\mathcal{B}(\mathcal{H}))$ and $P(\mathcal{B}(\mathcal{H}))$.

Theorem 3.1

$P(\mathcal{B}(\mathcal{H})) = V(\mathcal{B}(\mathcal{H})) = Z(\mathcal{B}(\mathcal{H})) \equiv \{\omega \in E(\mathcal{B}(\mathcal{H})) : \omega = \lambda\omega_1 + (1-\lambda)\omega_2,\ 0 \le \lambda \le 1,\ \omega_1$ -vector state, ω_2 -singular state on $\mathcal{B}(\mathcal{H})\}$.

Our aim is to generalize this theorem to the unbounded case, namely to $\mathcal{L}^+(\mathcal{D})$. To do this we will use a result of Wils /21/ which was generalized by Anderson /2/.

Theorem 3.2

There is a fixed sequence $(\psi_n) \subset \mathcal{H}$, $\|\psi_n\| = 1$ so that for any $\omega \in P(\mathcal{B}(\mathcal{H})) = Z(\mathcal{B}(\mathcal{H}))$ there exists an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ with

$$\omega(A) = \lim_{\mathcal{U}} \langle \psi_n, A\psi_n \rangle .$$

It is enough to take (ψ_n) to be norm-dense in the unit sphere of $\mathcal{H}$, so we can suppose $(\psi_n) \subset \mathcal{D}$. Call such a sequence (ψ_n) a Wils sequence.

In analogy to the bounded case we define
$Z = \{\omega \in E : \omega = \lambda\omega_1 + (1-\lambda)\omega_2,\ 0 \le \lambda \le 1,\ \omega_1 \in V_0,\ \omega_2\text{-singular}\}$.
Then the following theorem is valid.

Theorem 3.3

Let (ψ_n) be a fixed Wils sequence. For every $\omega \in Z$ there is a $C \in \mathcal{B}(\mathcal{D})$ and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ so that

$$\omega(A) = \lim_{\mathcal{U}} \langle C\psi_n, AC\psi_n \rangle$$

Proof:

Let $\omega \in Z$, $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $\omega_1 = \langle \phi, .\phi \rangle$, $\|\phi\| = 1$, $\phi \in \mathcal{D}$ and ω_2 singular.
Then ω, ω_1, ω_2 and the corresponding states $\tilde{\omega}$, $\tilde{\omega}_1$, $\tilde{\omega}_2$ on $\mathcal{B}(\mathcal{H})$ can be estimated by some seminorm $\|B \cdot B\|$ with $B \ge 0$, $B \in \mathcal{B}(\mathcal{D})$. Let $B = \int_0^\infty \lambda\, dE_\lambda$. From /15/ we know that $\tilde{\omega}_1 = \langle \phi, .\phi \rangle$, $\tilde{\omega}_2$ singular, i.e., $\tilde{\omega} = \lambda\tilde{\omega}_1 + (1-\lambda)\tilde{\omega}_2 \in Z(\mathcal{B}(\mathcal{H}))$.
For $n \in \mathbb{N}$ and fixed $0 < \alpha < 1$ put $P_n = \int_{1/n}^\infty dE_\lambda$, $B_n^{-\alpha} = \int_{1/n} \lambda^{-\alpha} dE_\lambda$. Then $B_n^{-\alpha} B = B^{1-\alpha} P_n$, moreover $B_n^{-\alpha} \in \mathcal{B}(\mathcal{H})$. In /15/ it was shown that $\tilde{\omega}(B_n^{-\alpha} A B_n^{-\alpha})$ is a Cauchy sequence for all $A \in \mathcal{B}(\mathcal{H})$ and $\rho(A) = \lim_{n\to\infty} \tilde{\omega}(B_n^{-\alpha} A B_n^{-\alpha})$ defines a positive linear functional on $\mathcal{B}(\mathcal{H})$.
Moreover

(1) $$\rho(B^\alpha A B^\alpha) = \tilde{\omega}(A).$$

The same can be done for $\tilde{\omega}_1$ and $\tilde{\omega}_2$ which leads to ρ_1, ρ_2. Furthermore

$$\rho(I) = \lim_{n\to\infty} \tilde{\omega}(B_n^{-2\alpha}) = \lambda \cdot \lim_{n\to\infty} \tilde{\omega}_1(B_n^{-2\alpha}) + (1-\lambda) \cdot \lim_{n\to\infty} \tilde{\omega}_2(B_n^{-2\alpha}) = \lambda\rho_1(I) + (1-\lambda)\rho_2(I).$$

Now let us distinguish some extreme cases. $\rho(I) = 0$ and the Cauchy-Schwarz inequality give $\rho \equiv 0$, hence $\tilde{\omega} \equiv 0$ (by (1)) which is a contradiction. If $\rho_1(I) = 0$ or $\rho_2(I) = 0$, then $\tilde{\omega}_1 \equiv 0$ or $\tilde{\omega}_2 \equiv 0$. But this means that ω is a singular state or a vector state on $\mathcal{L}^+(\mathcal{D})$. Then we are done. For singular states this representation theorem was given in /15/. For vector states the representation is obtained as follows. Take any $B \in \mathcal{B}(\mathcal{D})$ with $B\psi_k = \phi$ for some k (having in mind that the vector state is generated by ϕ). The ultrafilter $\mathcal{U}$ fixed at k leads to the desired representation.
So we may suppose $\tilde{\omega}(B_n^{-2\alpha}) > c$, $\tilde{\omega}_1(B_n^{-2\alpha}) > c$, $\tilde{\omega}_2(B_n^{-2\alpha}) > c$ for .

all $n \geq n_0$ and some $c > 0$. Then the following states are defined on $\mathcal{B}(\mathcal{H})$:

$$\rho^n(A) = \tilde{\omega}(B_n^{-\alpha} A B_n^{-\alpha}) / \tilde{\omega}(B_n^{-2\alpha}) \quad \text{for } n \geq n_0.$$

We show $\rho^n \in Z(\mathcal{B}(\mathcal{H}))$. This follows from the decomposition

$$\rho^n(A) = [\lambda \tilde{\omega}_1(B_n^{-\alpha} A B_n^{-\alpha}) + (1-\lambda)\, \tilde{\omega}_2(B_n^{-\alpha} A B_n^{-\alpha})] / \tilde{\omega}(B_n^{-2\alpha}) =$$

$$= \mu_n \rho_1^n(A) + (1-\mu_n)\, \rho_2^n(A)$$

with $\mu_n = \lambda \tilde{\omega}_1(B_n^{-2\alpha}) / \tilde{\omega}(B_n^{-2\alpha})$ and $\rho_i^n(A) = \tilde{\omega}_i(B_n^{-\alpha} A B_n^{-\alpha}) / \tilde{\omega}_i(B_n^{-2\alpha})$, $i = 1,2$.

From the properties of $\tilde{\omega}$ it follows that ρ_1^n are vector states and ρ_2^n are singular states on $\mathcal{B}(\mathcal{H})$, hence $\rho^n \in Z(\mathcal{B}(\mathcal{H}))$.

Moreover

$$\lim_{n \to \infty} \rho^n(A) = \rho(A) / \rho(I) \equiv \sigma(A) \quad \text{for all } A \in \mathcal{B}(\mathcal{H}),$$

so $\sigma \in Z(\mathcal{B}(\mathcal{H}))$ because this set is w^*-closed. By theorem 3.2 there is an ultrafilter $\mathcal{U}$ so that

$$\sigma(A) = \lim_{\mathcal{U}} \langle \psi_k, A \psi_k \rangle \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

From (1) it is seen that for all $A \in \mathcal{B}(\mathcal{H})$

$$\tilde{\omega}(A) = \rho(B^{\alpha} A B^{\alpha}) = \rho(I)\, \sigma(B^{\alpha} A B^{\alpha}) = \lim_{\mathcal{U}} \langle C\psi_k, AC\psi_k \rangle \quad \text{with}$$

$C = B^{\alpha}(\rho(I))^{1/2} \in \mathcal{B}(\mathcal{D})$. The standard estimation (cf./15/)for $X \in \mathcal{L}^+(\mathcal{D})$:

$$|\omega(X) - \lim_{\mathcal{U}} \langle C\psi_k, XC\psi_k \rangle| \leq |\omega(X) - \tilde{\omega}(P_n X P_n)| + |\tilde{\omega}(P_n X P_n) -$$

$$- \lim_{\mathcal{U}} \langle C\psi_k, XC\psi_k \rangle| \longrightarrow 0 \text{ for } n \longrightarrow \infty$$ leads to the desired result

$$\omega(X) = \lim_{\mathcal{U}} \langle C\psi_k, XC\psi_k \rangle .$$

q.e.d.

Corollary 3.4

The following inclusions are valid

$$P_0 \subset Z \subset V .$$

Especially, any pure state on $\mathcal{L}^+(\mathcal{D})$ has the representation

$$\omega(A) = \lim_{\mathcal{U}} \langle C\psi_k, AC\psi_k \rangle$$

for some ultrafilter $\mathcal{U}$ and $C \in \mathcal{B}(\mathcal{D})$, (ψ_k) as in Theorem 3.3.

Proof:

The first inclusion follows from /15/, where it was proved that pure states are either vector states or pure singular states. The second inclusion is an immediate consequence of the representation given in Theorem 3.3. The proof is the same as that of Corollary 4.6 in /15/.

q.e.d.

Lemma 3.5

The vector state space and the pure state space of $\mathcal{L}^+(\mathcal{D})$ coincide, i.e. $V = P$.

Proof:

Since any vector state is pure /15/, $V \subset P$ follows. On the other hand, let $\omega \in P$. For given $A_1, \dots, A_n \in \mathcal{L}^+(\mathcal{D})$, $\varepsilon > 0$ there is a pure state ω' so that $|\omega(A_i) - \omega'(A_i)| < \varepsilon/2$, $i = 1,\dots,n$.

If ω' is a vector state, we are done. If not, ω' must be a pure singular state. But the singular states are contained in the vector state space (/15/,Corollary 4.6), so there is a vector state ω'' with $|\omega''(A_i) - \omega'(A_i)| < \varepsilon/2$, $i = 1,\dots,n$, i.e. $|\omega(A_i) - \omega''(A_i)| < \varepsilon$ for all i. This means $P \subset V$, hence $P = V$.

q.e.d.

Now we prove the main result of the paper.

Theorem 3.6

For $\mathcal{L}^+(\mathcal{D})$ one has $V = P = Z$.

Proof:

According to Corollary 3.4 and Lemma 3.5 it remains to prove that $V \subset Z$. The proof uses the idea of Glimm's original proof /5/. Let $\omega \in V$ and $\omega = \lambda\omega_n + (1-\lambda)\omega_s$ the corresponding decomposition into normal and singular states (/15/,Theorem 3.4).If $\lambda = \omega_n(I) = 0$,then $\omega_n = 0$ and we are done, because the singular states are contained in V. So let $\lambda \neq 0$. It remains to prove that ω_n is a vector state. By /17/ ω_n is $\tau_{\mathcal{D}}^c$-continuous, i.e.

$$|\omega_n(A)| \leq \| CAC \| \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}) \text{ and some } C \in \mathcal{S}_\infty(\mathcal{D}).$$

Let $C = \int_0^\infty \lambda\, dE_\lambda$ and put $P_n = \int_{q_n}^{c} dE_\lambda$ (without restriction of generality we suppose $C \geq I$). The P_n are finite dimensional, $P_n\mathcal{H} \subset \mathcal{D}$. Since $\omega \in V$ there is a net of unit vectors $\{\varphi_\alpha, \alpha \in J\} \subset \mathcal{D}$ so that $\omega = \sigma_0\text{-}\lim \omega_\alpha$ with $\omega_\alpha = \langle \varphi_\alpha, \cdot\, \varphi_\alpha \rangle$. The first step in the proof is to construct a sequence $(\psi_k) \subset \mathcal{D}$ so that

i) $\omega_{\psi_k} = \lambda\omega_n(P_k \cdot P_k)$

ii) $P_k \psi_{k+1} = \psi_k$.

The set $\{P_1\varphi_\alpha, \alpha \in J\}$ is $\|\ \|$-bounded and contained in the finite-dimensional subspace $P_1\mathcal{H} \subset \mathcal{D}$. So there is a subnet $\{\varphi_{j(\alpha)}, \alpha \in J\}$ and $\psi_1 \in \mathcal{D}$ with $P_1\varphi_{j(\alpha)} \xrightarrow{\|\ \|} \psi_1 = P_1\psi_1$. Let us remark that $\{P_1\varphi_{j(\alpha)}\}$ is a t-bounded set and $P_1AP_1 \in \mathcal{S}_\infty(\mathcal{D}) \subset \mathcal{C}(\mathcal{D})$ for all $A \in \mathcal{L}^+(\mathcal{D})$. It is

$$\omega(P_1 \cdot P_1) = \lambda\omega_n(P_1 \cdot P_1) + (1-\lambda)\,\omega_s(P_1 \cdot P_1) = \lambda\omega_n(P_1 \cdot P_1) =$$

$$= \sigma_0\text{-}\lim \omega_{j(\alpha)}(P_1 \cdot P_1) = \sigma_0\text{-}\lim \langle P_1\varphi_{j(\alpha)}, \cdot\, P_1\varphi_{j(\alpha)} \rangle = \langle \psi_1, \cdot\, \psi_1 \rangle.$$

To avoid complicated notions let us denote the subnets again by $\{\varphi_\alpha, \alpha\in J\}$.

Let $\psi_k \in \mathcal{D}$ be chosen so that $P_k\varphi_\alpha \xrightarrow{\|\ \|} \psi_k$, then the same considerations as above show that in view of the boundedness of $\{P_{k+1}\varphi_\alpha\}$ we find a $\psi_{k+1}\in\mathcal{D}$ with $P_{k+1}\varphi_\alpha \xrightarrow{\|\ \|} \psi_{k+1}$. Moreover from $P_kP_{k+1} = P_k$ we see that $P_k\psi_{k+1} = P_k(\lim P_{k+1}\varphi_\alpha) = \lim P_kP_{k+1}\varphi_\alpha = \lim P_k\varphi_\alpha = \psi_k$.

Thus, the existence of the sequence (ψ_k) is established.

The second step is to prove that $\psi_k \xrightarrow{t} \psi\in\mathcal{D}$. Let $k > l$, then in view of $P_k\psi_k = \psi_k$, $P_l\psi_k = \psi_l$, $P_kP_l = P_l$:

$\|A(\psi_k - \psi_l)\|^2 = \|A(P_k-P_l)\psi_k\|^2 = \langle\psi_k, (P_k-P_l)A^+A(P_k-P_l)\psi_k\rangle =$

$= \omega_{\psi_k}((P_k-P_l)A^+A(P_k-P_l)) = \lambda\omega_n(P_k((P_k-P_l)A^+A(P_k-P_l)P_k)) =$

$= \lambda\omega_n((P_k-P_l)A^+A(P_k-P_l)) \leq \lambda\|C(P_k-P_l)A^+A(P_k-P_l)C\| \leq$

$\leq 2\lambda\|C(P_k-P_l)\|\cdot\|A^+AC\| \longrightarrow 0$ for $k,l\to\infty$.

This means (ψ_k) is a t-Cauchy sequence, so there is a $\psi\in\mathcal{D}$, $\psi_k \xrightarrow{t} \psi$. Moreover, $\|\psi\|^2 = \lim\limits_{k\to\infty}\|\psi_k\|^2 = \lim\omega_{\psi_k}(I) = \lim(\lambda\omega_n(P_kIP_k))$ $= \lambda\cdot\lim\omega_n(P_k) = \lambda\omega_n(I) = \lambda$.

Now it is easy to see that $\omega_n = \omega_\varphi$ with $\varphi = \psi/\|\psi\|$. Indeed using $\omega_n(P_kAP_k) \longrightarrow \omega_n(A)$ for all $A\in\mathcal{L}^+(\mathcal{D})$ and $\psi_k \xrightarrow{t} \psi$ it follows that

$|\omega_n(A) - \omega_\varphi(A)| \leq |\omega_n(A) - \omega_n(P_kAP_k)| + |\omega_n(P_kAP_k)$ $- (1/\lambda)\langle\psi, A\psi\rangle| = |\omega_n(A) - \omega_n(P_kAP_k)| + (1/\lambda)|\langle\psi_k, P_kAP_k\psi_k\rangle -$ $- \langle\psi, A\psi\rangle|$ which goes to zero for $k\to\infty$.

Thus ω_n is a vector state and therefore $\omega\in Z$.

q.e.d.

In the second part of this section we add some results concerning the σ_0-sequentially completeness and closedness of some sets of functionals. Since the σ_0-topology is not metrizable one has to work with nets to consider closedness or completeness. In the bounded case it is known that the set of normal functionals is weakly sequentially complete (/1/,/16/). This result is not valid for $\mathcal{L}^+(\mathcal{D})$ as we shall see. Let us start with a lemma which is a weaker variant of Theorem III.1 of /1/.

Lemma 3.7

Let $\omega_k = \omega_k^n + \omega^s$ be a sequence of $\tau_\mathcal{D}$-continuous functionals on $\mathcal{L}^+(\mathcal{D})$, where ω_k^n are normal, ω^s is fixed and singular. Suppose $\omega_k(A) \longrightarrow 0$ for all $A\in\mathcal{L}^+(\mathcal{D})$ (i.e. $\omega_k \xrightarrow{\sigma_0} 0$) Then $\omega^s \equiv 0$ and clearly $\omega_k^n \xrightarrow{\sigma_0} 0$.

Proof:

It is enough to prove that $\omega^s(P) = 0$ for all projections $P\in\mathcal{B}(\mathcal{D})$. Indeed, the linear space generated by the projections of $\mathcal{B}(\mathcal{D})$ is $\tau_\mathcal{D}$-dense in $\mathcal{B}(\mathcal{D})$ hence in $\mathcal{L}^+(\mathcal{D})$. So let $P\in\mathcal{B}(\mathcal{D})$, $P\mathcal{H} = \mathcal{H}_1 \subset \mathcal{D}$. Consider $\mathcal{B}(\mathcal{H}_1)$ as a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ and denote the restrictions of the functionals above by $\bar\omega_k, \bar\omega_k^n, \bar\omega^s$. Then they fulfil the conditions of Theorem III.1 /1/. That means $\bar\omega^s \equiv 0$, especially $\bar\omega^s(I_{\mathcal{H}_1}) = \omega^s(P) = 0$.

q.e.d.

This lemma can be used to prove a result which demonstrates in a nice way the difference between the convergence of nets and sequences.

Lemma 3.8

The set V_0 of vector states on $\mathcal{L}^+(\mathcal{D})$ is σ_0-sequentially complete.

Proof:

Let $(\omega_{\psi_n}) \subset V_0$ be a σ_0-Cauchy sequence. Then considering $\omega_{\psi_n}(A^+A)$ it is immediately seen that $\sup\limits_n\|A\psi_n\| < \infty$ for all $A\in\mathcal{L}^+(\mathcal{D})$, i.e. (ψ_n) is a t-bounded sequence. By Proposition 2.1 there is a $B \geq 0$, $B\in\mathcal{B}(\mathcal{D})$ and a sequence $(\varphi_n)\subset\mathcal{H}$, $\|\varphi_n\|\leq 1$ so that $B\varphi_n = \psi_n$.

This leads to

$|\omega_{\psi_n}(A)| = |\langle\psi_n, A\psi_n\rangle| = |\langle B\varphi_n, AB\varphi_n\rangle| \leq \|BAB\|$ for all $A\in\mathcal{L}^+(\mathcal{D})$, i.e. (ω_{ψ_n}) is a sequence of equicontinuous functionals. Because (ω_{ψ_n}) is a weak Cauchy sequence, the equation $\omega(A) = \lim\omega_{\psi_n}(A)$ defines a positive linear functional on $\mathcal{L}^+(\mathcal{D})$ and $\omega(I) = 1$. Moreover, due to the equicontinuity of $\{\omega_{\psi_n}\}$ ω is also $\tau_\mathcal{D}$-continuous. Therefore $\omega\in V = P = Z$ (cf. Theorem 3.6) and $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $\omega_1\in V_0$, ω_2-singular. The sequence $\omega_n = \omega - \omega_{\psi_n} = (\lambda\cdot\omega_1 - \omega_{\psi_n}) +$ $+ (1-\lambda)\omega_2$ fulfils the assumptions of Lemma 3.7. Hence $(1-\lambda)\omega_2 =$ $= 0$ and $\omega = \omega_1 = \omega_\psi$ for some $\psi\in\mathcal{D}$, $\|\psi\| = 1$.

q.e.d.

We add some simple remarks.

Remarks 3.9

i) The proof of Lemma 3.8 can be a little bit modified to show that the set of all positive vector functionals $\{\omega_\varphi = \langle\varphi, \cdot\,\varphi\rangle, \varphi\in\mathcal{D}\}$ is σ_0-sequentially complete.

ii) It is trivial that $\psi_n \xrightarrow{t} \psi$ implies $\omega_{\psi_n} \xrightarrow{\sigma_0} \omega_\psi$. What about the converse? Let $\|\psi_n\| = \|\psi\| = 1$, $\omega_{\psi_n} \longrightarrow \omega_\psi$ as in the Lemma above. Then (ψ_n) is t-bounded. The weak compactness of the unit ball of $\mathcal{H}$ implies the existence of a subsequence (ψ_{n_k}) which is weakly convergent, say to $\chi\in\mathcal{H}$, i.e.

$\langle \psi_{n_k}, \varphi\rangle \longrightarrow \langle \chi,\varphi\rangle$ for all $\varphi \in \mathcal{H}$, hence $\psi_{n_k} \xrightarrow{\| \ \|} \chi$.

Let $(\varphi_j) \subset \mathcal{D}$ be an orthonormal basis of $\mathcal{H}$, $P_j = \langle \varphi_j, \cdot\, \rangle \varphi_j$.
Then $|\langle \psi_{n_k}, \varphi_j\rangle|^2 \longrightarrow |\langle \chi,\varphi_j\rangle|^2$ and $|\langle \psi_{n_k}, P_j \psi_{n_k}\rangle|^2 \longrightarrow |\langle \psi, P_j \psi\rangle|^2 = |\langle \psi, \varphi_j\rangle|^2$, i.e. $\|\psi\|^2 = \|\chi\|^2 = 1$. Moreover put $\varphi = \psi$, then $|\langle \psi_{n_k}, \psi\rangle|^2 \longrightarrow |\langle \chi,\psi\rangle|^2$ and $\langle \psi_{n_k}, P_\psi \psi_{n_k}\rangle = |\langle \psi_{n_k}, \psi\rangle|^2 \longrightarrow \|\psi\|^2 = 1$, i.e. $|\langle \chi,\psi\rangle| = 1$ and consequently $\chi = \lambda\psi$ with $|\lambda| = 1$.

This can be interpreted as follows: $\omega_{\psi_n} \xrightarrow{\sigma_0} \omega_\psi$ implies that any $\| \ \|$-convergent subsequence of (ψ_n) converges to the same element $[\psi]$ in the projective space $[\mathcal{H}]$ associated with $\mathcal{H}$, i.e. $[\mathcal{H}] = \mathcal{H}/\sim$, where $\sim$ is the equivalence relation $\psi \sim \chi$ if and only if $\psi = \lambda\chi$ for some λ with $|\lambda| = 1$. This is quite natural because ω_ψ on $\mathcal{L}^+(\mathcal{D})$ determines only $[\psi]$ but not ψ uniquely.

For the next result we need a simple lemma.

Lemma 3.9

The vector functional $\omega_{\varphi,\psi}(A) = \langle \varphi, A\psi\rangle$, $\psi \in \mathcal{D}$, $\psi \neq 0$, $\varphi \in \mathcal{H}$ is $\tau_\mathcal{D}$-continuous if and only if $\varphi \in \mathcal{D}$.

Proof:

One direction is trivial. The other direction follows from /15/. But for completeness let us give a direct prove.
Suppose that $\omega_{\varphi,\psi}$ is $\tau_\mathcal{D}$-continuous, i.e. $|\langle\varphi, A\psi\rangle| \leq \|BAB\|$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and some $B \geq 0$, $B \in \mathcal{B}(\mathcal{D})$. If $\varphi \notin \mathcal{D}$, then there is an $A \in \mathcal{L}^+(\mathcal{D})$ with $\varphi \notin \mathcal{D}(A^*)$. But this means that the functional $\omega(\chi) = \langle \varphi, A\chi\rangle$ is not norm-continuous on $\mathcal{D}$. Thus there exists a sequence $(\chi_n) \subset \mathcal{D}$, $\|\chi_n\| = 1$ so that $|\langle\varphi, A\chi_n\rangle| \longrightarrow \infty$.
Consider the operators $P_{\psi,\chi_n} = \langle \psi, \cdot\, \rangle \chi_n \in \mathcal{L}^+(\mathcal{D})$. Then

$$|\langle \varphi, AP_{\psi,\chi_n}\psi\rangle| = \|\psi\|^2 |\langle\varphi, A\chi_n\rangle| \longrightarrow \infty .$$

On the other hand $|\langle\varphi, AP_{\psi,\chi_n}\psi\rangle| \leq \|BAP_{\psi,\chi_n}B\| \leq \|B\psi\|\cdot\|BA\|$
which is a contradiction.

q.e.d.

Lemma 3.10

The set of normal functionals on $\mathcal{L}^+(\mathcal{D})$ is σ_0-sequentially closed but not σ_0-sequentially complete.

Proof:

The first part follows from Lemma 3.7. Indeed, if $\omega_n \xrightarrow{\sigma_0} \omega$ and $\omega = \omega_1 + \omega_2$, ω_1-normal, ω_2-singular, then $\omega_n' = \omega - \omega_n = (\omega_1 - \omega_n) + \omega_2$ fulfils the conditions of Lemma 3.7. hence $\omega_2 = 0$ and ω is normal. To see the second part consider $\psi \neq 0$, $\psi \in \mathcal{D}$, $\varphi \in \mathcal{H} \setminus \mathcal{D}$. Then there is a sequence $(\varphi_n) \subset \mathcal{D}$, $\varphi_n \xrightarrow{\| \ \|} \varphi$.

The vector functionals $\omega_{\varphi_n,\psi}(A) = \langle\varphi_n, A\psi\rangle$ are $\tau_\mathcal{D}$-continuous, $\omega_{\varphi_n,\psi}(A) \longrightarrow \omega_{\varphi,\psi}(A) = \langle\varphi, A\psi\rangle$ for all $A \in \mathcal{L}^+(\mathcal{D})$. But $\omega_{\varphi,\psi}$ is not normal because it is not $\tau_\mathcal{D}$-continuous by Lemma 3.9.

q.e.d.

Remark 3.11

In the proof of Lemma 3.8 the equicontinuity of the set $\{\omega_{\psi_n}\}$ was important. In the bounded case the equicontinuity is automatically fulfilled for w^*-Cauchy sequences of normal functionals. This is not the case for $\mathcal{L}^+(\mathcal{D})$ as the sequence $(\omega_{\varphi_n,\psi})$ above shows. Indeed, suppose $|\langle\varphi_n, A\psi\rangle| \leq \|BAB\|$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and some $B \in \mathcal{B}(\mathcal{D})$.
This would imply that (φ_n) is t-bounded since for $P_{\psi,\varphi_n} \in \mathcal{L}^+(\mathcal{D})$:

$$|\langle\varphi_n, A^+AP_{\psi,\varphi_n}\psi\rangle| = \|\psi\|^2 \|A\varphi_n\|^2 \leq \|BA^+AP_{\psi,\varphi_n}B\| \leq \|BA^+A\|\,\|B\varphi_n\|\cdot\|\psi\|$$

But (φ_n) is $\| \ \|$-bounded, so $\|B\varphi_n\| \leq C$ because $B \in \mathcal{B}(\mathcal{D})$. Thus (φ_n) is t-bounded and $\varphi_n \xrightarrow{\| \ \|} \varphi$ which implies $\varphi_n \xrightarrow{t} \varphi$ by /14/.
This is a contradiction.
Now one could give several conditions which would imply that σ_0-Cauchy sequences of normal functionals have a normal functional as limit. But we will not push this further. Let us only give a Corollary to Lemma 3.10 for the case where $\mathcal{L}^+(\mathcal{D})[\tau_\mathcal{D}]$ is a bornological space. A large class of examples of such (F)-spaces $\mathcal{D}$ was given in /18/.

Corollary 3.12

Let $\mathcal{L}^+(\mathcal{D})[\tau_\mathcal{D}]$ be a bornological space. Then the set of positive normal functionals is σ_0-sequentially complete. Especially, the set of normal states is σ_0-sequentially complete.

Proof:

If $\mathcal{L}^+(\mathcal{D})$ is bornologically then $\tau_\mathcal{D}$ coincides with the order topology and every positive functional on $\mathcal{L}^+(\mathcal{D})$ is $\tau_\mathcal{D}$-continuous /18/. On the other hand, if $\omega_n(A) \longrightarrow \omega(A)$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and $\omega_n \geq 0$, then ω is positive, hence $\tau_\mathcal{D}$-continuous. So the assertion follows from Lemma 3.10.

q.e.d.

We close this section with two results about singular functionals. The first one is the analog to /1/, Theorem III.5 for $\mathcal{L}^+(\mathcal{D})$.

Lemma 3.13

Let (ω_k) be a sequence of singular functionals on $\mathcal{L}^+(\mathcal{D})$. Then any weak limit point of (ω_k) is singular.

Proof:

Let ω be a weak limit point of (ω_k) and let (ω_α) be a subnet of (ω_k), $\omega_\alpha \xrightarrow{\sigma_0} \omega = \omega^n + \omega^s$. Again (cf. Lemma 3.7) it is enough to prove that $\omega^n(P) = 0$ for all projections $P \in \mathcal{B}(\mathcal{D})$.

Let $\bar{\omega}$, $\bar{\omega}^{n}$, $\bar{\omega}^{s}$, be the restrictions of the functionals above to $\mathcal{B}(\mathcal{H}_1) \subset \mathcal{L}^+(\mathcal{D})$, $\mathcal{H}_1 = P\mathcal{H} \subset \mathcal{D}$. Then these functionals fulfil the conditions of Theorem III.5 /1/, so $\bar{\omega}^{n} = 0$, i.e. $\omega^{n}(P) = 0$.

q.e.d.

Corollary 3.14

The set of singular functionals on $\mathcal{L}^+(\mathcal{D})$ is σ_0 -sequentially closed.

4. The state space of Op^*-algebras

In this section we start the investigation of the state space of general Op^*-algebras.

Remember that in the C^*-theory the fact that the unit ball in the dual space is w^*-compact allows to apply the Krein-Milman theorem. This leads to the well-known result that the state space of a C^*-algebra is the w^*-closed convex hull of pure states.

In contrast to this in the unbounded case the state space is not w^*-compact, even not w^*-bounded if $\mathcal{A}(\mathcal{D})$ contains unbounded operators. This can be seen by the following simple example. Let $A = A^+ \in \mathcal{A}(\mathcal{D})$ be unbounded. Then there is a sequence $(\varphi_n) \subset \mathcal{D}$, $\|\varphi_n\| = 1$ with $\|A\varphi_n\| \longrightarrow \infty$. By $\omega_n = \langle \varphi_n, \cdot\, \varphi_n \rangle$ we get a sequence of well-defined vector states but $\omega_n(A^+A) = \|A\varphi_n\|^2 \longrightarrow \infty$. So (ω_n) is not w^*-bounded.

Nevertheless there can be derived some results which correspond to those in the bounded case. But one has to take into account some refinements. On the one hand the topology $\tau_{\mathcal{D}}$ does not play an exceptional role in what follows. On the other hand in the proofs it is essential that states are strongly positive functionals. For $\mathcal{L}^+(\mathcal{D})$ the positive and strongly positive $\tau_{\mathcal{D}}$-continuous functionals coincide /15/. But this is in general not the case for general Op^*-algebras or other topologies. So in what follows on $\mathcal{L}^+(\mathcal{D})$ or $\mathcal{A}(\mathcal{D})$ there can be taken any locally convex topology τ so that the vector states are τ -continuous, and states are supposed to be strongly positive, normed and τ -continuous linear functionals.

The first proposition we are going to prove is Lemma 3.4.iii) of /4/ for Op^*-algebras. $E(\mathcal{A})$ is now thought to be in the context of a topology τ mentioned above.

Proposition 4.1

Let $\mathcal{A}(\mathcal{D})$ be an Op^*-algebra, $Q \subset E(\mathcal{A})$ a subset with the property:

if $A \in \mathcal{A}(\mathcal{D})_h$ (hermitean part) and $\omega(A) \geq 0$ for all $\omega \in Q$, then $A \geq 0$.

Under these assumptions the w^* - closed convex hull of Q coincides with $E(\mathcal{A})$.

Proof:

The proof is the same as in /4/. Since $E(\mathcal{A})$ is convex and w^*-closed the w^*-closed convex hull of Q is contained in $E(\mathcal{A})$.

Now let Q^o be the polar of Q in $\mathcal{A}(\mathcal{D})_h$. If $A \in \mathcal{A}(\mathcal{D})_h$, then

$A \in Q^o$ if and only if $\omega(A) \leq 1$ for all $\omega \in Q$

if and only if $\omega(I-A) \geq 0$ for all $\omega \in Q$

if and only if $I-A \geq 0$ if and only if $\omega(A) \leq 1$ for all $\omega \in E(\mathcal{A})$

if and only if $A \in E(\mathcal{A})^o$.

Hence $Q^o = E(\mathcal{A})^o = Q_1^o$, where Q_1 is the w^*-closed convex hull of Q.

Then by the bipolar theorem: $Q^{oo} = E(\mathcal{A})^{oo} = Q_1^{oo} = \overline{co(Q_1 \cup \{0\})} = \overline{co(E(\mathcal{A}) \cup \{0\})}$. Since Q_1 and $E(\mathcal{A})$ are convex and w^*-closed this implies $co(Q_1 \cup \{0\}) = co(E(\mathcal{A}) \cup \{0\})$, hence $Q_1 = E(\mathcal{A})$.

q.e.d.

An example of such a set Q is $V_o(\mathcal{A})$, the set of vector states. So we obtain as a conclusion:

Corollary 4.2

Let $\mathcal{A}(\mathcal{D})$ be an Op^*-algebra. Then $E(\mathcal{A})$ is the w^*-closed convex hull of the set of vector states on $\mathcal{A}(\mathcal{D})$.

In the C^*-theory the following fact is well-known /4/. Let $\mathcal{A}$ be a C^*-algebra (say with unit) acting irreducibly on $\mathcal{H}$. Then the vector states are pure.

In contrast to the C^*-case in the unbounded case many different notions of irreducibility can be introduced. The weakest seems to be the triviality of the weak commutant $\mathcal{A}(\mathcal{D})' = \{T \in \mathcal{B}(\mathcal{H}) : \langle \varphi, TA\psi \rangle = \langle A^+\varphi, T\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$, $A \in \mathcal{A}(\mathcal{D})\}$ (cf./23/).

If $\mathcal{A}(\mathcal{D})$ is selfadjoint the weak commutant coincides with the strong commutant $\mathcal{A}(\mathcal{D})'_s = \{T \in \mathcal{B}(\mathcal{H}) : AT\varphi = TA\varphi$ for all $\varphi \in \mathcal{D}$, $A \in \mathcal{A}(\mathcal{D})\}$

Clearly, in this case $\mathcal{A}(\mathcal{D})' \subset \mathcal{L}^+(\mathcal{D})$.

We will here use the following notion of irreducibility:

Definition 4.3

An Op^*-algebra is said to be topologically irreducible if

$$\mathcal{D}_\varphi = \{A\varphi : A \in \mathcal{A}(\mathcal{D})\}$$

is $t_{\mathcal{A}}$ -dense in $\mathcal{D}$ for all non-zero $\varphi \in \mathcal{D}$ (i.e. any such φ is a strongly cyclic vector for $\mathcal{A}(\mathcal{D})$).

It is not our intention to analyse here the whole hierarchy of possible irreducibility notions. Let us only remark that topological irreducibility implies the triviality of the (weak) commutant for selfadjoint Op^*-algebras.

Proposition 4.4

Let $\mathcal{A}(\mathcal{D})$ be a selfadjoint, topologically irreducible Op*-algebra. Then every vector state on $\mathcal{A}(\mathcal{D})$ is pure.

Proof:

Suppose $0 \leq \omega \leq \omega_\varphi$, i.e. $\omega(A) \leq \langle\varphi, A\varphi\rangle$ for all $A \geq 0$, $A \in \mathcal{A}(\mathcal{D})$. On $\mathcal{D}_\varphi \times \mathcal{D}_\varphi$ consider the sesquilinear form

$$(\psi, \chi) = \omega(B^+A) \quad \text{for } \psi = B\varphi,\ \chi = A\varphi .$$

It is easy to check that $(\ ,\ )$ is correctly defined, $(\psi,\psi) \geq 0$ for all $\psi \in \mathcal{D}_\varphi$. Moreover:

$$|(\psi,\psi)| = |\omega(B^+B)| \leq |\omega_\varphi(B^+B)| \leq \|\psi\|^2 .$$

This estimation can be continued onto $\mathcal{H} \times \mathcal{H}$ because $\mathcal{D}_\varphi$ is $\|\ \|$-dense in $\mathcal{H}$.

Thus there exists an operator $T \in \mathcal{B}(\mathcal{H})$ with

$$(\psi, \chi) = \langle\psi, T\chi\rangle \quad \text{for all } \psi, \chi \in \mathcal{H} .$$

If $\psi = B\varphi \in \mathcal{D}_\varphi$, then from $(\psi,\psi) = \langle\psi, T\psi\rangle \geq 0$ it follows that $T \geq 0$ on $\mathcal{D}_\varphi$ hence on $\mathcal{H}$. Now let $\psi = B\varphi$, $\chi = A\varphi$, $C \in \mathcal{A}(\mathcal{D})$. Then the two equalities

$$\langle\chi, TC\psi\rangle = \langle B\varphi, TCA\psi\rangle = (B\varphi, CA\psi) = \omega(B^+CA)$$

and

$$\langle C^+\chi, T\psi\rangle = \langle C^+B\varphi, TA\varphi\rangle = (C^+B\varphi, A\varphi) = \omega(B^+CA)$$ imply that

$$\langle\chi, TC\psi\rangle = \langle C^+\chi, T\psi\rangle \quad \text{for all } \psi, \chi \in \mathcal{D}_\varphi$$

From the $t_{\mathcal{A}}$-density of $\mathcal{D}_\varphi$ we conclude that this last equation is valid for all $\psi, \chi \in \mathcal{D}$. Hence $T \in \mathcal{A}(\mathcal{D})' = \mathcal{A}(\mathcal{D})'_s$. From the triviality of $\mathcal{A}(\mathcal{D})'$ (cf. remark before the proposition) one gets $T = \lambda I$, and $0 \leq \lambda \leq 1$. This is the desired result because

$$\omega(A) = \omega(I \cdot A) = (\varphi, A\varphi) = \langle\varphi, TA\varphi\rangle = \lambda\langle\varphi, A\varphi\rangle = \lambda\,\omega_\varphi(A).$$

q.e.d.

Corollary 4.5

i) Let $\mathcal{A}(\mathcal{D})$ be a selfadjoint, topologically irreducible Op*-algebra. Then $E(\mathcal{A})$ is the w^*-closed convex hull of pure states.

ii) Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint, then E is the w^*-closed convex hull of pure states.

Remark 4.6

By quite other methods Corollary 4.5.ii) was obtained in /15/ for the case that $\mathcal{D}[t]$ is an (F)-space.

Let us further remark that the representation of positive functionals dominated by vector functionals (cf. Proposition 4.4) is not new. It is well-known in the bounded case and also used in the unbounded case in several version (for one possibility see /22/).

At the end let us summarize some of the structure properties of the state space E of $\mathcal{L}^+(\mathcal{D})$ in the case that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint, $\mathcal{D}[t]$ is an (F)-space and the topology on $\mathcal{L}^+(\mathcal{D})$ is $\tau_{\mathcal{D}}$.

Proposition 4.7

i) E is the w^*-closed convex hull of V_0. V_0 is w^*-sequentially complete. If $\mathcal{D}[t]$ is a Montel space then V_0 is w^*-closed.

ii) The normal states are w^*-sequentially closed and at the same time w^*-dense in E.

Proof:

i) The first two assertions are already proved. The last assertion follows from Theoreme 3.3 and 3.6 and the fact that in the case of Montel domains there are no non-trivial singular functionals on $\mathcal{L}^+(\mathcal{D})$.

ii) The first part is the content of Lemma 3.10, while the second part follows from the first assertion of i).

q.e.d.

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Received by Publishing Department
on October 9. 1985.

Леффлер Ф., Тиммерманн В. E5-85-727

О структуре пространства состояний максимальных Op* алгебр

Работа посвящена теории линейных, непрерывных функционалов на алгебрах неограниченных операторов. В частности исследуется пространство состояний на максимальной Op*-алгебре неограниченных операторов заданных на областях Фреше в гильбертовом пространстве. Показано, что слабые замыкания множеств векторных и чистых состояний на таких алгебрах совпадают. Кроме того, дано описание этого замыкания множеств векторных и чистых состояний подобно ограниченному случаю. Рассмотрен ряд свойств таких состояний. В частности, приведены результаты по последовательностям векторных, нормальных и сингулярных состояний. Дана характеристика пространства состояний на общих Op*-алгебрах. Использовались методы теории операторов, теории операторных алгебр и функционального анализа.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Löffler F., Timmermann W. E5-85-727

On the Structure of the State Space of Maximal Op*-Algebras

The paper is devoted to the study of linear, continuous functionals on algebras of unbounded operators. Especially, there is investigated the structure of the state space of maximal Op*-algebras defined on (F)-domains. It is proved that the vector state space and the pure state space coincide. Moreover there is given a description of these spaces which is similar to the bounded case. Several properties of such states are mentioned. Especially, there are proved results concerning sequences of vector, normal and singular states. In section 4 of the paper the state space of general Op*-algebras is characterized. There are used methods from operator theory, theory of operator algebras and functional analysis.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985