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HETEROPHASE RANDOM FIELDS.

Physical Background, Markov Property and Equilibrium Description

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INTRODUCTION

This paper is a continuation of the first part under the same title, hereafter referred to as I. Sections, theorems, and formulae are numbered consecutively, starting with section 5 below. References [1-15] are listed at the end of I.

5. FUNDAMENTAL FEATURES OF HETEROPHASE SYSTEMS

Let us present some basic conclusions about the nature of heterophase systems (see also [15]).

(I) A heterophase system describes a certain "mixture" of pure thermodynamical phases.

(II) The mixture in (I) should be understood (locally; cf. Remark 1) in the sense that configurations typical of the heterophase system consist of "pieces" of configurations, typical of the pure phases comprising that system.

(III) Though local fluctuations are possible, there exist definite concentrations with which the pieces of configurations mentioned in (II) are met in the infinite volume limit (see (2.2)).

(IV) A heterophase system itself should be macroscopically observable in the usual sense of equilibrium statistical mechanics.

These are the basic aspects of the "static" situation when dynamics is not taken into account. Though dynamical properties are of primary interest, one has to develop an equilibrium theory prior to dealing with dynamics. Furthermore, there are certain specific difficulties related to dynamics of heterophase systems.

In fact, much work has been done on dynamics of infinite particle systems (e.g., Glauber dynamics [19], exclusion with speed change [18,24]), and for the mentioned model mechanisms of time evolution the time-invariant measures coincide with Gibbs (=equilibrium) measures. However, the ergodic theorems describing the approach to equilibrium can be proved only when the time-invariant measure is unique. Hence, they do not apply to systems exhibiting a phase transition (i.e., non-uniqueness of the Gibbs measure).

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On the other hand, the concept of a heterophase system does not make sense at all when phase transition is absent. This makes the dynamical problems for heterophase systems rather difficult, unless we meet the following phenomenon: we have a system with a phase transition, but the heterophase system composed of the different pure phases has a unique Gibbs measure. A model system which exhibits such a surprising behaviour will be studied in the next paper.

A phase transition can be understood in two different ways:

(a) a symmetry preserving phase transition, i.e., non-uniqueness of the infinite-volume Gibbs measure for a given potential such that all these measures obey the same symmetry group;

(b) phase transition as a spontaneous symmetry breaking, i.e., existence of Gibbs measures with a given symmetry group as well as ones having lower (broken) symmetry.

The first interpretation is commonly employed in mathematically oriented literature on phase transitions (see[3,17,22]). The second one seems physically more relevant and natural (e.g., imagine a ferromagnetic phase with paramagnetic fluctuations[9,23]).

Of course, the two interpretations of a phase transition lead to different requirements on the concept of macroscopic observability. In the wide framework of stationary random fields I ergodicity is appropriate. This follows from the interpretation of the formula

$$P(\cup\{R(\phi):\phi\in R \bmod P\}) = 1, \quad P\in \mathfrak{M}(S) \tag{5.1}$$

as given in[18] (see also Section 19 of[13]). Within equilibrium systems of statistical mechanics we need another concept, for (a) the measures considered have special properties (they are Gibbs measures) and (b) they are not, in general, invariant so that we cannot speak about ergodicity with respect to the group $T = Z^d$. Here the results of I are not applicable, and it is the problem we start with.

6. HETEROPHASE RANDOM FIELDS AND LOCAL SPECIFICATION

Since our aim is to deal with random fields for which no symmetry group is singled out, we may consider the space S^T, where T is any countably infinite set. Recall from[20] (cf. also[17,22]) that a local specification is a family

$$q = \{q_{V,\phi}(A) : V\in\mathcal{A},\ \phi\in S^T,\ A\in\mathcal{S}^T\} \tag{6.1}$$

such that

(a) for any $V\in\mathcal{A}$ and $\phi\in S^T$, $q_{V,\phi}(\cdot):\mathcal{S}^V\to[0,1]$ is a probability measure,

(b) for any $V\in\mathcal{A}$ and $A\in\mathcal{S}^V$, $\phi\to q_{V,\phi}(A):S^T\to[0,1]$ is $\mathcal{S}^{\bar V}$-measurable, where $\bar V = T\setminus V$,

(c) if $V_1\subset V_2\in\mathcal{A}$, then for any $A\in\mathcal{S}^{V_1}$, $B\in\mathcal{S}^{V_2-V_1}$, and $\phi\in\mathcal{S}^T$

$$q_{V_2,\phi}(A\quad B) = \int_B q_{V_1,c(V_2;\phi',\phi)}(A)\, q_{V_2,\phi}(d\phi'), \tag{6.2}$$

where

$$c(V_2;\phi',\phi) = \begin{cases} \phi'(t) & \text{if } t\in V_2, \\ \phi(t) & \text{if } t\notin V_2. \end{cases}$$

A probability measure $P\in\mathcal{P}(S)$ is said to be a <u>specified random field</u> to the specification q in symbols, $P\in\mathfrak{R}(q)$, if

$$P(A\,|\,\mathcal{S}^{\bar V}) = q_{V,\phi}(A) \quad P\text{-a.e.},\ V\in\mathcal{A},\ A\in\mathcal{S}^V. \tag{6.3}$$

We suppose that $\mathfrak{R}(q)\neq\emptyset$. This is the case, e.g., if $|S|<\infty$ and the specification is continuous in the sense that

$$\lim_{W\uparrow T}\sup_{\phi'} |q_{V,\phi'}(A) - q_{V,c(W;\phi',\phi)}(A)| = 0.$$

Let

$$(\mathcal{S}^T)_\infty = \bigcap_{V\in\mathcal{A}} \mathcal{S}^{\bar V} \tag{6.4}$$

denote the tail σ-field. We say $P\in\mathcal{P}(S)$ has a <u>trivial tail</u> if

$$P(E)\in\{0,1\} \quad \text{for any } E\in(\mathcal{S}^T)_\infty. \tag{6.5}$$

As is shown in[20], the following assertions are equivalent:

- $P\in\mathfrak{R}(q)$ has a trivial tail,
- $P\in \mathrm{Ext}\,\mathfrak{R}(q)$, i.e., P is an extreme point of the convex set $\mathfrak{R}(q)$, and
- P is <u>regular</u>, that is, for any $A\in\mathcal{S}^T$,

$$\lim_{V\in\mathcal{A},\,V\uparrow T}\ \sup_{B\in\mathcal{S}^{\bar V}} |P(A\cap B) - P(A)P(B)| = 0. \tag{6.6}$$

In other words P has short-range correlations[21]. Of course, (6.6) is a reasonable formulation of macroscopic observability in case when a symmetry group is absent. This can be seen also from the fact that there exists a <u>decomposition at infinity</u> of any $P\in\mathfrak{R}(q)$, i.e., a decomposition of P into fields having a trivial tail. Formally, that decomposition can be parametrized again by configurations:

$$P(E) = \int_R Q_\phi(E)\, P(d\phi), \quad P \in \mathcal{R}(q), \quad E \in \mathcal{S}^T, \tag{6.7}$$

and analogues of all properties of ergodic decomposition, as well as an analogue of its interpretation, are valid [21]. Since canonical systems of measures are associated with an arbitrary Rokhlin measurable partition of a Lebesque space [6], we can process exactly in the same way as indicated in the proof of Theorem 3 and get the following result:

Theorem 4. Let $P \in \mathcal{R}(q) \cap \mathcal{H}(P^{(i)}, \gamma_i, I)$ (cf. (6.3) and (2.1)). Then

$$P\{\phi \in R : Q_\phi \in \mathcal{R}(q) \cap \mathcal{H}(P^{(i)}, \gamma_i, I)\} = 1. \tag{6.8}$$

In particular, if $\mathcal{R}(q) \cap \mathcal{H}(P^{(i)}, \gamma_i, I) \neq \emptyset$ then also

$\mathrm{Ext}\, \mathcal{R}(q) \cap \mathcal{H}(P^{(i)}, \gamma_i, I) \neq \emptyset$.

However, unlike Theorem 3 the conclusion of this assertion is rather conditional, for it presupposes that $\mathcal{R}(q) \cap \mathcal{H}(P^{(i)}, \gamma_i, I) \neq \emptyset$. We do not know general conditions for this, but we shall meet situations in which the following assertion is true:

"Theorem" 5. Let $\{P^{(i)} : i \in I\} \subset \mathcal{R}(q)$. Let $\gamma = (\gamma_i)_{i \in I} \in \Gamma$. Then there exists a local specification $\tilde{q}$ such that

$\mathcal{R}(\tilde{q}) \cap \mathcal{H}(P^{(i)}, \gamma_i, I) \neq \emptyset$.

7. HETEROPHASE RANDOM FIELDS AND MARKOV PROPERTY

Markov property is physically important because under rather general conditions Markov fields are equivalent to Gibbs fields. We shall consider the simplest state space

$$S = \{0, 1\}. \tag{7.1}$$

Consequently, any configuration $\phi \in \{0,1\}$ may be identified with a subset of T, namely, with

$$N(\phi) = \{t \in T : \phi(t) = 1\}. \tag{7.2}$$

Let $\|t' - t''\|$ denote the usual Euclidean distance between points $t', t'' \in T = \mathbf{Z}^d$. For any set $V \subset \mathbf{Z}^d$ we let ∂V denote the set of all nearest neighbours of V:

$$\partial V = \{t \in \mathbf{Z}^d \setminus V : \|t - V\| = 1\}. \tag{7.3}$$

More generally, if $R \geq 1$, put

$$\partial_R V = \{t \in \mathbf{Z}^d \setminus V : \|t - V\| \leq R\}. \tag{7.4}$$

Since $\min\{\|t' - t''\| : t', t'' \in \mathbf{Z}^d, t' \neq t''\} = 1$, we actually have $\partial_1 V = \partial V$.

Remark 4. We may in an obvious way define the concept of neighbours and R-neighbours also when instead of $\mathbf{Z}^d$ the set T is a countable graph. Then all considerations below will remain valid (see [21] for more details on Markov and Gibbs fields on countable graphs). However, for the sake of simplicity we shall work with $T = \mathbf{Z}^d$ throughout the rest of this section.

A $\{0,1\}$ - valued random field $X = (X_t, t \in T)$ defined on a probability space $(\Omega, \mathcal{F}, \mu)$ is said to be R-Markov ($R \geq 1$) if $P = \mathrm{dist}(X)$ is positive on all f.d. sets $E = \mathcal{S}^T$ ($\mathcal{S}$ = the power set of $\{0,1\}$), and if for any $V \in \mathcal{Q}$, $\phi_V \in \{0,1\}^V$ and $\bar{\phi}_{\bar{V}} \in \{0,1\}^{\bar{V}}$ (recall $\bar{V} = T \setminus V$)

$$\mu[(X_t ; t \in V) = \phi_V | (X_t ; t \in \bar{V}) = \bar{\phi}_{\bar{V}}] = \\ = \mu[(X_t ; t \in V) = \phi_V | (X_t ; t \in \partial_R V) = \bar{\phi}_{\partial_R V}]. \tag{7.5}$$

A 1-Markov field is called simply Markov. The measure $P = \mathrm{dist}(X) \in \mathcal{P}(\{0,1\})$ is also said to be R-Markov and Markov respectively.

Theorem 6. Let $\{X^{(i)} : i \in I\}$ be a set of Markov fields defined on a common probability space $(\Omega, \mathcal{F}, \mu)$. Let Z be an I-valued random field, independent of the set $\{X^{(i)} : i \in I\}$ and such that $\mathrm{dist}(Z) \in \mathcal{P}(I)$ is strictly positive for all f.d. sets $E \in \mathcal{S}^T$. Then the random field X, constructed via (3.9), is Markov. More generally, if $X^{(i)}$ is $R^{(i)}$-Markov and $\sup\{R^{(i)} : i \in I\} \leq R < \infty$, then X is at most R-Markov (i.e., X is R'-Markov for some $R' \leq R$).

Proof. We shall deal only with the case $R = 1$ and $I = \{1, 2\}$. An extension to $R \geq 1$ and to an arbitrary countable I is straightforward, however, the formulae involved are rather complex.

Let $P = \mathrm{dist}(X)$. If $E \in \{0,1\}^T$ is any f.d. set, $P(E) > 0$ since $P^{(i)}(E) > 0$ for each $i \in I$, and $\mathrm{dist}(Z)$ is positive on f.d. sets in $\mathcal{S}^T$, and P(E) does not depend on more coordinates than those which determine E because of the properties of the channel ν defined in the proof of Theorem 1.

Let $V \in \mathcal{Q}$, $\phi_V \in \{0,1\}^V$, $\bar{\phi}_{\bar{V}} \in \{0,1\}^{\bar{V}}$. We must show (7.5) with $R = 1$ (i.e., $\partial_R = \partial$). Since Z is not supposed stationary, finite-dimensional distributions $\mathrm{dist}[(Z_t ; t \in V)]$ depend not only on $V \in \mathcal{Q}$ but also on its location.

First consider one-dimensional fields ($d = 1$). Then for any $t \in \mathbf{Z}^d$ there is a number $0 < q_t < 1$ such that

$\mu[Z_t = 1] = q_t = 1 - \mu[Z_t = 2]$.

By redefining the space $(\Omega, \mathcal{F}, \mu)$ we can assume that the random fields $X^{(1)}$ and $X^{(2)}$ are stochastically independent, and Z is independent of the pair $(X^{(1)}, X^{(2)})$. In particular, finite-dimensional distributions of $X^{(1)}$ and $X^{(2)}$ are independent probability vectors, too. Consider the case $V=\{t\}$, and let $\bar{t}=\{\bar{t}\}=T\setminus\{t\}$. A direct calculation yields

$$\mu[(X_t^{(Z_t)}=\phi(z) \mid (X_{t'}^{(Z_{t'})};\ t'\in\bar{t})=\bar{\phi}_{\bar{t}}]=$$

$$= q_t\,\mu[X_t^{(1)}=\phi(t) \mid (X_{t'}^{(Z_{t'})};\ t'\in\bar{t})=\bar{\phi}_{\bar{t}}] + \tag{7.6}$$

$$+ (1-q_t)\,\mu[X_t^{(2)}=\phi(t) \mid (X_{t'}^{(Z_{t'})};\ t'\in\bar{t})=\bar{\phi}_{\bar{t}}].$$

If $t'=t-1$, then either $Z_{t'}=1$ and hence $X_{t'}^{(Z_{t'})}=X_{t-1}^{(1)}$ or, $Z_{t'}=2$ and $X_{t'}^{(Z_{t'})}=X_{t-1}^{(2)}$. Since $X^{(1)}$ and $X^{(2)}$ are independent, only the first case remains in the condition of the first summand in (7.6), and only the second case remains in the condition of the second summand. Similarly for the case $t'=t+1$. If $t'=t-2$, then either $Z_{t'}=1$ and $X_{t'}^{(Z_{t'})}=X_{t-2}^{(1)}$ or $Z_{t'}=2$ and $X_{t'}^{(Z_{t'})}=X_{t-2}^{(2)}$. In the first case we use the Markov property of $X_{t-2}^{(2)}$, in the second one independence of $X^{(1)}$ and $X^{(2)}$. This permits to exclude both cases from conditions in each of the two conditional probabilities in (7.6). Exactly the same argument applies to $t'=t+2$, $t-3$, $t+3$, etc. Consequently,

$$\mu[X_t^{(Z_t)}=\phi(t) \mid (X_{t'}^{(Z_{t'})};\ t'\in\bar{t})=\bar{\phi}_{\bar{t}}]=$$

$$= q_t\,\mu[X_t^{(1)}=\phi(t) \mid (X_{t-1}^{(1)}, X_{t-1}^{(1)})=(\bar{\phi}(t-1), \bar{\phi}(t+1))] +$$

$$+ (1-q_t)\,\mu[X_t^{(2)}=\phi(t) \mid (X_{t-1}^{(2)}, X_{t+1}^{(2)})=(\bar{\phi}(t-1), \bar{\phi}(t+1))]=$$

$$= \mu[X_t^{(Z_t)}=\phi(t) \mid (X_{t-1}^{(Z_{t-1})}, X_{t+1}^{(Z_{t+1})})=(\bar{\phi}(t-1), \bar{\phi}(t+1))] =$$

$$= \mu[X_t=\phi(t) \mid X_{t'};\ t'\in\partial\{t\})=\bar{\phi}_{\partial\{t\}}].$$

It is easy to see that any probability measure on $\mathcal{S}^T$ is uniquely determined by the values it takes on f.d. sets of form (1.5),

where $V\in\mathcal{A}$ is a connected set. In case $d=1$ it suffices to take the sets $V=\{-n,\dots,n\}$, $n=0,1,2,\dots$. The boundary of any of these sets consists again of exactly two points so that the preceding analysis applies and we are done in case $d=1$.

If $d>1$, then it is clear (although somewhat cumbersome to write down explicitly) that the conditional probability in (7.5) may be expressed as a combination of conditional probabilities

$$\mu[(X_t^{(i_t)};\ t\in V)=\phi_V \mid (X_t;\ t\in\bar{V})=\bar{\phi}_{\bar{V}}];\ (i_t;\ t\in V)\in\{1,2\}^V,$$

with coefficients being functions of the components of the probability vector

$$(\mu[Z_t;\ t\in V)=\psi_V];\ \psi_V\in\{1,2\}^V).$$

For any fixed $(i_t;\ t\in V)\in\{1,2\}^V$ the boundary effects can be evaluated as above, and some combinatorics lead to the desined conclusion.

By combining Theorem 6 with the end of the proof of Theorem 1 (starting with (3.15) and (3-16)) we get the following result:

Corollary 7. Let the hypothesis of Theorem 6 be satisfied, and let the random field Z satisfy (3.15) and (3.16) for some fixed $\gamma=(\gamma_i)_{i\in I}\in\Gamma$. Then the set of Markov fields in $\mathcal{H}(X^{(i)}, \gamma_i, I)$ is non-empty.

In other words, if the sibfields $\{X^{(i)}: i\in I\}$ are all Markov, then for any $\gamma\in\Gamma$ there exists a heterophase random field obeying the Markov property, too. Of course, the latter two assertions admit also a formulation in terms of Gibbs fields. To this end recall some concepts (cf. /17,21,22/). If $U:\mathcal{A}\to R^1$ is an arbitrary function, then due to the identification (7.2) we may put

$$U(A)=U(\phi),\quad A\in\mathcal{A},\ \phi\in\{0,1\}^T,\quad N(\phi)=A. \tag{7.7}$$

A set $A\subset T$ is said to be a simplex if for any two distinct points $t, \mu\in A$ we have $\|t-u\|=1$. Any function $u:\mathcal{A}\to R^1$ such that $U(\emptyset)=0$ is said to be a potential. Here the empty set corresponds, according to (7.2), to the configuration $\epsilon=(\epsilon(t);\ t\in T)$, where $\epsilon(t)\equiv 0$. Any potential $\mathcal{J}_U$ induces an interaction potential $\mathcal{J}_U:\mathcal{A}\to R^1$ via the Mobius formula

$$\mathcal{J}_U(A)=\sum_{X\subset A}(-1)^{|A\setminus X|}U(X),\quad A\in\mathcal{A}. \tag{7.8}$$

If $\mathcal{J}_U(A)\neq 0$ only when A is a simplex then, U is said to be a rearest neighbour potential. If $V, V'\in\mathcal{A}$ and $V'\supset V\cup\partial V$ (cf. (7.3)), we define the Gibbs distribution in V given the boundary conditions $\bar{\Phi}$ by the properties that

$$\pi_{V,\bar{\Phi}}(\phi) = Z^{-1}_{V,\bar{\Phi}} \exp U(\phi \bullet \bar{\Phi}), \quad \phi \in \{0,1\}^V, \tag{7.9}$$

where

$$Z_{V,\bar{\Phi}} = \sum_{\phi \in \{0,1\}^V} \exp U(\phi \bullet \bar{\Phi}), \tag{7.10}$$

Here, if $V, W \in \mathfrak{A}, V \cap W = \emptyset, \phi \in \{0,1\}$ and $\psi \in \{0,1\}^V$, $\phi \bullet \psi \in \{0,1\}^{V \cup W}$ is defined by (compare with Sect.6)

$$(\phi \bullet \psi)(t) = \begin{cases} \phi(t) & \text{if } t \in V, \\ \psi(t) & \text{if } t \in W. \end{cases} \tag{7.11}$$

A measure $P \in \mathcal{P}(\{0,1\})$ is said to be a Gibbs measure (and X, $\mathrm{dist}(X) = P$, a Gibbs field) to the nearest neighbour potential if for any V, V', ϕ and $\bar{\Phi}$ specified as above

$$\mu[(X_t ; t \in V) = \phi \mid (X_{t'} ; t' \in V' \setminus V) = \bar{\Phi}] = \pi_{V,\bar{\Phi}_{\partial V}}(\phi), \tag{7.12}$$

i.e.,

$$\pi_{V,\bar{\Phi}_{\partial V}}(\phi) = \frac{P\{\bar{\Phi} \in \{0,1\}^T : \bar{\Phi}_{V'} = \phi \bullet \bar{\Phi}\}}{P\{\bar{\Phi} \in \{0,1\}^T : \bar{\Phi}_{V' \setminus V} = \bar{\Phi}\}}. \tag{7.13}$$

The property of nearest neighbour interaction is contained in (7.13), for it can be rewritten in the form

$$\pi_{V,\bar{\Phi}}(\phi) = \pi_{V,\bar{\Phi}_{\partial V}}(\phi).$$

In what follows we shall deal only with nearest neighbour potentials. By Thm.4.1. of Preston[24], if $P \in \mathcal{P}(\{0,1\})$ is positive on all f.d. sets, then P is a Markov measure if and only if there exists a nearest neighbour potential $U : \mathfrak{A} \to \mathbf{R}^1$ such that P is a Gibbs measure to the potential U, in symbols, $P \in \mathcal{G}(U)$.

As is well-known, the set $\mathcal{G}(U)$ is non-empty in this case, and Gibbs and specified random fields are equivalent notions (in the sense that any nearest neighbour potential gives rise to a continuous specification and conversely, to any nearest neighbour potential one can define a specification such that $\mathcal{R}(q) = \mathcal{G}(U)$; cf.[24]), Corollary 7 may be reformulated in terms of specified random fields:

Corollary 8. Let $S = \{0,1\}$, and let $P^{(i)} \in \mathcal{R}(q^{(i)})$; $i \in I$, where the $q^{(i)}$'s are specifications determined by nearest neighbour potentials. Then for any $\gamma = (\gamma_i)_{i \in I} \in \Gamma$ there exists a specification q such that $\mathcal{R}(q) \cap \mathcal{H}(P^{(i)}, \gamma_i, I) \neq \emptyset$. In particular, Theorem 5 is true.

8. HETEROPHASE RANDOM FIELDS AND GIBBS PROPERTY

Let $V \in \mathfrak{A}$, and let $\phi \in \{0,1\}^T$ be such that $N(\phi) \cup \partial N(\phi) \subset V$ (see (7.2), (7.3)). If P is a Markov field, we can define a nearest neighbour potential U corresponding to P by

$$U(\phi) = \log \frac{P\{\bar{\Phi} \in \{0,1\}^T : \bar{\Phi}_V = \phi\}}{P\{\bar{\Phi} \in \{0,1\}^T : \bar{\Phi}_V = \epsilon_V\}}, \quad \phi \in \{0,1\}^V, \tag{8.1}$$

where $\epsilon(t) \equiv 0$. Consequently, Theorem 6 may be reformulated as follows:

Theorem 9. Let $\{P^{(i)} : i \in I\} \subset \mathcal{P}(\{0,1\})$ be a set of Gibbs measures, $P^{(i)} \in \mathcal{G}(U^{(i)})$, where $U^{(i)}$ is a nearest neighbour potential for each $i \in I$. Let X be a $\{0,1\}$-valued random field constructed via (3.9), where $\mathrm{dist}(X^{(i)}) = P^{(i)}$ for each $i \in I$, and let Z satisfy the assumptions of Theorem 6. Then there exists a nearest neighbour potential U such that $\mathrm{dist}(X) \in \mathcal{G}(U)$ (we shall write also $X \in \mathcal{G}(U)$).

Note that if $T = \mathbf{Z}$, the random fields $X^{(i)}$ in Theorem 9 are stationary, and Z satisfies the conditions (3.15) and (3.16), then we can assert the existence of a nearest neighbour potential U such that

$$\mathcal{G}(U) \cap \mathcal{M}(\{0,1\}) \cap \mathcal{H}(X^{(i)}, \gamma_i, I) \neq \emptyset \tag{8.2}$$

for any $\gamma = (\gamma_i)_{i \in I} \in \Gamma$. As mentioned above, a reasonable description of macroscopic observability provides the concept of regularity (cf. (6.6)). However, we shall deal only with the weaker property of X having a trivial tail (i.e., (6.5) is true for $P = \mathrm{dist}(X)$). Indeed, if $X \in \mathcal{G}(U)$, then the equivalences stated in Section 6 share their validity due to the above mentioned relations between specified random fields and Gibbs fields (this is shown in[3] for a large class of potentials including ours). Trivial tail enables us to avoid imposing too strong conditions on Z. We shall need only the following one. Let $\lambda = \mathrm{dist}(Z)$ and, for any $V \subset T$, $\lambda_V = \mathrm{dist}((Z_t ; t \in V))$. The field Z is said to be mixing if for any $V, W \in \mathfrak{A}$, $\psi \in I^V$ and $\bar{\psi} \in I^W$,

$$\lim_{t \to \infty} | \lambda_{(V-t) \cup W} (\psi \bullet \bar{\psi}) - \lambda_V(\psi) \lambda_W(\bar{\psi}) | = 0, \tag{8.3}$$

where $V + t = \{u + t : u \in V\}$ (cf. (7.11)). The next assertion is valid for an arbitrary countable discrete state space S.

Lemma 10. Let $\{P^{(i)}: i\in I\}\subset \mathcal{P}(S)$ be a set of regular measures, let $\mathrm{dist}(X^{(i)}) = P^{(i)}$, $i\in I$. Let X be constructed via (3.9), where Z is a stationary mixing random field, stochastically independent of the set $\{X^{(i)}: i\in I\}$. Then X has a trivial tail (i.e., (6.5) is true for $P = \mathrm{dist}(X)$).

Proof. (6.5) is a consequence of the fact that the σ-fields $\mathcal{S}^T$ and $(\mathcal{S}^T)_\infty$ (cf. (6.4)) are independent under P, and this in turn follows from the following mixing property: for any f.d. sets $E, \bar{E}\in \mathcal{S}^T$,

$$\lim_{t\to\infty} | P(T_t^{(S)} E \quad \bar{E}) - P(\tau_t^{(S)} E) P(\bar{E}) | = 0. \tag{8.4}$$

We shall prove (8.4). To this end fix two f.d. sets, say

$$E = \{ \phi\in S^T: \phi_V \in C\}, \quad \bar{E} = \{ \bar{\phi}\in S^T: \bar{\phi}_W \in \bar{C} \},$$

where $V, W\in \mathcal{Q}$, $C\subset S^V$ and $\bar{C}\subset S^W$. Let $\lambda = \mathrm{dist}(Z)$ and let ν denote the channel introduced in the proof of Theorem 1 (cf. (3.11)-(3.13) and note that the fact that ν is input historyless and nonanticipatory without and stationary assumption on the $X^{(i)}$'s, being a more consequence of the independence assumption (3.8)). Then

$$P(E) = \mu[(X_t; t\in V)\in C] = P[I^T\times\{\phi\in S^T: \phi_V\in C\}] =$$

$$= \int_{I^T} \nu_\psi(\{\phi\in S^T: \phi_V\in C\}\lambda(d\psi) = \sum_{\psi_V\in I^V} \lambda_V(\psi_V)\mu[(X_t^{(\psi_V(t))}; t\in V)\in C].$$

By expressing $P(\tau_t^{(S)}(E))$ and $P(\bar{E})$ in the same manner we see that the difference in (8.4) can be written as follows:

$$\Big| \sum_{\substack{\psi\in I^V\\ \bar{\psi}\in I^W}} \mu[(X_\mu^{(\psi(u))}; u\in V+t)\in C, \quad (X_\mu^{(\bar{\psi}(u))}; u\in W)\in \bar{C}]\times$$

$$\times \lambda_{(V+t)\cup W}(\psi \bullet \bar{\psi}) - \{ \sum_{\psi\in I^V} \mu[(X_\mu^{(\psi(u))}; u\in V+t)\in C]\cdot \lambda_{V+t}(\psi)\}\times$$

$$\times \{ \sum_{\psi\in I^W} \mu[(X_\mu^{(\bar{\psi}(u))}; u\in W)\in \bar{C}]\lambda_W(\bar{\psi})\}\Big|.$$

By changing the basic probability space if necessary we may suppose that the set $\{P^{(i)}: i\in I\}$ is jointly regular. From this and (8.3) we see that the latter difference approaches zero as $t\to\infty$.

Lemma 10 allows us to strengthen Theorem 9 in order to get the desired property of macroscopic observability for heterophase random fields:

Theorem 11. Let $\{P^{(i)}: i\in I\}$, $\{X^{(i)}: i\in I\}$, Z and X be as in Theorem 9. Suppose in addition that Z is mixing and $P^{(i)}\in \mathrm{Exp}\,\mathcal{G}(U^{(i)})$ for any $i\in I$. Then there exists a nearest neighbour potential U such that

$$P = \mathrm{dist}(X)\in \mathrm{Ext}\,\mathcal{G}(U). \tag{8.5}$$

In particular, for any $\gamma = (\gamma_i)_{i\in I}\in\Gamma$ there exists a nearest neighbour potential U such that

$$\mathrm{Ext}\,\mathcal{G}(U)\cap \mathcal{H}(P^{(i)}, \gamma_i, I) \neq \emptyset. \tag{8.6}$$

Similarly to the concepts of joint stationarity (cf. (3.5), (3.6)) and of joint weak mixing (cf. (3.5), (3.17)) we can define also the concept of joint mixing. Then we have the following result:

Lemma 12. Let $\{X^{(i)}: i\in I\}\subset \mathfrak{M}(S)$ be a jointly mixing set, and let Z be an I-valued random field independent of the set $\{X^{(i)}: i\in I\}$. Let X be defined via (3.9). If Z is ergodic [weak mixing, mixing] then X is ergodic [mixing, weak mixing].

Proof. By repeating the arguments of [1], from the fact that $\{X^{(i)}: i\in I\}$ is jointly mixing it follows that the channel ν (cf. (3.11), (3.13)) is a strongly mixing stationary channel. Using 1, Theorem, p.929 we get the claim.

Remark 5. Lemma 12 in a very transparent form illustrates the difference between "heterophase mixtures" and the usual mixtures

$$P = \sum_{i\in I} \gamma_i P^{(i)}. \tag{8.7}$$

Since $P^{(i)}\in \mathcal{E}(S)$, we have from (5.1) that

$$P[\bigcup_{i\in I} R(P^{(i)})] = 1, \tag{8.8}$$

where $R(P^{(i)}) = R(\phi)$ (cf. (4.3)) for $\phi\in R$ with $P_\phi = P^{(i)}$, $i\in I$. Consequently, the average $\langle f\rangle_P$ cannot be calculated as an ergodic (Cesaro) average, for it depends on the choice of ϕ in sets $R(P^{(i)})$. Since the averages $\langle f\rangle_P$ depend on the particular configuration observed, mixtures (8.7) cannot describe macroscopically observable systems. On the other hand, heterophase mixtures can possess, in principle, arbitrarily good ergodic properties.

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Гетерофазные случайные поля.
Физическое обоснование, марковское свойство
и описание равновесия

После представления короткого обзора физических заключений о свойствах гетерофазных систем изучается марковское свойство составных случайных полей и установлена возможность описания равновесия /гиббсовского/ этих систем.

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Heterophase Random Fields.
Physical Background, Markov Property,
and Equilibrium Description

After presenting a brief summary of physical conclusions about the nature of heterophase systems the Markov property of composite random fields and of heterophase random fields is studied, and the possibility of equilibrium (Gibbsian) description of such systems is established.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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