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SINGULAR STATES ON MAXIMAL OP*-ALGEBRAS

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1. Introduction

In the last 15 years the theory of topological algebras of unbounded operators has been substantially developed. This concerns the study of the topological and order structure, commutants, ideals, special classes of algebras and representations as well as applications to quantum statistics /10/.

As for states on topological operator algebras, satisfying results were obtained first of all for normal states (see /19/ for a short summary). The present paper is the first of a series devoted to the investigation of the structure of the state space of Op^*-algebras. We start here with the study of positive singular functionals on the maximal Op^*-algebra $\mathcal{L}^+(\mathcal{D})$ on (F)-domains. The paper is organized as follows. Section 2 contains the necessary definitions, notations and auxiliary results. In section 3 we describe a restriction-extension procedure which relates classes of $\tau_{\mathcal{D}}$-continuous functionals on $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{H})$. This gives a useful method to transform results from the bounded case ($\mathcal{B}(\mathcal{H})$) to the unbounded one ($\mathcal{L}^+(\mathcal{D})$). As an application we prove that states can be uniquely decomposed into the sum of normal and singular positive functionals. In section 4 a representation theorem for positive singular functionals is given (see also /19/), which has interesting applications. For example one gets that singular states are w^*-limits of vector states. This has the following important conclusion. Despite the fact that the state space of $\mathcal{L}^+(\mathcal{D})$ (i.e., the set of $\tau_{\mathcal{D}}$-continuous states) is not w^*-compact, it is the w^*-closed convex hull of vector states, hence of pure states. A more general result derived by quite other methods is given in /13/.

2. Preliminaries

For a dense linear manifold $\mathcal{D}$ in a separable Hilbert space $\mathcal{H}$ the set of linear operators $\mathcal{L}^+(\mathcal{D}) = \{A: A\mathcal{D} \subset \mathcal{D},\ A^*\mathcal{D} \subset \mathcal{D}\}$ is a $*$-algebra with respect to the usual operations and the involution $A \longrightarrow A^+ = A^* \restriction \mathcal{D}$. An Op^*-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing the identity operator I. The graph topology $t_{\mathcal{A}}$ on $\mathcal{D}$ induced by

$\mathcal{A}(\mathcal{D})$ is given by the family of seminorms $\varphi \longrightarrow \|A\varphi\|$ for all $A \in \mathcal{A}(\mathcal{D})$. Denote $t_{\mathcal{L}^+(\mathcal{D})}$ simply by t. This topologization of $\mathcal{D}$ gives rise to a canonical rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}'[t']$ and a canonical dual pair $(\mathcal{D}, \mathcal{D}')$. Here t' is the strong topology in $\mathcal{D}'$. Let $\sigma = \sigma(\mathcal{D}, \mathcal{D}')$ be the weak topology in $\mathcal{D}$. Remember that a sequence $(\varphi_n) \subset \mathcal{D}$ is σ-convergent to zero ($\varphi_n \xrightarrow{\sigma} 0$) if and only if $\{\varphi_n\}$ is t-bounded and $\langle\varphi, \varphi_n\rangle \longrightarrow 0$ for all $\varphi \in \mathcal{D}$ hence for all $\varphi \in \mathcal{H}$.
An Op^*-algebra $\mathcal{A}(\mathcal{D})$ is called

closed if $\mathcal{D} = \bigcap_{A \in \mathcal{A}} \mathcal{D}(\bar{A})$ or equivalently if $\mathcal{D}[t]$ is complete;

selfadjoint if $\mathcal{D} = \mathcal{D}_* \equiv \bigcap_{A \in \mathcal{A}} \mathcal{D}(A^*)$.

In Op^*-algebras there can be defined a lot of topologies (cf. e.g. /9/-/11/,/14/,/15/). We mention only those used here:

the uniform topology $\tau_{\mathcal{D}}$ given by the family of seminorms

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi \in \mathcal{M}} |\langle\varphi, A\psi\rangle|$$

where $\mathcal{M}$ runs over all $t_{\mathcal{A}}$-bounded sets of $\mathcal{D}$;

the topology $\tau^c_{\mathcal{D}}$ given by the family of seminorms

$$A \longrightarrow \|A\|_{\mathcal{N}} = \sup_{\varphi,\psi \in \mathcal{N}} |\langle\varphi, A\psi\rangle|,$$

where $\mathcal{N}$ runs over all relatively $t_{\mathcal{A}}$-compact subsets of $\mathcal{D}$.
Later on we will use the fact that $\tau_{\mathcal{D}}$ and $\tau^c_{\mathcal{D}}$ are not only defined on $\mathcal{L}^+(\mathcal{D})$ but also on $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ and hence on $\mathcal{B}(\mathcal{H})$ /10/,/5/.

The most important domains $\mathcal{D}$ are those where t is a Fréchet-topology, i.e. $\mathcal{D}[t]$ is complete and t can be given by $\{\|A_n\cdot\|, n \in \mathbb{N}\}$ with $A_n = A_n^+$, $I = A_0 \le A_1 \le \dots$. A special type of such domains is of the form

$$\mathcal{D} = \mathcal{D}^\infty(T) \equiv \bigcap_{n \ge 0} \mathcal{D}(T^n), \quad T = T^* \ge I.$$

In what follows we allways suppose that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint and $\mathcal{D}[t]$ is an (F)-space. We only remark that some of the results are also valid in more general situations.
To simplify notations we denote a bounded operator $A \in \mathcal{L}^+(\mathcal{D})$ and its closure $\bar{A} \in \mathcal{B}(\mathcal{H})$ by the same letter A. The following sets are two-sided $*$-ideals in $\mathcal{L}^+(\mathcal{D})$ and play an important role in the description of the topologies $\tau_{\mathcal{D}}$ and $\tau^c_{\mathcal{D}}$ (/4/,/5/,/11/,/18/,/19/):

$\mathcal{B}(\mathcal{D}) = \{T: T\mathcal{H} \subset \mathcal{D},\ T^*\mathcal{H} \subset \mathcal{D}\} = \{T: AT,\ AT^* \text{ bounded for all } A \in \mathcal{L}^+(\mathcal{D})\}$

$\mathcal{S}_\infty(\mathcal{D}) = \{T: T \in \mathcal{S}_\infty(\mathcal{H}) \cap \mathcal{B}(\mathcal{D})\} = \{T: AT,\ AT^* \in \mathcal{S}_\infty(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathcal{D})\}$.

Here $\mathcal{S}_\infty(\mathcal{H})$ denotes the $*$-ideal of compact operators on $\mathcal{H}$.

Proposition 2.1

Let $\mathcal{K}$ be the unit ball in $\mathcal{H}$.
i) The family $\{B\mathcal{K} : B \in \mathcal{B}(\mathcal{D}),\ B \ge 0,\ \operatorname{Ker} B = (0),\ \mathcal{R}(B)$ t-dense in $\mathcal{D}\}$
is a fundamental system of t-bounded sets. Hence $\tau_{\mathcal{D}}$ can be given by the seminorms $A \longrightarrow \|BAB\|$, B as above.
ii) The family $\{C\mathcal{K} : C \in \mathcal{S}_\infty(\mathcal{D}),\ C \ge 0,\ \operatorname{Ker} C = (0),\ \mathcal{R}(C)$ t-dense in $\mathcal{D}\}$
is a fundamental system of relatively t-compact sets. Hence $\tau^c_{\mathcal{D}}$ can be given by $A \longrightarrow \|CAC\|$, C as above.

An important consequence is the fact that $\mathcal{B}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, consequently in $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{H})$ /5/. In what follows we need some special assertions along this line and collect them in the next lemma.

Lemma 2.2

Let $B \in \mathcal{B}(\mathcal{D})$, $B \ge 0$, $B = \int_0^b \lambda\, dE_\lambda$. For $a > 0$ put $P_a = \int_a^b dE_\lambda$ and $A_a = P_a A P_a$ for $A \in \mathcal{L}^+(\mathcal{D})$. Then

i) P_a and A_a are elements of $\mathcal{B}(\mathcal{D})$.

ii) $\lim_{a \to 0} \|B^\alpha(A - A_a)B^\alpha\| = 0$ for all $\alpha > 0$.

Next we consider linear functionals on Op^*-algebras. In contrast to the C^*-case one has to distinguish two notions of positivity. Namely, a linear functional ω on an Op^*-algebra $\mathcal{A}(\mathcal{D})$ is said to be

positive if $\omega(A) \ge 0$ for all $A \in \mathcal{P}(\mathcal{A}(\mathcal{D})) = \{A = \sum A_i^+ A_i,\ A_i \in \mathcal{A}(\mathcal{D})\}$
where the sum above is finite;

strongly positive if $\omega(A) \ge 0$ for all $A \in \mathcal{K}(\mathcal{A}(\mathcal{D})) = \{A \in \mathcal{A}(\mathcal{D}):$
$\langle\varphi, A\varphi\rangle \ge 0$ for all $\varphi \in \mathcal{D}\}$.

Remark that these definitons hold for any $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$, it is not necessary that they contain I. The problem of $\tau_{\mathcal{D}}$-continuity of strongly positive linear functionals was investigated in /14/.
Clearly, $\mathcal{P}(\mathcal{A}(\mathcal{D})) \subset \mathcal{K}(\mathcal{A}(\mathcal{D}))$ and the inverse inclusion depends on the fact whether or not $\mathcal{A}(\mathcal{D})$ contains the square root of its positive operators.
In /7/ it was shown that $B \in \mathcal{B}(\mathcal{D})$ implies $B^\alpha \in \mathcal{B}(\mathcal{D})$ for all $\alpha > 0$. Hence $\mathcal{P}(\mathcal{B}(\mathcal{D})) = \mathcal{K}(\mathcal{B}(\mathcal{D}))$. Moreover, considerations as in Lemma 2.2 show that the following facts are true.

Lemma 2.3

i) $\mathcal{P}(\mathcal{B}(\mathcal{D}))$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{P}(\mathcal{L}^+(\mathcal{D}))$ and $\mathcal{H}(\mathcal{L}^+(\mathcal{D}))$.

ii) On $\mathcal{L}^+(\mathcal{D})$ the sets of $\tau_{\mathcal{D}}$-continuous positive and $\tau_{\mathcal{D}}$-continuous strongly positive linear functionals coincide.

In view of this lemma it is unambiguously to speak about $\tau_{\mathcal{D}}$-continuous states on $\mathcal{L}^+(\mathcal{D})$. This set is denoted by $E = E(\mathcal{L}^+(\mathcal{D})) = \{\omega : \omega \geq 0, \ \omega(I) = 1, \ \tau_{\mathcal{D}}\text{-continuous}\}$.

Let $\mathcal{L}^+(\mathcal{D})'$ denote the dual space to $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ and let $\sigma_* = \sigma(\mathcal{L}^+(\mathcal{D})', \mathcal{L}^+(\mathcal{D}))$ be the w*-topology in $\mathcal{L}^+(\mathcal{D})'$.

We need the following subsets of E:

vector states: $V_0 = V_0(\mathcal{L}^+(\mathcal{D})) = \{\omega \in E: \ \omega(A) = \langle \varphi, A\varphi \rangle, \ \varphi \in \mathcal{D}, \ \|\varphi\| = 1\}$

pure states : $P_0 = P_0(\mathcal{L}^+(\mathcal{D})) = \{\omega \in E: \ \omega_1 \in \mathcal{L}^+(\mathcal{D})', \ \omega_1 \geq 0, \ \omega_1 \leq \omega$ implies $\omega_1 = \lambda\omega, \ \lambda \in [0,1]\}$

Clearly, $\omega \in P_0$ if and only if ω cannot be represented as a nontrivial convex combination of $\omega_1, \omega_2 \in E$.

As in the bounded case one defines:

vector state space $V = V(\mathcal{L}^+(\mathcal{D})) = \sigma_*$-closure of V_0

pure state space $P = P(\mathcal{L}^+(\mathcal{D})) = \sigma_*$-closure of P_0

These set of states will be investigated in detail in /13/.

To define normal and singular functionals on $\mathcal{L}^+(\mathcal{D})$ let us introduce the following subsets of $\mathcal{L}^+(\mathcal{D})$ which are two-sided *-ideals

$\mathcal{J}_1(\mathcal{D}) = \{T \in \mathcal{L}^+(\mathcal{D}): AT, AT^* \in \mathcal{J}_1(\mathcal{H})$ for all $A \in \mathcal{L}^+(\mathcal{D})\}$,

$\mathcal{F}(\mathcal{D}) = \{F \in \mathcal{L}^+(\mathcal{D}): \ \dim F\mathcal{D} < \infty\}$.

Here $\mathcal{J}_1(\mathcal{H})$ stands for the set of nuclear operators on $\mathcal{H}$.

It appears that the $\tau_{\mathcal{D}}$-closure of $\mathcal{F}(\mathcal{D})$, $\mathcal{C}(\mathcal{D}) \equiv \overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$ is a very appropriate generalization of the set of compact operators to the unbounded case. Among other things the following assertions are valid /6/,/12/:

Proposition 2.4

i) $A \in \mathcal{C}(\mathcal{D})$ if and only if $(A\varphi_n)$ is t-convergent to zero for any sequence (φ_n) which is σ-convergent to zero.

ii) If $\mathcal{D}[t]$ is not a Montel space, then $\mathcal{C}(\mathcal{D})$ is the only non-trivial, $\tau_{\mathcal{D}}$-closed two-sided *-ideal in $\mathcal{L}^+(\mathcal{D})$.

iii) $\mathcal{D}[t]$ is a Montel space if and only if $\mathcal{C}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})$.

These properties give rise to the following definition.

Definition 2.5

A linear functional ω on $\mathcal{L}^+(\mathcal{D})$ is said to be

normal if $\omega(A) = \operatorname{Tr} AT$ for some $T \in \mathcal{J}_1(\mathcal{D})$ and all $A \in \mathcal{L}^+(\mathcal{D})$;

singular if ω is $\tau_{\mathcal{D}}$-continuous and $\omega(C) = 0$ for all $C \in \mathcal{C}(\mathcal{D})$.

Let us remark that while normal functionals are automatically $\tau_{\mathcal{D}}$-continuous (even $\tau^c_{\mathcal{D}}$-continuous) we have included the $\tau_{\mathcal{D}}$-continuity in the definition of singular functionals.

Proposition 2.4 iii) shows that if $\mathcal{D}[t]$ is a Montel space there does not exist a non-trivial singular functional.

The general procedure of finding the normal part of $\omega \in \mathcal{L}^+(\mathcal{D})'$ (cf. e.g. /15/) gives immediately the following lemma.

Lemma 2.6

Every $\omega \in \mathcal{L}^+(\mathcal{D})'$ can be uniquely decomposed into $\omega = \omega_n + \omega_s$, where ω_n, ω_s are respectively $\tau_{\mathcal{D}}$-continuous normal and singular functionals.

In section 3, Theorem 3.4 we give a more strong result.

3. A restriction-extension procedure

The $\tau_{\mathcal{D}}$-density of $\mathcal{B}(\mathcal{D})$ in $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{H})$ leads to the following procedure.

Let $\omega \in \mathcal{L}^+(\mathcal{D})'$ be given. By restriction to $\mathcal{B}(\mathcal{D})$ and extension to $\mathcal{B}(\mathcal{H})$ one gets a unique $\tau_{\mathcal{D}}$-continuous linear functional on $\mathcal{B}(\mathcal{H})$; notation: $\omega \longrightarrow \tilde{\omega}$.

Conversely, let ω be a $\tau_{\mathcal{D}}$-continuous linear functional on $\mathcal{B}(\mathcal{H})$. Restriction to $\mathcal{B}(\mathcal{D})$ and extension to $\mathcal{L}^+(\mathcal{D})$ gives again a unique $\tau_{\mathcal{D}}$-continuous linear functional on $\mathcal{L}^+(\mathcal{D})$; notation: $\omega \longrightarrow \hat{\omega}$.

Obviously: $\omega = \hat{\tilde{\omega}} \in \mathcal{L}^+(\mathcal{D})'$, $\omega = \tilde{\hat{\omega}} \in \mathcal{B}(\mathcal{H})'$, ω - $\tau_{\mathcal{D}}$-continuous.

This procedure will be frequently used (see also /13/). So we collect some of its properties in the next lemmata. Before doing so let us mention that in the context of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ one would call this an extension-restriction procedure (e.g. $\mathcal{L}^+(\mathcal{D}) \longrightarrow \mathcal{L}(\mathcal{D}, \mathcal{D}') \longrightarrow \mathcal{B}(\mathcal{H})$ and so on). But because we do not want to leave $\mathcal{H}$ we prefer this one.

A direct consequence of the definition and of Lemma 2.2 is the following lemma.

Lemma 3.1

i) If $\omega \in \mathcal{L}^+(\mathcal{D})'$, $|\omega(A)| \leq \|BAB\|$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and some $B \in \mathcal{B}(\mathcal{D})$, then
$|\tilde{\omega}(A)| \leq \|BAB\|$ for all $A \in \mathcal{B}(\mathcal{H})$ and moreover

$$\tilde{\omega}(A) = \lim_{a \to 0} \omega(P_a A P_a).$$

and conversely

ii) If $\omega \in \mathcal{B}(\mathcal{H})'$, $\tau_{\mathcal{D}}$-continuous, $|\omega(A)| \leq \|BAB\|$ for all $A \in \mathcal{B}(\mathcal{H})$, then $|\hat{\omega}(A)| \leq \|BAB\|$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and $\hat{\omega}(A) = \lim_{a \to 0} \omega(P_a A P_a)$.

iii) The same assertions are true for $\tau^c_{\mathcal{D}}$-continuous linear functionals on $\mathcal{L}^+(\mathcal{D})$, $\mathcal{B}(\mathcal{H})$ respectively.

In the next lemma we will see which sets of $\tau_{\mathcal{D}}$-continuous linear functionals mutually correspond in this procedure.

Lemma 3.2

In the restriction-extension procedure described above the following sets of $\tau_{\mathcal{D}}$-continuous linear functionals mutually correspond:

i) positive functionals on $\mathcal{L}^+(\mathcal{D}) \longleftrightarrow$ positive functionals on $\mathcal{B}(\mathcal{H})$;

ii) normal functionals on $\mathcal{L}^+(\mathcal{D}) \longleftrightarrow$ normal functionals on $\mathcal{B}(\mathcal{H})$, more exactly: $\omega(A) = \mathrm{Tr}\, AT$, $T \in \mathcal{J}_1(\mathcal{D})$ implies $\tilde{\omega}(A) = \mathrm{Tr}\, AT$, and conversely, $\omega(A) = \mathrm{Tr}\, AT$, $A \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{J}_1(\mathcal{H})$, ω-$\tau_{\mathcal{D}}$-continuous implies $T \in \mathcal{J}_1(\mathcal{D})$ and $\hat{\omega}(A) = \mathrm{Tr}\, AT$.

iii) singular functionals on $\mathcal{L}^+(\mathcal{D}) \longleftrightarrow$ singular functionals on $\mathcal{B}(\mathcal{H})$

iv) pure states on $\mathcal{L}^+(\mathcal{D}) \longleftrightarrow$ pure states on $\mathcal{B}(\mathcal{H})$.

Proof:

i) follows from Lemma 2.3.

ii) Let $\omega(A) = \mathrm{Tr}\, AT$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and some $T \in \mathcal{J}_1(\mathcal{D})$. Because $T \in \mathcal{J}_1(\mathcal{H})$ it follows immediately that $\tilde{\omega}(A) = \mathrm{Tr}\, AT$ for all $A \in \mathcal{B}(\mathcal{H})$. On the other hand, if ω on $\mathcal{B}(\mathcal{H})$ is $\tau_{\mathcal{D}}$-continuous and $\omega(A) = \mathrm{Tr}\, AT$ for all $A \in \mathcal{B}(\mathcal{H})$ and some $T \in \mathcal{J}_1(\mathcal{H})$, then $\hat{\omega}$ is $\tau_{\mathcal{D}}$-continuous on $\mathcal{L}^+(\mathcal{D})$ and can be decomposed into a linear combination of positive $\tau_{\mathcal{D}}$-continuous linear functionals which are normal if restricted to $\mathcal{F}(\mathcal{D})$ /15/. Hence $\hat{\omega}(F) = \mathrm{Tr}\, FS$ for all $F \in \mathcal{F}(\mathcal{D})$ and some $S \in \mathcal{J}_1(\mathcal{D})$. Since $\omega(F) = \hat{\omega}(F)$ we get $S = T$. Thus $\hat{\omega}$ is normal on $\mathcal{L}^+(\mathcal{D})$ (using the fact that $\omega \longrightarrow \hat{\omega}$ is unique).

iii) Let ω be singular on $\mathcal{L}^+(\mathcal{D})$, then $\omega(F) = 0$ for all $F \in \mathcal{F}(\mathcal{D})$. But $\mathcal{F}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{F}(\mathcal{H})$ hence in $\mathcal{J}_\infty(\mathcal{H})$, so $\tilde{\omega}(C) = 0$ for all $C \in \mathcal{J}_\infty(\mathcal{H})$, i.e., $\tilde{\omega}$ is singular. The other direction follows similarly.

iv) Let ω be pure on $\mathcal{B}(\mathcal{H})$. Suppose $\hat{\omega} = \lambda\omega_1 + (1-\lambda)\omega_2$, ω_1, ω_2 - positive and $\tau_{\mathcal{D}}$-continuous on $\mathcal{L}^+(\mathcal{D})$. This would imply $\omega = \tilde{\hat{\omega}} = \lambda\tilde{\omega}_1 + (1-\lambda)\tilde{\omega}_2$ with $\tilde{\omega}_i \in \mathcal{B}(\mathcal{H})'$, $\tilde{\omega}_i$ - $\tau_{\mathcal{D}}$-continuous and positive according to i). This is a contradiction. In the same way the other direction can be proved.

q.e.d.

It is necessary to add the following remark. The notion "pure state" on $\mathcal{B}(\mathcal{H})$ is not unambiguous, because we can consider decompositions within the set of all states or only within the $\tau_{\mathcal{D}}$-continuous states. We have in mind the $\tau_{\mathcal{D}}$-continuous states alone. To consider pureness in the context of all states and derive the direction $\longrightarrow$ in Lemma 3.2 iv) one would need a result of the following type: if $\omega_1 \leq \omega$, ω - $\tau_{\mathcal{D}}$-continuous on $\mathcal{B}(\mathcal{H})$, ω_1 positive, then ω_1 is $\tau_{\mathcal{D}}$-continuous, too.

Nevertheless normal pure states (i.e., vector states) on $\mathcal{L}^+(\mathcal{D})$ lead to pure states (vector states) on $\mathcal{B}(\mathcal{H})$ also in the context of continuity with respect to the norm in $\mathcal{B}(\mathcal{H})'$ as the next corollary shows.

Corollary 3.3

i) A vector functional $\omega_{\varphi,\psi}(A) = \langle \varphi, A\psi \rangle$ on $\mathcal{B}(\mathcal{H})$ is $\tau_{\mathcal{D}}$-continuous if and only if $\varphi, \psi \in \mathcal{D}$.

ii) The vector states $\omega_\varphi(A) = \langle \varphi, A\varphi \rangle$ on $\mathcal{L}^+(\mathcal{D})$ are pure states.

iii) Let ω be a state on $\mathcal{L}^+(\mathcal{D})$ so that $\omega(C) = \omega_\varphi(C)$ for all $C \in \mathcal{C}(\mathcal{D})$. Then $\omega = \omega_\varphi$ on $\mathcal{L}^+(\mathcal{D})$.

Proof:

i) follows from Lemma 3.2 ii) and the fact that $P_{\varphi,\psi} = \langle \varphi, \cdot \rangle \psi \in \mathcal{J}_1(\mathcal{D})$ if and only if $\varphi, \psi \in \mathcal{D}$.

ii) Lemma 3.2.ii) gives that $\omega = \omega_\varphi$ on $\mathcal{L}^+(\mathcal{D})$ implies $\tilde{\omega} = \omega_\varphi$ on $\mathcal{B}(\mathcal{H})$. Because vector states on $\mathcal{B}(\mathcal{H})$ are pure, the assertion follows from Lemma 3.2.iv).

iii) It follows that $\tilde{\omega}$ on $\mathcal{B}(\mathcal{H})$ has the property $\tilde{\omega}(C) = \tilde{\omega}_\varphi(C) = \omega_\varphi(C)$ for all $C \in \mathcal{J}_\infty(\mathcal{H})$. Then by /21/ $\tilde{\omega} = \tilde{\omega}_\varphi = \omega_\varphi$ on $\mathcal{B}(\mathcal{H})$, hence $\omega = \hat{\tilde{\omega}} = \omega_\varphi$ by Lemma 3.2.ii).

q.e.d.

Next we prove the decomposition theorem for positive $\tau_{\mathcal{D}}$-continuous linear functionals.

Theorem 3.4

Let ω be a positive $\tau_{\mathcal{D}}$-continuous linear functional on $\mathcal{L}^+(\mathcal{D})$. Then there is a unique decomposition

$$\omega = \omega_n + \omega_s$$

such that $\omega_n \geq 0$, normal, $\omega_s \geq 0$ singular.

Proof:

By Lemma 2.6 we have a unique decomposition $\omega = \omega^\circ_n + \omega^\circ_s$ with $\omega^\circ_n \geq 0$, normal, ω°_s singular. So it remains to prove that $\omega^\circ_s \geq 0$.

Let $\tilde{\omega}$, $\tilde{\omega}^\circ_n$, $\tilde{\omega}^\circ_s$ be the corresponding functionals on $\mathcal{B}(\mathcal{H})$. Then

$\tilde{\omega} = \tilde{\omega}_n^\circ + \tilde{\omega}_s^\circ$. On the other hand $\tilde{\omega}$ can be uniquely decomposed into $\tilde{\omega} = \omega_1 + \omega_2$ with $\omega_1 \geq 0$, normal and $\omega_2 \geq 0$ singular. Since the normal parts again coincide on $\mathfrak{F}(\mathcal{D})$, the corresponding trace class operators are the same, hence $\tilde{\omega}_n^\circ = \omega_1$, i.e., $\omega_2 = \tilde{\omega}_s^\circ \geq 0$. So the assertion follows from Lemma 3.2.i), namely $\omega_s^\circ \geq 0$.

q.e.d.

Corollary 3.5

Let $\omega \in P_o$, i.e., ω is a pure state on $\mathcal{L}^+(\mathcal{D})$. Then either ω is a vector state or ω is a pure singular state.

Proof:

Let $\omega \in P_o$, then $\omega = \omega_n + \omega_s = \omega_n(I)(\omega_n / \omega_n(I)) + \omega_s(I)(\omega_s / \omega_s(I))$ is a convex linear combination of states. Since ω is pure, either $\omega_n(I) = 0$, i.e., $\omega_n = 0$ and $\omega = \omega_s$ is a pure singular state, or $\omega_s(I) = 0$, i.e., $\omega_s = 0$ and $\omega = \omega_n$ is a pure normal state, hence a vector state.

q.e.d.

4. Singular positive functionals on $\mathcal{L}^+(\mathcal{D})$

In this section we start with some simple equivalent characterizations of singular positive functionals which correspond to those of the bounded case /16/,/17/. Then there is given a representation theorem of singular states on $\mathcal{L}^+(\mathcal{D})$. This result has important conclusions, for example about the structure of the state space $E(\mathcal{L}^+(\mathcal{D}))$, cf. Theorem 4.8.

Lemma 4.1

Let ω be a positive, $\tau_{\mathcal{D}}$-continuous linear functional on $\mathcal{L}^+(\mathcal{D})$. Then the following statements are equivalent:

i) ω is singular.

ii) The only positive normal functional ϱ with $\varrho \leq \omega$ is $\varrho = 0$.

iii) For any projection $E \in \mathcal{L}^+(\mathcal{D})$ there is a projection $F \neq 0$, $F \in \mathcal{L}^+(\mathcal{D})$ so that $F \leq E$ and $\omega(F) = 0$.

Proof:

We show i) $\longleftrightarrow$ ii), i) $\longleftrightarrow$ iii).

i) $\longrightarrow$ ii): suppose ϱ is normal and $\varrho \leq \omega$, then $\varrho(F) = 0$ for all $F \in \mathfrak{F}(\mathcal{D})$, hence $\varrho = 0$.

ii) $\longrightarrow$ i): If ω is not singular, then by Theorem 3.4 $\omega = \omega_n + \omega_s$ with $\omega_n \geq 0, \omega_n \neq 0$. Thus $\omega_n \leq \omega$.

i) $\longrightarrow$ iii): If $\omega(E) = 0$ then take $E = F$. If $\omega(E) \neq 0$, then there is a finite dimensional $F \leq E$ so that $F \in \mathcal{L}^+(\mathcal{D})$. Clearly, $\omega(F) = 0$ follows.

iii) $\longrightarrow$ i): Take E one-dimensional, $E \in \mathfrak{F}(\mathcal{D})$. Then $\omega(E) = 0$, so $\omega(F) = 0$ for all $F \in \mathfrak{F}(\mathcal{D})$ and consequently $\omega(C) = 0$ for all $C \in \mathcal{C}(\mathcal{D})$. This means ω is singular.

q.e.d.

Lemma 4.2

Let ω be singular and positive on $\mathcal{L}^+(\mathcal{D})$. Then for all projections $P \in \mathcal{L}^+(\mathcal{D})$, $\omega(P) = 1$ there is a projection $Q \in \mathcal{L}^+(\mathcal{D})$, $Q \leq P$ such that $P-Q$ is infinite dimensional and $\omega(Q) = 1$.

Proof:

It is easy to see that $P\mathcal{D}$ is a t-closed subspace of $\mathcal{D}$ (i.e., again an (F)-space). Moreover one has the characterization:

$P \in \mathcal{C}(\mathcal{D})$ if and only if $P\mathcal{D}$ is a Montel space if and only if $P\mathcal{D}$ does not contain an infinite dimensional Hilbert space $\mathcal{H}_o$ (i.e., $P\mathcal{D}$ is of type I /8/). So $\omega(P) = 1$ means $P \notin \mathcal{C}(\mathcal{D})$ and consequently there is such an $\mathcal{H}_o \subset P\mathcal{D} \subset \mathcal{D}$. Let P_o be the orthoprojection onto $\mathcal{H}_o$. If $\omega(P_o) = 0$, then we are done taking $Q = P-P_o$ which must be infinite dimensional. Suppose $\omega(P_o) = b \neq 0$, i.e., $\omega(P-P_o) = 1-b$. Applying the reasoning of /1/,p.305 to $\mathcal{H}_o$ and P_o one gets a P_1 so that $P_1 \leq P_o$, $P_o - P_1$ is infinite dimensional and $\omega(P_1) = b$. Putting $Q = P_1 + (P-P_o)$ we get the desired result.

q.e.d.

Our next aim is to describe all positive singular functionals on $\mathcal{L}^+(\mathcal{D})$. Let us remember the situation in the bounded case. It has often been mentioned that $\mathcal{B}(\mathcal{H})$, $\mathcal{J}_1(\mathcal{H})$ and $\mathcal{J}_\infty(\mathcal{H})$ are the non-commutative analogs of l^∞ , l^1 and c_o (the zero-sequences). The complex homomorphisms of l^∞ are given by the elements of $\beta\mathbb{N}$ (the Stone-Čech-compactification of $\mathbb{N}$) which can be identified with ultrafilters on $\mathbb{N}$. There are two types of ultrafilters: fixed (containing all subsets of $\mathbb{N}$ which contain a fixed element of $\mathbb{N}$) and free ultrafilters giving the elements of $\beta\mathbb{N} \setminus \mathbb{N}$. The formula

$$\omega_{\mathcal{U}}((x_n)) = \lim_{\mathcal{U}} x_n \quad , (x_n) \in l^\infty , \quad \mathcal{U}\text{-ultrafilter}$$

gives the complex homomorphisms of l^∞. If $\mathcal{U}$ is fixed at $k \in \mathbb{N}$, then

$$\omega_{\mathcal{U}} = \omega_k \quad \text{with} \quad \omega_k((x_n)) = x_k \ .$$

If $\mathcal{U}$ is free, then

$$\omega_{\mathcal{U}}((x_n)) = 0 \text{ for all } (x_n) \in c_o,$$

i.e., free ultrafilters give rise to singular states. This is a guide to construct singular states on $\mathcal{B}(\mathcal{H})$. Let (φ_n) be a sequence of unit vectors in $\mathcal{H}$ weakly converging to zero. Then $\omega_{\mathcal{U}}$ defined by

(1) $\omega_{\mathcal{U}}(A) = \lim_{\mathcal{U}} \langle \psi_n, A\psi_n \rangle$

gives a state on $\mathcal{B}(\mathcal{H})$ which is singular if and only if $\mathcal{U}$ is free. Moreover, if one takes any sequence (ψ_n) of unit vectors, then exactly those free ultrafilters $\mathcal{U}$ give singular states for which

$$\lim_{\mathcal{U}} \langle \psi_n, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{H} \quad (/1/).$$

Furthermore, the following theorem was established by Wils /20/ and extended to the non-separable case by Anderson /1/.

Theorem 4.3

There is a fixed sequence (ψ_n) of unit vectors of $\mathcal{H}$ so that any singular state on $\mathcal{B}(\mathcal{H})$ has the form

$$\omega_{\mathcal{U}}(A) = \lim_{\mathcal{U}} \langle \psi_n, A\psi_n \rangle$$

for an appropriate free ultrafilter $\mathcal{U}$ (namely such one with $\lim_{\mathcal{U}} \langle \psi_n, \varphi \rangle = 0$ for all $\varphi \in \mathcal{H}$).
Call (ψ_n) a Wils sequence.

Remark that this sequence (ψ_n) can be taken to be arbitrary $\|\cdot\|$ -dense in the unit sphere. A little bit more is known. While the only normal pure states on $\mathcal{B}(\mathcal{H})$ are the vector states, pure singular states are obtained if one takes in (1) instead of (ψ_n) an orthonormal system (φ_n). But the question whether the converse statement is also true seems to be open.
Now we turn to the unbounded case. Let us mention the following simple fact.

Lemma 4.4

Let $(\varphi_n) \subset \mathcal{D}$ be a t-bounded sequence, $\mathcal{U}$ a free ultrafilter. Then
$\omega_{\mathcal{U}}$: $\omega_{\mathcal{U}}(A) = \lim_{\mathcal{U}} \langle \varphi_n, A\varphi_n \rangle$
defines a positive, $\tau_{\mathcal{D}}$-continuous functional on $\mathcal{L}^+(\mathcal{D})$ which is singular if and only if $\lim_{\mathcal{U}} \langle \varphi_n, \psi \rangle = 0$ for all $\psi \in \mathcal{D}$.

Proof:

The positivity and $\tau_{\mathcal{D}}$-continuity are trivial. The proof of the remaining part is as in the bounded case. Let $\varphi, \psi \in \mathcal{D}$, $P_{\varphi,\psi} = \langle \varphi, \cdot \rangle \psi$. Then the finite linear combinations of the $P_{\varphi,\psi}$'s are $\tau_{\mathcal{D}}$-dense in $\mathcal{C}(\mathcal{D})$. Thus $\omega_{\mathcal{U}}$ is singular if and only if $\omega_{\mathcal{U}}(P_{\varphi,\psi}) = 0$ for all $\varphi, \psi \in \mathcal{D}$. But this is the case if and only if $\lim_{\mathcal{U}} \langle \varphi, \psi_n \rangle \langle \varphi_n, \psi \rangle = 0$ for all $\varphi, \psi \in \mathcal{D}$. This implies the assertion.

q.e.d.

Now let (ψ_n) be a Wils sequence, $B \in \mathcal{B}(\mathcal{D})$, $\mathcal{U}$ a free ultrafilter. Then

$\omega_{B,\mathcal{U}}$: $\omega_{B,\mathcal{U}}(A) = \lim_{\mathcal{U}} \langle B\psi_n, AB\psi_n \rangle$, $A \in \mathcal{L}^+(\mathcal{D})$

gives a positive singular functional on $\mathcal{L}^+(\mathcal{D})$. The positivity is clear, the $\tau_{\mathcal{D}}$-continuity follows from the estimation

$$|\omega_{B,\mathcal{U}}(A)| = |\lim_{\mathcal{U}} \langle B\psi_n, AB\psi_n \rangle| \le \sup_{\varphi,\psi \in \mathcal{M}} |\langle \varphi, A\psi \rangle| \quad \text{where } \mathcal{M} = \{B\psi_n\}.$$

The singularity follows from the fact that $S \in \mathcal{C}(\mathcal{D})$ implies $B^*SB \in \mathcal{J}_\infty(\mathcal{H})$ for all $B \in \mathcal{B}(\mathcal{D})$ /12/. To get states on $\mathcal{L}^+(\mathcal{D})$ one takes for example such pairs $(\mathcal{U}, B)$ that $\lim_{\mathcal{U}} \langle B\psi_n, B\psi_n \rangle = 1$.
In the way just described one gets all positive singular functionals.

Theorem 4.5

The positive singular functionals on $\mathcal{L}^+(\mathcal{D})$ are given by

$\omega_{C,\mathcal{U}}$: $\omega_{C,\mathcal{U}}(A) = \lim_{\mathcal{U}} \langle C\psi_n, AC\psi_n \rangle$ for all $A \in \mathcal{L}^+(\mathcal{D})$,

where (ψ_n) is a Wils sequence, $\mathcal{U}$ an appropriate free ultrafilter and $C \in \mathcal{B}(\mathcal{D})$ which can be taken to be positive, having t-dense range and Ker $C = (0)$.

We will give two different proofs. One uses explicitly the extension procedure described in section 3.

First proof:

Let ω be a positive singular functional on $\mathcal{L}^+(\mathcal{D})$, then
$|\omega(A)| \le \|BAB\|$
with $B \in \mathcal{B}(\mathcal{D})$ which can be assumed to be positive, invertible and with t-dense range.
The set $\mathcal{A}_1 = \{BAB;\ A \in \mathcal{L}^+(\mathcal{D})\}$ is a $*$-algebra which can be considered as a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Consider the linear functional ω_1 on $\mathcal{A}_1$ given by $\omega_1(X) = \omega(A)$ for all $X = BAB \in \mathcal{A}_1$.
The properties of B yield that ω_1 is correctly defined, positive and norm-continuous on $\mathcal{A}_1$. Therefore ω_1 can be continued to the C^*-algebra $\widetilde{\mathcal{A}}_1 \subset \mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}_1$. Denote this extension also by ω_1. Again the properties of B give that $\{BFB;\ F \in \mathcal{F}(\mathcal{D})\}$ is norm-dense in $\mathcal{F}(\mathcal{H})$, hence $\mathcal{J}_\infty(\mathcal{H}) \subset \widetilde{\mathcal{A}}_1$. In the usual way ω_1 can be extended to the C^*-algebra $\widetilde{\mathcal{A}}$ obtained by adjoining the identity to $\widetilde{\mathcal{A}}_1$. This gives again a positive norm-continuous functional, also denoted by ω_1. Since $I \in \widetilde{\mathcal{A}}$, ω_1 can be extended to a positive linear functional $\widetilde{\omega}_1$ on $\mathcal{B}(\mathcal{H})$. This functional is singular because $\widetilde{\omega}_1(BFB) = \omega(F) = 0$ for all $F \in \mathcal{F}(\mathcal{D})$, hence by continuity $\widetilde{\omega}_1(C) = 0$ for all $C \in \mathcal{J}_\infty(\mathcal{H})$. Put $\widetilde{\omega} = \widetilde{\omega}_1 / \widetilde{\omega}_1(I)$ and apply Theorem 4.3, then

$$\widetilde{\omega}(X) = \lim_{\mathcal{U}} \langle \psi_n, X\psi_n \rangle \quad \text{for all } X \in \mathcal{B}(\mathcal{H}).$$

If $X = BAB \in \mathcal{A}_1$, we get

$$\widetilde{\omega}(X) = \lim_{\mathcal{U}} \langle \psi_n, BAB\psi_n \rangle$$

Thus,

$$\omega(A) = \widetilde{\omega}_1(I) \cdot \lim_{\mathcal{U}} \langle \psi_n, BAB\psi_n \rangle = \lim_{\mathcal{U}} \langle C\psi_n, AC\psi_n \rangle$$

with $C = (\widetilde{\omega}_1(I))^{1/2} B$.

q.e.d.

Second proof:

Let $|\omega(A)| \le \|BAB\|$ with $B = \int_0^b \lambda \, dE_\lambda$ as above. Then the corresponding functional $\tilde{\omega}$ on $\mathfrak{B}(\mathcal{H})$ is positive, singular and fulfills (cf.sect.3)

(2) $\tilde{\omega}(X) = \lim_{a \to 0} \omega(P_a X P_a)$, $|\tilde{\omega}(X)| \le \|BXB\|$

Fix $\alpha \in (0,1)$ and put $B_a^{-\alpha} = \int_a^b \lambda^{-\alpha} dE_\lambda$, $a > 0$. Then $B B_a^{-\alpha} = B^{1-\alpha} P_a$ and $B_a^{-\alpha} \in \mathfrak{B}(\mathcal{H})$. From (2) it follows easily that $\tilde{\omega}(B_a^{-\alpha} X B_a^{-\alpha})$ is a Cauchy sequence (having in mind $a \to 0$) for any $X \in \mathfrak{B}(\mathcal{H})$. Put

$$\tilde{\omega}_1(X) = \lim_{a \to 0} \tilde{\omega}(B_a^{-\alpha} X B_a^{-\alpha})$$

This gives a positive functional on $\mathfrak{B}(\mathcal{H})$. Moreover $\tilde{\omega}_1$ is singular because for $X \in \mathcal{J}_\infty(\mathcal{H})$ the operators $B_a^{-\alpha} X B_a^{-\alpha} \in \mathcal{J}_\infty(\mathcal{H})$, too and $\tilde{\omega}(B_a^{-\alpha} X B_a^{-\alpha}) = 0$ for all $a > 0$.

For arbitrary $Y \in \mathfrak{B}(\mathcal{H})$ and $X = B^\alpha Y B^\alpha$ one gets

$$\tilde{\omega}_1(B^\alpha Y B^\alpha) = \lim_{a \to 0} \tilde{\omega}(B_a^{-\alpha} B^\alpha Y B^\alpha B_a^{-\alpha}) = \tilde{\omega}(Y).$$

The last equality follows from the estimation

$$|\tilde{\omega}(B_a^{-\alpha} B^\alpha Y B^\alpha B_a^{-\alpha}) - \tilde{\omega}(Y)| \le \| B(B_a^{-\alpha} B^\alpha Y B^\alpha B_a^{-\alpha} - Y)B \| =$$
$= \|B(P_a A P_a - A)B\| \longrightarrow 0$ for $a \to 0$ (see Lemma 2.2.ii)).

Next put $\tilde{\omega}_2 = \tilde{\omega}_1 / \tilde{\omega}_1(I)$, $\tilde{\omega}_2$ is a singular state on $\mathfrak{B}(\mathcal{H})$ (the case $\tilde{\omega}_1(I) = 0$ can be excluded since this would imply $\omega = 0$).
Applying Theorem 4.3 we get

$$\tilde{\omega}_2(Y) = \lim_{\mathcal{U}} \langle \psi_n, Y \psi_n \rangle \quad \text{for all } Y \in \mathfrak{B}(\mathcal{H}).$$

Consequently

$$\tilde{\omega}(Y) = \tilde{\omega}_1(B^\alpha Y B^\alpha) = \tilde{\omega}_1(I)\, \tilde{\omega}_2(B^\alpha Y B^\alpha) = \tilde{\omega}_1(I) \lim_{\mathcal{U}} \langle \psi_n, B^\alpha Y B^\alpha \psi_n \rangle$$
$$= \lim_{\mathcal{U}} \langle C \psi_n, Y C \psi_n \rangle, \text{ with } C = (\tilde{\omega}_1(I))^{1/2} B^\alpha \in \mathfrak{B}(\mathcal{D}).$$

Now we return to $\mathcal{L}^+(\mathcal{D})$ by observing that

$$\tilde{\tilde{\omega}}(A) = \omega(A) = \lim_{a \to 0} \tilde{\omega}(P_a A P_a) = \lim_{a \to 0} \lim_{\mathcal{U}} \langle C \psi_n, (P_a A P_a) C \psi_n \rangle$$

for all $A \in \mathcal{L}^+(\mathcal{D})$. Then

$$| \lim_{\mathcal{U}} \langle C \psi_n, (P_a A P_a) C \psi_n \rangle - \lim_{\mathcal{U}} \langle C \psi_n, A C \psi_n \rangle | \le \sup_n | \langle \psi_n, C(P_a A P_a -$$
$-A) C \psi_n \rangle | \le \tilde{\omega}_1(I) \| B^\alpha (P_a A P_a - A) B^\alpha \|$ which goes to zero for $a \to 0$.

This gives us the desired result

$$\omega(A) = \lim_{\mathcal{U}} \langle C \psi_n, A C \psi_n \rangle \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}).$$

q.e.d.

From this theorem we get the following trivial but important corollary.

Corollary 4.6

Any positive singular functional on $\mathcal{L}^+(\mathcal{D})$ is the weak limit of positive vector functionals. Especially, the singular states are contained in the vector state space of $\mathcal{L}^+(\mathcal{D})$.

Proof:

Let ω be singular and positive on $\mathcal{L}^+(\mathcal{D})$, $A_1, \dots, A_j \in \mathcal{L}^+(\mathcal{D})$, $\varepsilon > 0$ be given. From Theorem 4.5 we have

$$\omega(A) = \lim_{\mathcal{U}} \langle C \psi_n, A C \psi_n \rangle .$$

Hence there are $U_i \in \mathcal{U}$ so that

$$| \omega(A) - \langle C \psi_n, A_i C \psi_n \rangle | < \varepsilon \text{ for all } n \in U_i, \ i = 1, \dots, j$$

i.e., $|\omega(A) - \langle C \psi_n, A_i C \psi_n \rangle| < \varepsilon$ for all $n \in U = \bigcap_i U_i \in \mathcal{U}$, $i = 1, \dots, j$.

This proves the first assertion.

Let ω be a state, then if necessary replace ψ_n by $\lambda_n \psi_n = \psi_n'$ for all $n \in U \in \mathcal{U}$ for some U and $\lim_{\mathcal{U}} \lambda_n = 1$, so that $\langle C \psi_n', C \psi_n' \rangle = 1$ for all $n \in U$. Then the statement follows from the considerations above.

q.e.d.

Finally we derive from Theorem 4.5 an important result about the state space E of $\mathcal{L}^+(\mathcal{D})$.

Remember that in the case of C^*-algebras the fact that the unit ball in the dual space is w^*-compact allows to apply the Krein-Milman theorem. This leads to the well-known result that the state space of a C^*-algebra is the w^*-closed convex hull of the pure states. In contrast to this the state space of an Op^*-algebra is not w^*-compact if the algebra contains unbounded operators. But nevertheless for $\mathcal{L}^+(\mathcal{D})$ one can derive the analogous result . Here we will prove this almost constructively using Theorems 3.4 and 4.5. In /13/ there is given another proof in the context of more general considerations.

Theorem 4.8

The state space E of $\mathcal{L}^+(\mathcal{D})$ is the w^*-closed convex hull of the vector states, hence of the pure states.

Proof:

Because of Corollary 3.3ii) it is enough to prove that E is contained in the w^*-closed convex hull of the vector states. Thus let $\omega \in E$ and $A_1, \dots, A_k \in \mathcal{L}^+(\mathcal{D})$, $\varepsilon > 0$ be given. Without restriction of generality let $A_1 = I$. According to Theorem 3.4 : $\omega = \omega_n' + \omega_s'$, where $\omega_n' \ge 0$ normal and $\omega_s' \ge 0$ singular. $\omega(I) = \omega_n'(I) + \omega_s'(I) = 1$ implies that

(3) $\omega = t\, \omega_n + (1-t)\, \omega_s$ with $t = \omega_n'(I)$ and $\omega_n = t^{-1} \omega_n'$, $\omega_s = (1-t)^{-1} \omega_s'$ is a convex combination of states.

By Corollary 4.6 there is a vector state ω_φ on $\mathcal{L}^+(\mathcal{D})$ with

$$(4) \qquad |\omega_1(A_i) - \omega_\varphi(A_i)| < \varepsilon \quad \text{for } i = 1,\dots,k$$

Since $\omega_n(A_i) = \operatorname{Tr} A_i T = \sum t_n \langle \varphi_n, A_i \varphi_n \rangle$ for some $T \in \mathcal{S}_1(\mathcal{D})$, $T\varphi_n = t_n \varphi_n$, (φ_n) an orthonormal system in $\mathcal{D}$ and $\sum t_n = 1$, there is an $N \in \mathbb{N}$ so that

$$(5) \quad \Big| \sum_{n>N} t_n \langle \varphi_n, A_i \varphi_n \rangle \Big| < \varepsilon \quad \text{for } i = 1,\dots,k.$$

Then:

$$\omega_1 = \sum_{n=1}^{N} s_n \langle \varphi_n, \cdot\, \varphi_n \rangle \text{ with } s_n = t_n \Big(\sum_{n=1}^{N} t_n \Big)^{-1} \text{ is a con-}$$

vex combination of vector states and

$$(6) \quad |\omega_n(A_i) - \omega_1(A_i)| \le \Big| \sum_{n=1}^{N} (t_n - s_n) \langle \varphi_n, A_i \varphi_n \rangle \Big| + \Big| \sum_{n>N} t_n \langle \varphi_n, A_i \varphi_n \rangle \Big| \le$$

$$\le \Big(1 - \sum_{n=1}^{N} t_n\Big) + \varepsilon/2 \le \varepsilon .$$

Here we used (5) and $A_1 = I$. Combining (3),(4) and (6) we get that

$$\rho = \sum_{n=1}^{N+1} r_n \rho_n \quad \text{with } r_n = t \cdot s_n, \ \rho_n = \langle \varphi_n, \cdot\, \varphi_n \rangle, n = 1,\dots,N$$
$$r_{N+1} = (1-t), \rho_{N+1} = \omega_\varphi$$

is a convex combination of vector states satisfying

$$|\omega(A_i) - \rho(A_i)| < \varepsilon, \quad i = 1,\dots,k.$$

q.e.d.

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Лёффлер Ф., Тиммерманн В. E5-85-700

Сингулярные состояния на максимальных Op^*-алгебрах

Работа посвящена теории линейных, непрерывных функционалов. Исследуются различные структурные вопросы таких функционалов и их взаимосвязь с линейными, непрерывными функционалами на алгебрах ограниченных операторов.

Построен явный вид всех положительных сингулярных функционалов на максимальной Op^*-алгебре неограниченных операторов, заданных на областях Фреше в гильбертовом пространстве. Рассмотрен целый ряд свойств таких функционалов и даны эквивалентные характеристики положительных сингулярных функционалов. Показывается, что каждое состояние разлагается на сумму положительных нормальных и сингулярных функционалов. Кроме того дано описание пространства состояний на таких алгебрах. Использовались методы теории операторов, теории операторных алгебр и функционального анализа.

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Löffler F., Timmermann W. E5-85-700

Singular States on Maximal Op^*-Algebras

The paper is devoted to the study of linear continuous functionals. There are investigated several structural questions of functionals and their relation with functionals on algebras of bounded operators.

In the paper the explicit form is given of positive singular functionals on maximal Op^*-algebras defined on (F)-domains. Several properties of such states are mentioned and there are given equivalent characterizations of positive singular functionals. It is proved that any state can be decomposed into the sum of a positive normal and a positive singular functional.

Moreover there is given a description of the state space of such maximal Op^*-algebras. Methods from operator theory, theory of operator algebras and functional analysis are used.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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