объединенный
институт
ядерных
исследований
дубна

E5-85-663

Š.Šujan

HETERОPHASE RANDOM FIELDS.
Fields on Infinite Homogeneous Trees

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INTRODUCTION

This paper is a continuation of the first and second parts under the same title, hereafter referred to as [1,11]. Sections, theorems, and formulae are numbered consecutively, starting with Section 9 below. References [1-15] are listed at the end of [1]; references [16-24], at the end of [11], respectively.

9. A ONE-DIMENSIONAL MODEL

All results obtained so far in [1,11] are merely existence theorems, and the question on explicit construction of the potential U of a heterophase system from the knowledge of potentials $U^{(i)}$, $i \in I$, and concentrations $\gamma = (\gamma_i)_{i \in I}$ is still open. Here we present an example illustrating that a "heterophase potential", at least qualitatively, possesses the structure predicted on the base of physical considerations [9,23], i.e., the resulting interaction potential (cf. Section 7) contains an "operator" term (corresponding to interaction) and a "scalar" term (corresponding to concentrations of pure phases).

Let $S = \{0,1\}$, $I = \{1,2\}$ and $d = 1$. For $i = 1,2$ let $P^{(i)}$ be a stationary (and uniformy mixing) Markov chain determined by the stationary distribution $p^{(i)} = (p_0^{(i)}, p_1^{(i)})$ and the entry-wise positive transition probability matrix $\Pi^{(i)} = (\pi_{r,s}^{(i)})_{r,s=0,1}$. As is pointed out in [25], pp. 14-15, $P^{(i)}$ is a Gibbs measure to the interaction potential

$$\mathfrak{J}^{(i)}(\phi_V) = \begin{cases} -\log \pi^{(i)}_{\phi(t)\phi(t+1)} & \text{if} \quad V = \{t, t+1\} \text{ for some } t \in \mathbf{Z}, \\ 0 & \text{otherwise}. \end{cases} \tag{9.1}$$

Let Z be a $\{1,2\}$-valued sequence of independent and identically distributed random variables such that

$$\mu[Z_0 = i] = \gamma, \quad i = 1,2, \quad 0 < \gamma_i < 1. \tag{9.2}$$

By redefining the basic probability space if necessary we may suppose that the corresponding processes $X^{(1)}$ and $X^{(2)}$ are stochastically independent (and hence jointly Markov), and Z is independent of the pair $(X^{(1)}, X^{(2)})$. According to Theorems 6 and 9, the sequence $X=(X_t;t\in \mathbf{Z})$ (cf. (3.9)) is again a Markov chain so that a formula like (9.1) must be valid for X, too. Let $p=(p_0,p_1)$, $\Pi=(\pi_{r,s})_{r,s=0,1}$ and $\mathcal{J}(\cdot)$ denote the objects associated with X. From (3.9) it follows by a straightforward calculation that

$$p_r=\gamma_1 p_r^{(1)}+(1-\gamma_1)p_r^{(2)}; \quad r=0,1. \tag{9.3}$$

(Note that (9.3) is the same prescription of p as that one resulting from (8.7); however, for two-point probabilities the "heterophase" prescription will differ already from that one given by the usual mixture). Using the above-mentioned independence properties a direct calculation yields:

$$p_r\pi_{r,s}=\mu[(X_t,X_{t+1})=r,s]=\gamma_1^2 p_r^{(1)}\pi_{r,s}^{(1)}+\gamma_2^2 p_r^{(2)}\pi_{r,s}^{(2)}; \quad r,s\in\{0,1\}. \tag{9.4}$$

Consequently, on account of (9.3) and (9.4),

$$\pi_{r,s}=\frac{\gamma_1^2 p_r^{(1)}\pi_{r,s}^{(1)}+\gamma_2^2 p_r^{(2)}\pi_{r,s}^{(2)}}{\gamma_1 p_r^{(1)}+\gamma_2 p_r^{(2)}}; \qquad r,s\in\{0,1\}. \tag{9.5}$$

It follows that $\check{\mathcal{J}}(\phi_V)=0$ unless $V=\{t,t+1\}$ for some $t\in\mathbf{Z}$. But if $V=\{t,t+1\}$, then

$$\begin{aligned}\check{\mathcal{J}}(\phi_V)=\log[\gamma_1 p_{\phi(t)}^{(1)}+\gamma_2 p_{\phi(t)}^{(2)}]-\log\{\exp[-\check{\mathcal{J}}^{(1)}(\phi_V)+\\+\log(\gamma_1^2 p_{\phi(t)}^{(1)})]+\exp[-\check{\mathcal{J}}^{(2)}(\phi_V)+\log(\gamma_1^2 p_{\phi(t)}^{(2)})]\}.\end{aligned} \tag{9.6}$$

This example shows rather complicated relations between original potentials $U^{(i)}$ and the resulting potential U of the heterophase system. Furthermore, it is not possible to relate this example to physical reality. Indeed, since a heterophase system is related to the concept of phase transition (cf. Section 5), more realistic models should describe heterophase mixtures of two or more pure phases, corresponding to - single potential. Such a model cannot be constructed based on our example. Indeed, it is well-known (cf., e.g.,[21,26]) that short-range potentials on a one-dimensional lattice do not exhibit a phase transition.

10. HETEROPHASE RANDOM FIELDS ON INFINITE TREES

The problem of phase transitions usually cannot be solved explicitly[25]. This means that the existence of more than a single Gibbs measure is indicated with the help of some quantity (e.g., low and high density Gibbs measures for potentials

of the Ising type). An explicit derivation of finite-dimensional distributions of pure phases usually leads to serious combinational difficulties, connected with the interaction between neighbouring sites on the boundaries of finite volumes. The difficulties come from the fact that with growing volume V the number of mutually interacting sites on its boundary ∂V also increases. Usual techniques (correlation functions[16,21], contours[25]) yeild only estimates which can be used to deduce the existence of different pure phases without calculating their distributions exactly.

Difficulties of this type disappear when instead of the lattice $\mathbf{Z}^d$ we are working with a countably infinite homogeneous tree T_d such that each of its nodes terminates exactly d edges. In this case ∂V also increases with increasing V, nevertheless, the number of interacting boundary sites in ∂V remains bounded. Hence for large volumes and for nearest neighbour potentials the interactions on the boundary will have a vanishingly small influence. Consequently, exact calculation of finite-dimensional distributions for Gibbs measures is possible[21]. Thus, it appears reasonable to study heterophase random fields on such trees, at least, for the purpose of getting a manageable example of a heterophase system. To this end let us start with some definitions (cf.[21], Chpt. 10 and[27]).

A probability measure P on the space $\{0,1\}^{T_d}$ (and the corresponding random field $X=(X_t;t\in T_d)$ is said to be a <u>Markov random field</u> (MRF) if

(i) P is strictly positive on all f.d. subsets of $\{0,1\}^{T_d}$.

(ii) the conditional probabilities

$$P[X_t=1\,|\,\phi_{\{\bar t\}}]; \quad \{\bar t\}=T_d-\{t\}, \tag{10.1}$$

depend only on the values of ϕ at sites neighbouring t (i.e., on those nodes $u\in T_d$ such that there exists an edge joining u with T),

and

(iii) the conditional probabilities (10.1) are invariant with respect to any isomorphism of the graph T_d.

We shall consider the case $d=3$ (this is the minimal value of d for which one can prove nonuniqueness of MRF given conditional probabilities (10.1); cf.[21,27]. We fix a point $t\in T_3$, and use t_1, t_2, t_3 as the generic notation for its nearest neighbours. From (iii) it follows that the conditional probabilities of a MRF on T_3 are uniquely determined by the parameter vector $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, where

$$\alpha_k=P[X_t=1\,|\,|\{j:X_{t_j}=1\}|=k], \quad 0\leq k\leq 3. \tag{10.2}$$

Using equivalence of MRF and Gibbs fields it is possible to show (cf.[27], Thm. 1) that a vector α determines the conditional probabilities of a MRF if and only if there exist positive numbers x and y such that

$$\alpha_k = (1 + y \cdot x^{2k-3})^{-1}, \quad 0 \le k \le 3. \tag{10.3}$$

In more detail, let U be a homogeneous binary nearest neighbour potential, i.e., let

$$U(s,t) = \begin{cases} v_0 & \text{if} \quad s = t, \\ v_1 & \text{if} \quad s,t \quad \text{are neighbours}, \\ 0 & \text{otherwise}; \quad s,t \in T_3. \end{cases} \tag{10.4}$$

Then any $P \in \mathcal{G}(U)$ will possess conditional probabilities

$$\alpha_k = [1 + \exp(\frac{v_0}{2} + kv_1)]^{-1}, \quad 0 \le k \le 3. \tag{10.5}$$

It is easy to see, these are of the form (10.3) if we take

$$x = \exp(v_1/2); \quad y = \exp[(v_0 + 3v_1)/2]. \tag{10.6}$$

The class of all MRF having conditional probabilities α (cf. (10.3)) will be denoted by $\mathcal{G}_\alpha$. For any stochastic matrix

$$M = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}, \quad 0 < p,q < 1. \tag{10.7}$$

Preston[21], p.p.97-99 and Spitzer[27], def.3 and Thm.2 constructed a special MRF P_M (which behaves as a Markov chain along any infinite path in our tree). For any vector α there exists a matrix M of type (10.7) such that $P_M \in \mathcal{G}_\alpha$. As is shown in[27], $P_M \in \mathcal{G}_\alpha$ if and only if

$$P_M[X_t = 1 \,\|\, |\{j : X_{t_j} = 1\}| = 3] = (1 + yx^3)^{-1},$$

$$P_M[X_t = 1 \,\|\, |\{j : X_{t_j} = 1\}| = 2] = (1 + yx)^{-1}.$$

This fact will play a central role in our considerations. The MRF P_M can be described as a Gibbs field to the potential (of the form (10.4))

$$U(s,t) = \begin{cases} \log[(\frac{1-q}{p})^3 (\frac{1-p}{1-q})]; & \text{if} \quad s = t, \\ \frac{1}{2}\log[\frac{pq}{(1-p)(1-q)}], & \text{if} \quad s,t \quad \text{are neighbours}, \\ 0 & \text{otherwise}. \end{cases} \tag{10.8}$$

(cf.[21], Prop. 10.6). Conversely, if there is given a potential U of the form (10.4), then by solving the system of equations obtained by equating the right-hand sides of (10.4) and (10.8) we may construct the corresponding matrix M. A phase transition occurs if there exist two different solutions $(p^{(1)}, q^{(1)})$ and $(p^{(2)}, q^{(2)})$ giving rise to two different matrices $M^{(1)}, M^{(2)}$ (cf.[21], Prop. 10.7). Note that if $\alpha^{(1)}$, $\alpha^{(2)}$ denote the vectors (10.3) for the MRF's $P_{M^{(1)}}$ and $P_{M^{(2)}}$ then

$$\alpha_k^{(1)} = \alpha_k^{(2)}, \quad 0 \le k \le 3. \tag{10.9}$$

Indeed, α_k's are uniquely determined by v_0 and v_1 via (10.5), independently of the fact whether v_0, v_1 give rise to one or to more than one MRF. So, suppose v_0, v_1 are chosen so that a phase transition occurs[21]. Let $M^{(1)} \ne M^{(2)}$ denote the corresponding matrices, and let

$$\pi^{(1)} = (\pi_0^1, \pi_1^1), \quad \pi^{(2)} = (\pi_0^2, \pi_2^2). \tag{10.10}$$

denote their stationary probability vectors. The associated MRF's will be denote by

$$X^{(i)} = (X_t^{(i)} ; \; t \in T_3), \quad i = 1,2. \tag{10.11}$$

Let $Z = (Z_t ; \; t \in T_3)$ be a family of independent and identically distributed $\{1,2\}$ -valued random variables with

$$\Pr[Z_t = 1] = \gamma_1 = 1 - \Pr[Z_t = 2], \quad 0 < \gamma_1 < 1. \tag{10.12}$$

In particular, the distribution of Z is invariant under any isomorphism of the graph T_3. Suppose that $X^{(1)}, X^{(2)}$ are stochastically independent, and Z is independent of the pair $(X^{(1)}, X^{(2)})$. Then it is easy to check that the random field X, where

$$X_t = X_t^{(Z_t)}, \quad t \in T_3. \tag{10.13}$$

Is again a MRF. Consequently, its conditional probabilities are again determined by a parameter vector, say $\tilde{\alpha} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$. The following formula is proved in the appendix:

<u>Lemma 13</u>. The parameter $\tilde{\alpha}_3$ of the field X (cf. (10.13)) is expressed by the formulae

$$\tilde{\alpha}_3 = [\gamma_1 \pi_1^1 + (1 - \gamma_1)\pi_1^2]^{-3} \times$$

$$\times \{\gamma_1^4 (\pi_1^1)^3 \alpha_3 + 3\gamma_1^3 (1 - \gamma_1)(\pi_1^1)^2 \pi_1^2 \times$$

$$\times [\pi_0^1 a_3 + \pi_1^1 a_3] + 3\gamma_1^2 (1-\gamma_1)\pi_1^1 (\pi_1^2)^2 \times$$
$$\times [(\pi_0^1)^2 a_1 + 2\pi_0^1 \pi_1^1 a_2 + (\pi_1^1)^2 a_3] +$$
$$+ \gamma_1 (1-\gamma_1)^3 \pi_1^1 (\pi_1^2)^3 + \gamma_1^3 (1-\gamma_1)(\pi_1^1)^3 \pi_1^2 +$$
$$+ 3\gamma_1^2 (1-\gamma_1)^2 (\pi_1^1)^2 \pi_1^2 [(\pi_0^2)^2 a_1 + 2\pi_0^2 \times \tag{10.14}$$
$$\times \pi_1^2 a_2 + (\pi_1^2)^2 a_3] + 3\gamma_1 (1-\gamma_1)^3 \pi_1^1 (\pi_1^2)^2 \times$$
$$\times [\pi_0^2 a_2 + \pi_1^2 a_3] + (1-\gamma_4)^4 (\pi_1^2)^3 a_3 \}.$$

Now let U be a potential of type (10.4), where $v_0 = 0$ and $v_1 > 0$. By /21,27/, there exists a phase trandition for U. As mentioned above, the vector α is uniquely determined by (hence, by v_1), but there exists two different matrices, $M^{(1)}$, and $M^{(2)}$ such that $P_{M^{(1)}}, P_{M^{(2)}} \in \mathcal{G}(U)$. Let $\pi^{(1)}$, $\pi^{(2)}$ denote the corresponding stationary probability vectors (see (10.10)). By Lemma 13 (and similar expressions for the remaining parameters $\tilde{a}_i$, $0 \le i \le 2$) we can think of the "heterophase potential" $\tilde{U}$ as of the function $(\gamma_1, v_1) \to \tilde{U}(\gamma_1, v_1)$. We use Lemma 13 to prove the following unexpected result:

Theorem 14. There exists a $v^* > 0$ such that for any $0 < v_1 < v^*$, there is a $\gamma = (\gamma_1, 1-\gamma_1) \in \Gamma$ for which $|\mathcal{G}[\tilde{U}(\gamma_1, v_1)]| = 1$. In other words, the corresponding heterophase system has a unique Gibbs distribution.

Proof. For fixed $a, \pi^{(1)}$ and $\pi^{(2)}$ we see from (10.14) that

$$\tilde{a}_3 = f_3(\gamma_1; a_1 \pi^{(1)}, \pi^{(2)}), \tag{10.15}$$

where $\gamma_1 \to f_3(\gamma_1; a_1\pi^{(1)}, \pi^{(2)})$ is a continuous function satisfying

$$\lim_{\gamma_1 \to {0+ \atop 1-}} f_3(\gamma_1; a_1 \pi^{(1)}, \pi^{(2)}) = a_3. \tag{10.16}$$

Since $X^{(1)}$ and $X^{(2)}$ are MRF's, it follows from (10.5) that

$$a_3 = (1 + e^{3v_1})^{-1}. \tag{10.17}$$

Also $\tilde{a} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ must be expressible in the form (10.5) due to the fact that X (cf. (10.13)) is MRF, too. Since self-interaction in $X^{(1)}$ and $X^{(2)}$ is excluded (i.e., $v_0 = 0$), and

since Z is an independent and identically distributed random field, construction (10.13) cannot give rise to self-interaction. Hence $\tilde{v}_0 = 0$ and from (10.5) we get

$$\tilde{a}_3 = (1 + e^{3\tilde{v}_1})^{-1}. \tag{10.18}$$

On the other hand ($a, \pi^{(1)}, \pi^{(2)}$ are fixed!), using (10.15) and (10.16) we can express $\tilde{a}_3$ in the form

$$\tilde{a}_3 = (1 + e^{3\tilde{v}_1})^{-1} + \epsilon_3(\gamma_1), \tag{10.19}$$

where

$$\lim_{\gamma_1 \to 0+} \epsilon_3(\gamma_1) = \lim_{\gamma_1 \to 1-} \epsilon_3(\gamma_1) = 1. \tag{10.20}$$

Consequently (omitting the argument of ϵ_3),

$$1 + e^{3\tilde{v}_1} = (1 + e^{3v_1})[1 + \epsilon_3 (1 + e^{3v_1})]^{-1}.$$

It follows from this that

$$\tilde{v}_1 = \frac{1}{3} \log \left[\frac{1 + e^{3v_1}}{1 + \epsilon_3 (1 + e^{3v_1})} - 1 \right]. \tag{10.21}$$

If $\epsilon_3 \sim 0$ (i.e., if either $\gamma_1 \sim 0$ or $\gamma_1 \sim 1$), then it follows from (10.19) and (10.20) that

$$\tilde{v}_1 \sim \frac{1}{3} \log (1 + e^{3v_1} - 1) = \frac{1}{3} \log e^{3v_1} = v_1. \tag{10.22}$$

Now, the value $\tilde{v}_1$ is critical, for if $\tilde{v}_1 < 0$ ($\tilde{v}_1 > 0$) then there is no (there is) phase transition. Consequently, we must show when it is possible to have $\tilde{v}_1 < 0$. This can be true only if the argument of logarithm in (10.21) will be less than one, i.e., if

$$\epsilon_3 > \frac{e^{3v_1} - 1}{2(1 + e^{3v_1})} (> 0). \tag{10.23}$$

Suppose for a moment that (10.23) is true. Sinse the function $\gamma_1 \to \epsilon_3(\gamma_1)$ is continuous and has the properties (10.20), for any $\sigma > 0$ there is a $r > 0$ such that if either $0 < \gamma_1 < r$ or $1 - r < \gamma_1 < 1$, then $\epsilon_3(\gamma_1) < \sigma$. Find a value γ_1 such that $\epsilon_3(\gamma_1) = \sigma/2$, say. Since

$$\lim_{v_1 \to 0} (e^{3v_1} - 1)/[2(1 + e^{3v_1})] = 0, \tag{10.24}$$

we can find $\delta > 0$ so small that if $0 < v_1 < \delta$ then

$$(e^{3v_1} - 1)/[2(1 + e^{3v_1})] < \sigma/2 = \epsilon_3(\gamma_1). \qquad (10.25)$$

In other words, if we can prove simultaneously $\epsilon_3(\gamma_1) < \sigma$ and (10.23), there will be no phase transition for $\tilde{U}(\gamma_1, v_1)$ for γ_1 sufficiently distant from 0 and 1 for v_1 sufficiently close to 0. But in order to prove these last assertions, it suffices to prove $\epsilon_3(\gamma_1) > 0$, i.e., $\tilde{a}_3 > a_3$ for any concentrations $(\gamma_1, 1-\gamma_1) \in \Gamma$ (note that $\tilde{a}_3 = a_3$ for $\gamma_1 \in \{0,1\}$). Since $v_0 = 0$ and $v_1 > 0$, (10.5) yields

$$a_3 < a_2 < a_1 < a_0 ; \qquad (10.26)$$

that is, in our system there is a tendency of placing a particle at a site t (i.e., $X_t^{(1)} = 1$) when the neighbouring sites are vacant. Consequently, the unconditioned probabilities must satisfy

$$\pi_1^1 > a_3, \quad \pi_1^2 > a_3. \qquad (10.27)$$

Using (10.14) and (10.26) to lower-estimate $\tilde{a}_3$ we get the inequality

$$\tilde{a}_3 > a_3 + [\gamma_1 \pi_1^1 + (1-\gamma_1)\pi_1^2]^{-3} \times$$
$$\times \gamma_1 (1-\gamma_1)[(1-\gamma_1)^2 (\pi_1^2)^3 (\pi_1^1 - a_3) + \gamma_1^2 (\pi_1^1)^3 (\pi_1^2 - a_3)]. \qquad (10.28)$$

By (10.27) the term in square brackets in (10.28) is positive. Hence $\tilde{a}_3 > a_3$, regardless of which concentrations $(\gamma_1, 1-\gamma_1) \in \Gamma$ were chosen.

11. CONCLUSIONS

Our results suggest rather complicated relations between the original potential and the resulting "heterophase" potential. Even in the simplest case discussed in Section 10 we were able to get only qualitative results. Exact calculations do not appear manageable analytically due to occurrence of transcendent equations. Thus, the problem of calculating heterophase potentials turns out to be open and very accute. More specifically (and this may be the simplest problem to start with), it is of interest to relate the construction (3.9) with calculations not of the probabilities themselves, but merely of observables of physical interst (mean values of "spin", free energy, entropy, etc.). And, above all, to investigate the dynamical properties.

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APPENDIX

Here we shall prove Lemma 13. By definition,

$$\tilde{a}_3 = \Pr[X_t = 1 \mid X_{t_1} = X_{t_2} = X_{t_3} = 1]. \qquad (A.1)$$

Let

$$A(j) = [Z_t = j] \cap [X_t^{(j)} = 1], \quad j = 1,2. \qquad (A.2)$$

Note that $A(1) \cap A(2) = \emptyset$ since $[Z_t = 1] \cap [Z_t = 2] = \emptyset$.
The condition in (A.1) will be expressed as the union of the following mulually disjoint events:

$$\begin{aligned}
E(1) &= [Z_{t_1,t_2,t_3} = (1,1,1)] \cap [X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1]; \\
E(2) &= [Z_{t_1,t_2,t_3} = (2,1,1)] \cap [X_{t_1}^{(2)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1]; \\
E(3) &= \quad \ldots \quad (1,2,1) \quad \ldots \quad ; \\
E(4) &= \quad \ldots \quad (1,1,2) \quad \ldots \quad ; \\
E(5) &= \quad \ldots \quad (2,2,1) \quad \ldots \quad ; \\
E(6) &= \quad \ldots \quad (2,1,2) \quad \ldots \quad ; \\
E(7) &= \quad \ldots \quad (1,2,2) \quad \ldots \quad ; \\
E(8) &= [Z_{t_1,t_2,t_3} = (2,2,2)] \cap [X_{t_1}^{(2)} = X_{t_2}^{(2)} = X_{t_3}^{(2)} = 1].
\end{aligned} \qquad (A.3)$$

Now

$$\tilde{a}_3 = \Pr[A(1) \cup A(2) \mid X_{t_1} = X_{t_2} = X_{t_3} = 1] =$$
$$= \Pr[A(1) \mid X_{t_1} = X_{t_2} = X_{t_3} = 1] + \Pr[A(2) \mid X_{t_1} = X_{t_2} = X_{t_3} = 1] =$$
$$= \gamma_1 \Pr[X_t^{(1)} = 1 \mid X_{t_1} = X_{t_2} = X_{t_3} = 1] + (1-\gamma_1)\Pr[X_t^{(2)} = 1 \mid X_{t_1} = X_{t_2} = X_{t_3} = 1].$$

Here we used the definition of $A(j)$ and independence of Z and $(X^{(1)}, X^{(2)})$, according to which

$$\Pr[A(j)] = \Pr[(Z_t = j) \wedge (X_t^{(j)} = 1)] = \Pr[X_t^{(j)} = 1 \mid Z_t = j] \cdot \Pr[Z_t = j]$$

$$= \Pr[X_t^{(j)} = 1] \Pr[Z_t = j] .$$

Hence

$$\tilde{a}_3 = \gamma_1 \Pr[X_t^{(1)} = 1 \mid \bigcup_{\ell=1}^{8} E(\ell)] + (1-\gamma_1) \Pr[X_t^{(2)} = 1 \mid \bigcup_{\ell=1}^{8} E(\ell)] . \qquad (A.4)$$

In the first step, we evaluate the probability

$$\Pr[\bigcup_{\ell=1}^{8} E(\ell)] = \sum_{\ell=1}^{8} \Pr[E(\ell)] . \qquad (A.5)$$

Using independence of Z and $X^{(1)}$, independence of the random variables Z_u, Z_t $(u \neq t)$, the fact that $X^{(1)}$ is a MRF and that the sites t_1, t_2, t_3 are not nearest neighbours of each other, we get

$$\Pr[E(1)] = \Pr\{[Z_{t_1, t_2, t_3} = (1,1,1)] \cap [X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1]\} =$$

$$= \Pr[Z_{t_1, t_2, t_3} = (1,1,1)] \times \Pr[X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1 \mid Z_{t_1, t_2, t_3} = (1,1,1)] =$$

$$= \gamma_1^3 \Pr[X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] ,$$

i.e.,

$$\Pr[E(1)] = \gamma_1^3 (\pi_1^1)^3 . \qquad (A.6)$$

In a similar way, on the base of symmetry properties,

$$\Pr[E(2)] = \Pr[E(3)] = \Pr[E(4)] = \gamma_1^2 (1-\gamma_1) (\pi_1^1)^2 \pi_1^2 , \qquad (A.7)$$

$$\Pr[E(5)] = \Pr[E(6)] = \Pr[E(7)] = \gamma_1 (1-\gamma_1)^2 \pi_1^1 (\pi_1^2)^2 . \qquad (A.8)$$

$$\Pr[E(8)] = (1-\gamma_1)^3 (\pi_1^2)^3 . \qquad (A.9)$$

Summing (A.6) up to (A.9) we get

$$\Pr[\bigcup_{\ell=1}^{8} E(\ell)] = (\gamma_1 \pi_1^1 + (1-\gamma_1) \pi_1^2)^3 . \qquad (A.10)$$

Consequently,

$$\tilde{a}_3 = (\gamma_1 \pi_1^1 + (1-\gamma_1)\pi_1^2)^{-3} \times \{\gamma_1 \sum_{\ell=1}^{8} \Pr\{[X_t^{(1)} = 1] \cap E(\ell)\} +$$

$$+ (1-\gamma_1) \sum_{\ell=1}^{8} \Pr\{[X_t^{(2)} = 1] \cap E(\ell)\}\} . \qquad (A.11)$$

because of (A.10) and the relation

$$\tilde{a}_3 = \gamma_1 \frac{\Pr\{[X_t^{(1)} = 1] \cap (\bigcup_{\ell=1}^{8} E(\ell))\}}{\Pr[\bigcup_{\ell=1}^{8} E(\ell)]} + (1-\gamma_1) \times$$

$$\times \frac{\Pr\{[X_t^{(2)} = 1] \cap (\bigcup_{\ell=1}^{8} E(\ell))\}}{\Pr[\bigcup_{\ell=1}^{8} E(\ell)]} .$$

Next we evaluate the probabilities entering into the formula (A.11).

$$\Pr\{[X_t^{(1)} = 1] \cap E(1)\} =$$

$$= \Pr[X_t^{(1)} = 1, X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1, Z_{t_1, t_2, t_3} = (1,1,1)] =$$

$$= \Pr[X_t^{(1)} = 1, X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1 \mid Z_{t_1, t_2, t_3} = (1,1,1)] \times$$

$$\times \Pr[Z_{t_1, t_2, t_3} = (1,1,1)] = \gamma_1^3 \Pr[X_t^{(1)} = 1 \mid X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] \times$$

$$\times \Pr[X_{t_1}^{(1)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] ,$$

i.e.,

$$\Pr\{[X_t^{(1)} = 1] \cap E(1)\} = \gamma_1^3 (\pi_1^1)^3 a_3 . \qquad (A.12)$$

Next

$$\Pr\{[X_t^{(1)} = 1] \cap E(2)\} = \gamma_1^2 (1-\gamma_1) \Pr[X_t^{(1)} = 1 \mid X_{t_1}^{(2)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] \times$$

$$\times \Pr[X_{t_1}^{(2)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] = \gamma_1^2 (1-\gamma_1) \pi_1^2 \Pr[X^{(1)} = 1 \mid X_{t_1}^{(2)} = X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] .$$

Since $X^{(1)}$ and $X^{(2)}$ are independent random fields, the event $[X_{t_1}^{(2)} = 1]$ can be eliminated from the condition without changing the conditional probability, i.e.,

$$\Pr\{[X_t^{(1)} = 1] \cap E(2)\} =$$

$$= \gamma_1^2 (\pi_1^1)^2 (1-\gamma_1) \pi_1^2 \Pr[X_t^{(1)} = 1 \mid X_{t_2}^{(1)} = X_{t_3}^{(1)} = 1] .$$

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Savin I.A., Smirnov G.I. In: JINR Rapid Communications, N2-84, Dubna, 1984, p.3.

Шуян Ш. E5-85-663

Гетерофазные случайные поля.
Поля на бесконечных однородных деревьях

В работе предложен простой результат, касающийся формы гетерофазного потенциала для однородной системы. Остальная часть работы посвящена изучению марковских полей на бесконечных однородных деревьях.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Šujan Š. E5-85-663

Heterophase Random Fields.
Fields on Infinite Homogeneous Trees

We present a simple result concerning the form of heterophase potential for a one-dimensional system. The rest of the paper is devoted to the study of Markov fields on infinite homogeneous trees.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985