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P.Šťovíček

THE CONNECTION BETWEEN THE SELF-DUAL EQUATION IN THE J FORMULATION AND THE ADHM CONSTRUCTION

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1. Introduction

The Atiyah-Drinfeld-Hitchin-Manin construction reduced the problem of self-dual solutions to the purely algebraic one. But this topic still attracts a deal of interest. Most probably the reason is that methods of the algebraic geometry used in the ADHM construction are not commonly known and besides attempts to apply more general methods developed for nonlinear equations are being made. The starting point of these attempts is the re-formulation of the self-dual equation due to Yang, Belavin and Zakharov /1,2/. Some remarkable results have been achieved : the construction of Kac-Moody algebra (without the central extension) of symetries /3,4,5/ and the explicit solution of the initial value problem in the complexified Euclidian space /6/.

The aim of this paper is to point out the connection between the self-dual equation in the J formulation (7) (we adopt the terminology of Ref./4/) and the ADHM construction. Further the conformal invariance of the J equation is discussed and as an example the explicit form of 2-instanton solution is obtained.

We restrict ourselves to the symplectic groups Sp(n) (considered as subgroups of SU(2n)), but orthogonal and unitary groups can be treated in a similar way.

2. Formulation due to Yang, Belavin and Zakharov

Let us consider a gauge field A_μ, $\mu=1,..,4$ taking values in the Lie algebra su(n) with the strength field

$$F_{\mu\nu} = [\nabla_\mu,\nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

($\nabla_\mu = \partial_\mu + A_\mu$ stands for covariant derivation) which is self-dual

$$F_{\mu\nu} = {}^*F_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,F_{\alpha\beta} \; . \qquad (1)$$

We furnish the real Euclidian space $\mathbb{R}^4$ with a complex structure by means of identification $\mathbb{R}^4=\mathbb{C}^2$, the holomorphic coordinates are $y=x^4-ix^3$, $z=-x^2-ix^1$. The Lie algebra of the gauge group is complexified ($su(n)^c = sl(n,\mathbb{C})$) and the reality is guaranteed by conditions

$$A_{\bar y} = -A_y^{+} \; , \; A_{\bar z} = -A_z^{+} \qquad (2)$$

($^+$ stands for Hermitian adjoint of matrices). The equation (1) can be rewritten /1,2/

$$F_{yz} = F_{\bar y \bar z} = 0, \qquad (3)$$

$$F_{y\bar y} + F_{z\bar z} = 0. \qquad (4)$$

The relation (3) means that the field strength is of type (1,1), i.e. only mixed components $F_{a\bar b}$ are nonzero. The equations (3),(4) can be further rewritten adding an arbitrary complex parameter λ /2/

$$[\nabla_z - \lambda\nabla_{\bar y} \; , \; \nabla_y + \lambda\nabla_{\bar z}] = 0 \; . \qquad (5)$$

This equation can be considered as compability condition for linear equations /2/

$$(\nabla_z - \lambda\nabla_{\bar y})W = 0, \; (\nabla_y + \lambda\nabla_{\bar z})\,W = 0. \qquad (6)$$

We remind a theorem playing a crucial role in the first part of the ADHM construction /8/.

Theorem. Let $(E,\pi,X,\langle\;,\;\rangle)$ be a Hermitian vector bundle over a complex manifold X, A a Hermitian gauge field on E such that the field strength is of type (1,1). Then there exists a holomorphic structure on (E,π,X) such that - if expressed in a holomorphic basis (s_i) -

$$A_{\bar k} = 0, \; A_k = h^{-1}\partial_k h \; ,$$

where $h=(h_{ij})=(\langle s_i,s_j\rangle)$ is the Gram's matrix.

If we want to work in a holomorphic basis (in the case of $X=\mathbb{C}^2$) we must find a gauge transformation $\bar D: \mathbb{C}^2 \to SL(n,\mathbb{C})$ for which

$$A_{\bar y} = \bar D^{-1}\partial_{\bar y}\bar D \; , \; A_{\bar z} = \bar D^{-1}\partial_{\bar z}\bar D \; .$$

If we put $D = (\bar D^+)^{-1}$, it holds $A_y = D^{-1}\partial_y D$, $A_z = D^{-1}\partial_z D$.

Having performed the gauge transformation we obtain

$$A_{\bar y} = A_{\bar z} = 0, \; A_y = J^{-1}\partial_y J \; , \; A_z = J^{-1}\partial_z J \; , \text{ where } J = DD^+$$

is the Gram's matrix. The equations (3) are guaranteed by the holomorphic structure and only the equation (4) in the form

$$\partial_{\bar y}(J^{-1}\partial_y J) + \partial_{\bar z}(J^{-1}\partial_z J) = 0 \qquad (7)$$

remains. In accordance with Ref./4/ we shall call (7) the self-dual J equation.

3. The ADHM construction

We shall consider the parameter λ in (5) as another independent complex variable and extend the gauge field trivially: $A_\lambda = A_{\bar\lambda} = 0$. The complex structure can be found on $\mathbb{C}^3$ (y,z,λ will not be holomorphic coordinates) in such a way that the field strength will be of type (1,1) and all information about the self-dual field will be coded in the holomorphic structure of the vector bundle. More precisely, we shall make use of the fibration $\pi: P_3(\mathbb{C}) \to S^4$ (in fact this is Penrose transform, the sphere S^4 being the one-point conformal compactification of $\mathbb{R}^4$) and we shall deal with the pull-back of the vector bundle $\pi^* E$. This procedure was initiated by R. Ward /7/ and fully developed by Atiyah, Hitchin and Singer /8/.

The sphere S^4 can be identified with the quaternionic projective space $P_1(\mathbb{H})$, $\mathbb{H} = \mathbb{R}+i\mathbb{R}+j\mathbb{R}+k\mathbb{R}$. $\mathbb{H}$ can be considered as a 2-dimensional complex space (multiplication by scalar is from the right) : $\mathbb{H}=\mathbb{C}+j\mathbb{C}$: $\mathbb{C}=\mathbb{R}+i\mathbb{R}\subset\mathbb{H}$. Analogically $\mathbb{H}^m \cong \mathbb{C}^{2m}$. The coordinates y,z are combined in this way to the quaternion $y+jz$. The stereographic projection $S^4 \to \mathbb{R}^4$ reads

$$\mathrm{span}\left\{\begin{pmatrix} x \\ 1 \end{pmatrix}\right\} \to x \; .$$

The multiplication by j from the right induces a real structure on $\mathbb{C}^{2m}=\mathbb{H}^m$. The relation $S^4 = P_1(\mathbb{H})$ can be reformulated: the points of S^4 are in one-to-one correspondence with real projective lines in $P_3(\mathbb{C})$. We denote (Z_k) homogeneous coordinates on $P_3(\mathbb{C})$.

$$\xi = Z_1/\,Z_4 \; , \; \eta = Z_2/\,Z_4 \; , \; \zeta = Z_3/\,Z_4$$

are local holomorphic coordinates on $P_3(\mathbb{C})$. The transformation of coordinates reads

$$y = (\xi\bar{\zeta} + \bar{\eta})/(1 + \zeta\bar{\zeta}) \qquad \xi = \lambda y - \bar{z}$$
$$z = (\eta\bar{\zeta} - \bar{\xi})/(1 + \zeta\bar{\zeta}) \qquad \eta = \lambda z + \bar{y}$$
$$\lambda = \zeta \qquad \zeta = \lambda$$

We add to (5) two trivially fulfiled equations

$$[\nabla_{\bar{\lambda}}, \nabla_z - \lambda\nabla_{\bar{y}}] = [\nabla_{\bar{\lambda}}, \nabla_y + \lambda\nabla_{\bar{z}}] = 0$$

and rewrite them all in the holomorphic coordinates

$$[\nabla_a, \nabla_b] = [\nabla_{\bar{a}}, \nabla_{\bar{b}}] = 0 \quad , \quad a,b = \xi,\eta,\zeta . \tag{8}$$

This means that the field strength is of type (1,1).

Drinfeld and Manin converted the problem to the algebraic one /9,10/. We remind the final solution for m-instantons with the group Sp(n) /12/.

Let B,C be $(n+m)\times m$ quaternionic matrices for which $R^+R \in GL(m,\mathbb{R})$, where $R=R(x)= Bx_1 + Cx_2$, $x=(x_1,x_2)\in \mathbb{H}^2\setminus\{0\}$.

The fiber over span(x) $\in S^4$ is ker $R^+(x)$, the self-dual field is obtained from the trivial flat one by means of orthogonal projection /11/. The gauge transformations correspond to the equivalence

$B\sim B' = SBT$, $C\sim C' = SCT$, $S\in Sp(n+m)$, $T\in GL(m,\mathbb{R})$.

The construction can be reformulated /11/. We can suppose $m \geq n$ (in the case $m < n$ the solution with Sp(n) can be reduced to Sp(m)). The $2m\times 2m$ quaternionic matrix

$$Q = \begin{bmatrix} B^+B & B^+C \\ C^+B & C^+C \end{bmatrix}$$

can be looked upon as a $4m\times 4m$ complex matrix consisting of 2×2 blocks. Any quaternion can be considered as a linear operator in $\mathbb{C}^2 \cong \mathbb{H}$ and hence some 2×2 matrix (in the standard basis) corresponds to it (from this also follows $Sp(1) \cong SU(2)$). Fixing a basis in $\mathbb{C}^4\otimes\mathbb{C}^m$ we identify this space with $\mathbb{C}^{4m}$. Let

$(e_1,\tau(e_1),e_2,\tau(e_2))$ be the standard basis in $\mathbb{C}^4 \cong \mathbb{H}^2$ (τ is the real structure), $(e_1',\dots,e_m')$ the standard basis in $\mathbb{C}^m$, then we choose the following basis in $\mathbb{C}^4\otimes\mathbb{C}^m$:

$$\left((e_1\otimes e_j', \tau(e_1)\otimes e_j')_{j=1,\dots,m} \ , \ (e_2\otimes e_j', \tau(e_2)\otimes e_j')_{j=1,\dots,m} \right)$$

(in this order). We denote by $\langle\ ,\ \rangle$ the Hermitian metrics in $\mathbb{C}^{4m}$ (antilinear in the first argument) for which the Gram's matrix in the fixed basis is just Q. It holds

$\langle\ ,\ \rangle \geq 0$, the dimension of the null-space K is $2(n-m)$,

$\langle\ ,\ \rangle > 0$ on $I_Z = \mathrm{span}(Z\otimes e_1',\dots,Z\otimes e_m')$ for all $Z\in P_3(\mathbb{C})$,

$\langle\tau(Z)\otimes f_1, Z\otimes f_2\rangle = 0$ for all $Z\in\mathbb{C}^4$, $f_1,f_2\in\mathbb{C}^m$.

We denote by H_Z the ogthogonal complement of $I_Z\oplus I_{\tau(Z)}$ for $Z\in P_3(\mathbb{C})$. The fibre over $\pi(Z)\in S^4$ is H_Z/K, the self-dual field is again obtained from the trivial flat one by means of the orthogonal projection.

We are now going to describe the holomorphic structure on the vector bundle π^*E. We introduce a bilinear form on $\mathbb{C}^{4m}$: $(X,Y) = \langle\tau(X),Y\rangle$. If $\tilde{H}_Z$ denotes the complement of $I_Z\subset\mathbb{C}^{4m}$ with respect to this bilinear form and $\tilde{K}_Z = I_Z\oplus K$, then there exists a natural isomorphism $H_Z/K \to \tilde{H}_Z/\tilde{K}_Z$. Explicitly, denoting by P_Z the projection operator on H_Z , the following mappings

$$\begin{aligned} H_Z/K &\to \tilde{H}_Z/\tilde{K}_Z : [V] \to [V] \\ \tilde{H}_Z/\tilde{K}_Z &\to H_Z/K : [V] \to [P_Z V] \end{aligned} \tag{9}$$

are mutually inverse isomorphisms. There exists a natural holomorphic structure on the vector bundle $Z \to \tilde{H}_Z/\tilde{K}_Z$ which is by means of this isomorphism transferred to π^*E.

4. The self-dual J equation

In this point we shall show that the equations (6) can be looked upon as a search for a gauge transformation to a holomorphic basis, the existence of which (at least locally) is guaranteed by the theorem mentioned in part 2. So we are going to solve the following problem: on $\mathbb{C}^3$ (with holomorphic coordinates ξ,η,ζ) the gauge field is given by

$$A_{\bar{y}} = A_{\bar{z}} = A_{\bar{\lambda}} = A_\lambda = 0,\ A_y = J^{-1}\partial_y J,\ A_z = J^{-1}\partial_z J$$

and we have to find (at least locally) a gauge transformation to a holomorphic basis. That means to solve the system of equations

$$A_{\bar{a}} = G^{-1}\partial_{\bar{a}} G \,, \qquad \bar{a} = \bar{\xi}, \bar{\eta}, \bar{\zeta} \,.$$

Putting $G^{-1} = J^{-1}W$ we obtain from this

$$\begin{aligned} \partial_z W &= \lambda(\partial_{\bar{y}} + J\partial_{\bar{y}} J^{-1})\, W \\ -\partial_y W &= \lambda(\partial_{\bar{z}} + J\partial_{\bar{z}} J^{-1})\, W \\ \partial_{\bar{\lambda}} W &= 0 \,. \end{aligned} \qquad (10)$$

Two of these equations are only another form of (6), the last one means that W is holomorphic in λ. We can suppose $W_0 = W(\lambda=0) = I$ (if it is not the case we can put $W' = W_0^{-1}W$, $J'^{-1} = W_0^{+}J^{-1}W_0$ because of $\partial_y W_0 = \partial_z W_0 = 0$). Now we perform this gauge transformation and obtain the Gram's matrix with respect to this holomorphic basis:

$$h = W^{+}J^{-1}W \qquad \Rightarrow \quad J^{-1} = h(\lambda=0) \,. \qquad (11)$$

The relation (11) represents the main result and a solution of the above stated problem can be derived from it. We summarize:

If we express the Gram's matrix in a holomorphic basis and put $\lambda = \zeta = 0$ we obtain a solution of the self-dual J equation (7). Moreover it is possible to extract from (11) the matrix function W as well (and that vice versa enables to write down the G).

The last statement can be seen as follows. The complex variable λ represents two real variables and the matrix function h is real analytic in these variables (W is holomorphic, J does not depend on λ). So it can be analyticaly continued to two complex variables ($\mathbb{R}^2 \subset \mathbb{C}^2$) in some neighbourhood of the origin. That means we can deal with $\lambda, \bar{\lambda}$ as with two independent complex variables and put $\bar{\lambda}=0$ ($\Rightarrow W^{+} = I$).

Let us illustrate this idea on the most simple example - 1-instanton solution with the gauge group Sp(1):

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

a holomorphic basis $P_Z(V_1)$, $P_Z(V_2)$ can be chosen ($\mathbb{C}^4 \cong \mathbb{H}^2$)

$$V_1 = \begin{pmatrix} 1 \\ -\eta \end{pmatrix}, \quad V_2 = \begin{pmatrix} j \\ \xi \end{pmatrix}.$$

$$J^{-1} = \frac{1}{1+x^2}\begin{pmatrix} 1+(2+x^2)y\bar{y} & (2+x^2)y\bar{z} \\ (2+x^2)\bar{y}z & 1+(2+x^2)z\bar{z} \end{pmatrix}, \quad x^2 = y\bar{y} + z\bar{z} \,.$$

5. Conformal invariance

The conformal group on $S^4 = P_1(\mathbb{H})$ can be identified with the matrix group $GL(2,\mathbb{H})$ factorised by its centre $\{rI;\ r \in \mathbb{R} \setminus \{0\}\}$. The self-dual equation (1) is conformally invariant, the conformal group acts on the space of m-instanton solutions. This action is easily seen in the framework of the ADHM construction. Supposing a pair of matrices B,C determines a self-dual solution,

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{H})$, then a new solution is determined by

$$B' = Ba + Cc, \quad C' = Bb + Cd \,.$$

The conformal invariance of the J equation (7) is not so transparent. The reason is that generally conformal transformations don't respect the fixed complex structure on $\mathbb{R}^4 = \mathbb{C}^2$ (holomorphic coordinates are y,z). So the equation is directly invariant only with respect to conformal transformations which are holomorphic, i.e. with respect to the subgroup of matrices

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad d \in \mathbb{C}.$$

If it is the case we obtain a new solution $\tilde{J}(x) = J(g.x)$. In the case of a general transformation $g \in GL(2,\mathbb{H})$ the new solution will be of the form $\tilde{J}^{-1}_{(x)} = (F^{+}J^{-1}F)(g.x)$, where the matrix F must be found.

The explicit form of F is obtained utilizing the covering action of the conformal group on $P_3(\mathbb{C})$. Entries of 2x2 quaternionic matrices can be again considered as 2x2 complex blocks, i.e., $GL(2,\mathbb{H}) \subset GL(4,\mathbb{C})$ and so $GL(2,\mathbb{H})$ acts on $\mathbb{C}^4$, the conformal group on $P_3(\mathbb{C})$ and the transformations are holomorphic. We obtain a new solution from (11) putting $\zeta = Z_3 = 0$ in $\tilde{h}(Z) = h(g.Z)$.

Hence for $g = \mathrm{diag}(1, \exp(-\omega j))$, $\omega \in \mathbb{C}$,

$$\tilde{J}^{-1}_{(x)} = (W^{+}J^{-1}W)(g.x),$$

where W is the solution of (10) for $\lambda = \omega \,\mathrm{tg}\,|\omega| / |\omega|$.

The elements of the sugroup of matrices

$$g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad c \in \mathbb{H},$$

act on $\mathbb{R}^4$ only locally and that's why we restrit ourselves to the infinitesimal case $c = \varepsilon(u + jv)$, $u,v \in \mathbb{C}$, $\varepsilon \in \mathbb{R}$, $\varepsilon \to 0$. The result is

$$\tilde{J}^{-1}(x) = (1 + \varepsilon T^{+}) J^{-1}(g.x)(1 + \varepsilon T), \ \varepsilon \to 0,$$
$$T = (vy + \bar{u}z) W_1(x), \quad W_1 = \partial_\lambda W \big|_{\lambda=0} .$$

It is worth to note that equations (10) guarantee in some sense the conformal invariance of the J equation (7).

6. 2-instanton solutions with Sp(1)

We shall make use of the conformal invariance and shall seek only the explicit form of a 1-parametric set of solutions intersecting every orbit of the conformal group action at least once. After a tedious calculations we find the following proposition the proof of which we omit.

Every orbit of the conformal group action on the manifold of 2-instanton solutions (modulo gauge transformations) contains exactly one point determined by matrices of the form

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ r & r \end{pmatrix}, \quad r \in (0,1\rangle .$$

From this fact and utilizing gauge equivalence in the ADHM construction, we find that the 1-parametric set of Hermitian matrices can be chosen as

$$Q = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & r \end{pmatrix}, \quad r > 1.$$

The local holomorphic basis $(|\eta| < 1, \ \mathbb{C}^8 \cong \mathbb{H}^4)$:

$$V_1 = \sqrt{\frac{r}{(r-1)(1-\eta^2)}} \begin{pmatrix} \zeta + j \\ 0 \\ \zeta + j\eta \\ 0 \end{pmatrix}, \quad V_2 = \sqrt{\frac{1}{r(r-1)(1-\eta^2)}} \begin{pmatrix} -r + \eta^2 \\ r\eta \\ -\eta \\ 1-\eta^2 \end{pmatrix}$$

In the perametrisation /3/

$$J^{-1} = \frac{1}{\phi} \begin{pmatrix} 1 & \bar{\rho} \\ \rho & \phi^2 + |\rho|^2 \end{pmatrix}, \quad \phi \text{ real}, \ \rho \text{ complex},$$

$$\phi = \frac{|1-y^2|}{r} \left(1 + (r-1) \frac{1+|y|^2 + |z|^2}{D} \right)$$

$$\rho = -\frac{r-1}{rD} \left[(1+|y|^2+|z|^2) y + y + \bar{y} \right] \bar{z},$$

where $D = (1+|y|^2+|z|^2)^2 - (y+\bar{y})^2$.

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Штовичек П.
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Связь между автодуальными решениями в J формулировке и АДХМ конструкцией

Цель работы - установить соответствие между автодуальным уравнением в формулировке, принадлежащий Янгу, Белавину и Захарову, и конструкцией Атия-Дринфельда-Хитгина-Манина. Показано, что линейные уравнения Белавина и Захарова можно интерпретировать как уравнения для калибровочной трансформации к голоморфному базису. Как сопутствующий результат можно извлечь решения J-уравнения прямо из АДХМ конструкции. Конформная инвариантность J-уравнения продемонстрирована в явном виде. В качестве примера рассмотрены 2-инстантонные решения.

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Števíček P.
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The Connection between the Self-Dual Equation in the J Formulation and the ADHM Construction

The paper attempts to point out the connection between the self-dual equation in the formulation due to Yang, Belavin and Zakharov and the Atiyah-Drinfeld-Hitchin-Manin construction. The linear equations due to Belavin and Zakharov are interpreted as equations for a gauge transformation to a holomorphic basis. Utilizing this fact we are able to obtain solutions of the J equation directly from the ADHM construction. The conformal invariance of the J equation is discussed. As an example 2-instanton solutions are constructed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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