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ON THE IRREDUCIBILITY OF GENERALIZED CALKIN REPRESENTATIONS

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1. Introduction

In his classical paper Calkin /1/ constructed a class of faithful representations of the quotient C^*-algebra $\mathcal{B}(\mathcal{H})/\mathcal{S}_\infty(\mathcal{H})$ (Calkin algebra) of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a separable Hilbert space $\mathcal{H}$ by the closed two-sided $*$-ideal $\mathcal{S}_\infty(\mathcal{H})$ of all compact operators. Only almost 30 years later Reid /12/ investigated the problem of irreducibility of those representations of the Calkin algebra.

Recently it was pointed out that there is a very natural generalization of the Calkin algebra to algebras of unbounded operators and faithful representations were described for certain classes of those algebras (/5/,/10/,/13/).

The aim of this paper is to settle the question which representations of the generalized Calkin algebra are irreducible. It turns out that this can be done along the line of Reid's paper. Section 2 contains the necessary definitions, notations and auxiliary results (cf.also /7/,/9/,/10/), while the main results are given in section 3.

2. Preliminaries

For a dense linear manifold $\mathcal{D}$ in a separable Hilbert space $\mathcal{H}$ let $\mathcal{L}^+(\mathcal{D})$ denote the $*$-algebra of all operators A with $A\mathcal{D} \subset \mathcal{D}$ and $A^*\mathcal{D} \subset \mathcal{D}$. The involution is given by $A \longrightarrow A^+ = A^* \restriction \mathcal{D}$. An Op^*-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ with unit I (identity operator). The graph topology $t_{\mathcal{A}}$ on $\mathcal{D}$ is given by the seminorms $\varphi \longrightarrow \|A\varphi\|$ for all $A \in \mathcal{A}(\mathcal{D})$. We denote $t_{\mathcal{L}^+(\mathcal{D})}$ simply by t. This gives rise to a canonical rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}'[t']$ and a canonical dual pair $(\mathcal{D}, \mathcal{D}')$ where t' denotes the strong topology in $\mathcal{D}[t]'$. Let $\sigma = \sigma(\mathcal{D}, \mathcal{D}')$ be the weak topology in $\mathcal{D}$. Remember that a sequence $(\varphi_n) \subset \mathcal{D}$ is σ-convergent to zero ($\varphi_n \xrightarrow{\sigma} 0$) if and only if (φ_n) is t-bounded and $\langle\varphi, \varphi_n\rangle \longrightarrow 0$ for all $\varphi \in \mathcal{D}$, hence for all $\varphi \in \mathcal{H}$ /10/. The uniform topology $\tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$ is given by the family of seminorms

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\varphi, \psi \in \mathcal{M}} |\langle\varphi, A\psi\rangle| ,$$

where $\mathfrak{M}$ runs over all t-bounded subsets of $\mathfrak{D}$. $\mathcal{L}^+(\mathfrak{D})[\tau_{\mathfrak{D}}]$ is a topological $*$-algebra. Moreover, $\tau_{\mathfrak{D}}$ is not only defined on $\mathcal{L}^+(\mathfrak{D})$ but also on $\mathcal{L}(\mathfrak{D},\mathfrak{D}')$ and hence on $\mathcal{B}(\mathcal{H})$, too /9/. An Op*-algebra $\mathcal{A}(\mathfrak{D})$ is called selfadjoint if $\mathfrak{D} = \mathfrak{D}_* \equiv \bigcap_{A \in \mathcal{A}} \mathfrak{D}(A^*)$.

The most important domains $\mathfrak{D}$ are those where t is a Frechét-topology, i.e., $\mathfrak{D}[t]$ is complete and t can be given by $\{\|A_n \cdot\|, n \in \mathbb{N}\}$ with $A_n = A_n^+$, $I = A_0 \leq A_1 \leq A_2 \leq \ldots$. In what follows we allways suppose that $\mathcal{L}^+(\mathfrak{D})$ is selfadjoint and $\mathfrak{D}[t]$ is an (F)-space.
Let $\mathcal{B}(\mathfrak{D}) = \{T: T\mathcal{H} \subset \mathfrak{D},\ T^*\mathcal{H} \subset \mathfrak{D}\} = \{T: AT,\ AT^*$ bounded for all $A \in \mathcal{L}^+(\mathfrak{D})\}$ /15/, and $\mathcal{F}(\mathfrak{D}) = \{F \in \mathcal{L}^+(\mathfrak{D}): \dim F < \infty\}$. The following result was obtained by Kürsten /4/ in its final form (see also /8/,/3/).

> The family of sets $\{B\mathcal{K} : B \in \mathcal{B}(\mathfrak{D}), B \geq 0, \operatorname{Ker} B = (0)\}$ where $\mathcal{K}$ is the unit ball in $\mathcal{H}$, forms a fundamental systems of t-bounded sets in $\mathfrak{D}$. Hence the topology $\tau_{\mathfrak{D}}$ can be given by the seminorms $A \longrightarrow \|BAB\|$, B as above.

An important consequence is the fact that $\mathcal{B}(\mathfrak{D})$ is $\tau_{\mathfrak{D}}$-dense in $\mathcal{L}(\mathfrak{D},\mathfrak{D}')$, consequently also in $\mathcal{L}^+(\mathfrak{D})$ and $\mathcal{B}(\mathcal{H})$ /4/. We will here use only the following assertion which can be easily obtained using for example that $B \in \mathcal{B}(\mathfrak{D})$ implies $B^{1/2} \in \mathcal{B}(\mathfrak{D})$ /15/.

<u>Lemma 2.1</u>

Let $B \in \mathcal{B}(\mathfrak{D})$, $B \geq 0$, $B = \int_0^b \lambda\, dE_\lambda$, $P_a = \int_a^b dE_\lambda$, $a > 0$. Then

$$\lim_{a \to 0} \| A(I-P_a)B \| = 0 \text{ for all } A \in \mathcal{L}^+(\mathfrak{D}).$$

In /14/ several subsets of $\mathcal{L}^+(\mathfrak{D})$ (e.g., Vol(t,t), Vol(t,$\|\ \|$), Com(t,$\|\ \|$)) were introduced to generalize the notion of compact or completely continuous operators in the context of $\mathcal{L}^+(\mathfrak{D})$. We collect some results from /4/,/10/.

<u>Lemma 2.2</u>

i) $\mathcal{C}(\mathfrak{D}) \equiv \overline{\mathcal{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}} = \mathrm{Vol}(t,t) = \mathrm{Vol}(t,\|\ \|)$ is a two-sided $*$-ideal.

ii) $A \in \mathcal{C}(\mathfrak{D})$ if and only if $\psi_n \xrightarrow{\sigma} 0$ implies $A\psi_n \xrightarrow{\|\ \|} 0$.

iii) If $\mathfrak{D}[t]$ is a Montel space, then $\mathcal{L}^+(\mathfrak{D}) = \mathcal{C}(\mathfrak{D})$. If $\mathfrak{D}[t]$ is not a Montel space then $\mathcal{C}(\mathfrak{D})$ is the only non-trivial $\tau_{\mathfrak{D}}$-closed two-sided $*$-ideal in $\mathcal{L}^+(\mathfrak{D})$.

These properties of $\mathcal{C}(\mathfrak{D})$ lead to the following definition /16/.

<u>Definition 2.3</u>

A $\tau_{\mathfrak{D}}$-continuous linear functional ω on $\mathcal{L}^+(\mathfrak{D})$ is said to be singular if $\omega(A) = 0$ for all $A \in \mathcal{C}(\mathfrak{D})$.

Remember that a singular state ω on l^∞ is a positive (hence continuous), normed linear functional on the C*-algebra l^∞ with $\omega((x_n)) = 0$ for all sequences $(x_n) \in c_0$, the space of null sequences.
For a t-bounded sequence $(\psi_n) \subset \mathfrak{D}$ and a state ω on l^∞ let $\psi_n \xrightarrow{\sigma_\omega} 0$ mean $\omega((\langle \psi_n, \chi \rangle)) = 0$ for all $\chi \in \mathfrak{D}$ (hence for all $\chi \in \mathcal{H}$). Consider the following subset of $\mathcal{L}^+(\mathfrak{D})$:

$$\mathcal{C}_\omega(\mathfrak{D}) = \{A \in \mathcal{L}^+(\mathfrak{D}) : \psi_n \xrightarrow{\sigma_\omega} 0 \text{ implies } \omega((\|A\psi_n\|)) = 0\}$$

<u>Lemma 2.4</u>

For an arbitrary singular state ω on l^∞ the equality $\mathcal{C}_\omega(\mathfrak{D}) = \mathcal{C}(\mathfrak{D})$ holds.

<u>Proof:</u>

Clearly, $\mathcal{C}_\omega(\mathfrak{D})$ is a right ideal. The estimation $\|A^+\psi_n\|^2 = \langle AA^+\psi_n, \psi_n \rangle \leq \|AA^+\psi_n\| \cdot \|\psi_n\| \leq C\|AA^+\psi_n\|$ with $C = \sup\|\psi_n\|$ implies that $\mathcal{C}_\omega(\mathfrak{D})$ is a $*$-ideal, hence a left ideal. Next we show that $\mathcal{C}_\omega(\mathfrak{D})$ is $\tau_{\mathfrak{D}}$-closed. Let A be in the $\tau_{\mathfrak{D}}$-closure of $\mathcal{C}_\omega(\mathfrak{D})$, $\psi_n \xrightarrow{\sigma_\omega} 0$, then for all $B \in \mathcal{L}^+(\mathfrak{D})$: $\|A\psi_n\|^2 = \langle A\psi_n, A\psi_n \rangle = \langle (A-B)\psi_n, A\psi_n \rangle + \langle B\psi_n, A\psi_n \rangle \leq \sup_{\varphi,\psi \in \mathcal{N}} |\langle (A-B)\varphi, \psi \rangle| + C\|B\psi_n\|$, where $\mathcal{N}$ is the t-bounded set $\{A\psi_n, \psi_m\}$, $C = \sup\|A\psi_n\|$. For arbitrary $\varepsilon > 0$ let $B \in \mathcal{C}_\omega(\mathfrak{D})$ so that $\|A-B\|_{\mathcal{N}} < \varepsilon$, then $\|A\psi_n\|^2 \leq \varepsilon + C\|B\psi_n\|$ and consequently $\omega((\|A\psi_n\|^2)) \leq \varepsilon$, i.e., $\omega((\|A\psi_n\|^2)) = 0$ and so $\omega((\|A\psi_n\|)) = 0$.
Because $\mathcal{C}_\omega(\mathfrak{D}) \neq (0)$ and $I \notin \mathcal{C}_\omega(\mathfrak{D})$, if $\mathfrak{D}[t]$ is not a Montel space, the assertion follows from Lemma 2.2iii).

q.e.d.

An important class of singular states on l^∞ is generated by free ultrafilters $\mathcal{U}$ via $\omega_{\mathcal{U}}((x_n)) = \lim_{\mathcal{U}} x_n$. Remember that an ultrafilter $\mathcal{U}$ is said to be free if $\bigcap_{U \in \mathcal{U}} U = \emptyset$. For the $\sigma_{\omega_{\mathcal{U}}}$-convergence we use the symbol $\xrightarrow{\sigma_{\mathcal{U}}}$. Because in section 3 we need some special classes of ultrafilters and maps from $\mathbb{N}$ to $\mathbb{N}$, let us repeat the definitions and related facts. For more details see /12/ and the references there.
A set $M \subset \mathbb{N}$ (the natural numbers) is said to be contained almost in $K \subset \mathbb{N}$ if $M \setminus K$ is finite. An ultrafilter $\mathcal{U}$ with the property that given any countable collection of sets in $\mathcal{U}$, there exists a set of $\mathcal{U}$ almost contained in every set of the collection, is called δ-stable. It suffices to consider just decreasing sequences of sets in $\mathcal{U}$, and for free ultrafilters they can also supposed to have void intersection.

An ultrafilter $\mathcal{U}$ is called rare if given any partition of $\mathbb{N}$ into disjoint finite sets $N_1, N_2, \ldots$ there exists a set of $\mathcal{U}$ meeting each of the N_i's in at most one point. It is sufficient to consider partitions into finite intervals.
Finally, an ultrafilter $\mathcal{U}$ which is both σ-stable and rare is said to be absolute.
A map $f: \mathbb{N} \longrightarrow \mathbb{N}$ with the property that there is a set of $\mathcal{U}$ on which f is injective is called a $\mathcal{U}$-equivalence. For given $f: \mathbb{N} \longrightarrow \mathbb{N}$ and $\mathcal{U}$ we can define the image $f(\mathcal{U})$ of $\mathcal{U}$ under f to be the ultrafilter generated by $\{f(U): U \in \mathcal{U}\}$. $f(\mathcal{U})$ consists of all sets V such that $f^{-1}(V) \in \mathcal{U}$. Two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are called equivalent, if there exists a bijection $g: \mathbb{N} \longrightarrow \mathbb{N}$ such that $\mathcal{V} = g(\mathcal{U})$. Moreover, if $f: \mathbb{N} \longrightarrow \mathbb{N}$ has the property that restricted to some set of $\mathcal{U}$, $f^{-1}(n)$ is finite for all $n \in \mathbb{N}$, then f is called $\mathcal{U}$-finite.
Some connections between the classes of ultrafilters and maps mentioned above are collected in the next lemma.

<u>Lemma 2.5</u>

i) Let $f: \mathbb{N} \longrightarrow \mathbb{N}$, then $\mathcal{U}$ and $f(\mathcal{U})$ are equivalent if and only if f is a $\mathcal{U}$-equivalence.

ii) $\mathcal{U}$ is absolute if and only if for each map $f: \mathbb{N} \longrightarrow \mathbb{N}$ either $f(\mathcal{U})$ is a fixed ultrafilter or $f(\mathcal{U})$ is equivalent to $\mathcal{U}$.

iii) $\mathcal{U}$ is rare if and only if any $\mathcal{U}$-finite map is a $\mathcal{U}$-equivalence.

iv) If $\mathcal{U}$ is a free ultrafilter and f is $\mathcal{U}$-finite, then $f(\mathcal{U})$ is not a fixed ultrafilter.

3. The main results

In this section we consider two classes of Calkin-representations of $\mathcal{L}^+(\mathfrak{D})$ (and $\mathcal{L}^+(\mathfrak{D})/\mathcal{C}(\mathfrak{D})$), /5/,/10/,/13/. Our aim is to investigate under which conditions they coincide and to give criteria for the irreducibility of these representations.
Let us repeat shortly the construction of these representations. In what follows the ultrafilters are supposed to be free if not otherwise specified. For fixed ultrafilters the considerations are trivial.

i) Let $\hat{\hat{\mathfrak{D}}} = \{(\psi_n) = \hat{f}: \psi_n \in \mathfrak{D}, \psi_n \xrightarrow{\sigma} 0\}$. For a given ultrafilter $\mathcal{U}$ on $\mathbb{N}$ consider the sesquilinear forms

$$(\hat{f}, \hat{g})_k = \lim_{\mathcal{U}} \langle A_k \psi_n, A_k \varphi_n \rangle, \quad \hat{f} = (\psi_n), \ \hat{g} = (\varphi_n), \ k=0,1,2,\ldots$$

Let $\sim_{\mathcal{U}}$ be the following equivalence relation on $\hat{\hat{\mathfrak{D}}}$:

$$\hat{f} \sim_{\mathcal{U}} \hat{g} \quad \text{if and only if} \quad \lim_{\mathcal{U}} \|\psi_n - \varphi_n\|^2 = 0 .$$

Then $\hat{\mathfrak{D}}_{\mathcal{U}} \equiv \hat{\hat{\mathfrak{D}}} | \sim_{\mathcal{U}}$ is endowed with the scalar products $(\ ,\ )_k$ (cf./10/). For $(\ ,\ )_0^{1/2}$ we use the symbol $|||\cdot|||$.

ii) Let $\check{\check{\mathfrak{D}}}_{\mathcal{U}} = \{(\psi_n) = \check{f}: \psi_n \in \mathfrak{D}, \psi_n \xrightarrow{\sigma_{\mathcal{U}}} 0\}$. On $\check{\check{\mathfrak{D}}}_{\mathcal{U}}$ the sesquilinear forms $(\ ,\ )_k$ and the relation $\underset{\mathcal{U}}{\sim}$ are also defined. Moreover $\hat{\hat{\mathfrak{D}}} \subset \check{\check{\mathfrak{D}}}_{\mathcal{U}}$, hence $\hat{\hat{\mathfrak{D}}} | \underset{\mathcal{U}}{\sim} \equiv \hat{\mathfrak{D}}_{\mathcal{U}} \subset \check{\mathfrak{D}}_{\mathcal{U}} \equiv \check{\check{\mathfrak{D}}}_{\mathcal{U}} | \underset{\mathcal{U}}{\sim}$.

<u>Remark 3.1</u>

i) If $\mathfrak{D}[t]$ is a Montel space then $\check{\mathfrak{D}}_{\mathcal{U}} = (0)$. This follows from Lemma 2.4 since in this case $\mathcal{C}_{\omega_{\mathcal{U}}}(\mathfrak{D}) = \mathcal{L}^+(\mathfrak{D})$.

ii) If $\mathfrak{D} = \mathcal{H}$, then $\hat{\mathfrak{D}}_{\mathcal{U}} = K^0_{\mathcal{U}}$, $\check{\mathfrak{D}}_{\mathcal{U}} = K_{\mathcal{U}}$ in the notations of Reid /12/.

Let us denote the elements of $\hat{\mathfrak{D}}_{\mathcal{U}}, \check{\mathfrak{D}}_{\mathcal{U}}$ by $[\hat{f}]_{\mathcal{U}}$, $[\check{g}]_{\mathcal{U}}$ or simply by $[\hat{f}]$, $[\check{g}]$ or $(\hat{\psi}_n)$, $(\check{\varphi}_n)$ if the used ultrafilter is clear and $\hat{f} = (\psi_n)$, $\check{g} = (\varphi_n)$. For $B \in \mathcal{L}^+(\mathfrak{D})$ put

$$\pi_1(B)\,[\check{f}]_{\mathcal{U}} = [\check{g}]_{\mathcal{U}} \quad \text{with } \check{f} = (\psi_n), \ \check{g} = (B\psi_n).$$

Then the following is true /13/:

<u>Theorem 3.2</u>

i) π_1 is a $*$-representation of $\mathcal{L}^+(\mathfrak{D})$.

ii) $\ker \pi_1 = \mathcal{C}(\mathfrak{D})$.

iii) $\pi_1^{-1}: \pi_1(\mathcal{L}^+(\mathfrak{D}))[\tau_{\check{\mathfrak{D}}_{\mathcal{U}}}] \xrightarrow{\text{onto}} \mathcal{L}^+(\mathfrak{D})/\mathcal{C}(\mathfrak{D})[\hat{\tau}]$

is continuous, where $\hat{\tau}$ is the factor topology induced by $\tau_{\mathfrak{D}}$.

Let t_{π_1} be the graph topology on $\check{\mathfrak{D}}_{\mathcal{U}}$ induced by $\pi_1(\mathcal{L}^+(\mathfrak{D}))$. According to /6/ and /13/ $\check{\mathfrak{D}}_{\mathcal{U}}[t_{\pi_1}]$ is an (F)-space. Furthermore it is easy to see that $\hat{\mathfrak{D}}_{\mathcal{U}}$ is a π_1-invariant subspace of $\check{\mathfrak{D}}_{\mathcal{U}}$. Let π denote the restriction of π_1 to $\hat{\mathfrak{D}}_{\mathcal{U}}$ (more exactly: $\pi(A) = \pi_1(A) | \hat{\mathfrak{D}}_{\mathcal{U}}$ for all $A \in \mathcal{L}^+(\mathfrak{D})$). Then π defines also a $*$-representation of $\mathcal{L}^+(\mathfrak{D})$, $\hat{\mathfrak{D}}_{\mathcal{U}}[t_\pi]$ is an (F)-space and $t_\pi = t_{\pi_1} | \hat{\mathfrak{D}}_{\mathcal{U}}$.
Next we are going to give a necessary and sufficient condition for $\hat{\mathfrak{D}}_{\mathcal{U}} = \check{\mathfrak{D}}_{\mathcal{U}}$. For this we need some notations. Let $(\rho_n) \subset \mathfrak{D}$ be an orthonormal basis of $\mathcal{H}$ and denote by P_k the projections on the linear span of $(\rho_1, \ldots, \rho_k)$ for all k. For a given t-bounded sequence $(\varphi_n) \subset \mathfrak{D}$ put

$$S_{k,p}(\varphi_n) = \{n \in \mathbb{N} : \|P_k \varphi_n\| < 1/p\}.$$

Then the following Lemma is an easy consequence of the definition of the $\sigma_{\mathcal{U}}$- and σ-convergence.

Lemma 3.3

i) $\varphi_n \xrightarrow{\sigma_{\mathcal{U}}} 0$ if and only if $S_{k,p}(\varphi_n) \in \mathcal{U}$ for all k,p.

ii) $\varphi_n \xrightarrow{\sigma} 0$ if and only if for all $k \in \mathbb{N}$, $\varepsilon > 0$ there is an n_0 so that $\| P_k \varphi_n \| < \varepsilon$ for all $n \geq n_0$.

Lemma 3.4

A t-bounded sequence $(\varphi_n) \subset \mathcal{D}$ can be represented as $(\varphi_n) = (\psi_n) + (\chi_n)$ with (ψ_n), (χ_n) - t-bounded, $\psi_n \xrightarrow{\sigma} 0$ and $\lim_{\mathcal{U}} \|\chi_n\| = 0$, i.e., $(\hat{\varphi}_n) \in \hat{\mathcal{D}}_{\mathcal{U}}$ if and only if for each $p \in \mathbb{N}$ there exists a set $U_p \in \mathcal{U}$ such that U_p is almost contained in $S_{k,p}(\varphi_n)$ for every k.

Proof:

The proof of Reid's Proposition 1 /12/ can be repeated word-by-word if one has in mind that all sequences constructed there are t-bounded because they are formed starting with a t-bounded sequence (φ_n).

Theorem 3.5

Let $\mathcal{D}[t]$ be not a Montel space. Then $\hat{\mathcal{D}}_{\mathcal{U}} = \check{\mathcal{D}}_{\mathcal{U}}$ if and only if $\mathcal{U}$ is δ-stable.

Proof:

Let $\mathcal{U}$ be δ-stable, $\check{f} = (\psi_n) \in \check{\check{\mathcal{D}}}_{\mathcal{U}}$. Then by Lemma 3.3 $\{S_{k,p}(\psi_n), k,p \in \mathbb{N}\}$ is a countable collection of sets of $\mathcal{U}$. According to the δ-stability there is a $U \in \mathcal{U}$ almost contained in $S_{k,p}$ for all k,p. Thus, Lemma 3.4 implies $\hat{\mathcal{D}}_{\mathcal{U}} = \check{\mathcal{D}}_{\mathcal{U}}$.

The proof of the other direction is a modification of Reid's construction. So let $\hat{\mathcal{D}}_{\mathcal{U}} = \check{\mathcal{D}}_{\mathcal{U}}$, (U_n) a countable collection of sets of $\mathcal{U}$. According to section 2 we can suppose that i) $U_1 = \mathbb{N}$, ii) (U_n) is decreasing, $\bigcap_n U_n = \emptyset$. Because $\mathcal{D}$ is not a Montel space, there is an infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathcal{D}$. Let (ϱ_n) be an orthonormal basis of $\mathcal{H}$ contained in $\mathcal{D}$ such that for some subsequence (k_n) of $\mathbb{N}$ (ϱ_{k_n}) is an orthonormal basis of $\mathcal{H}_0$. P_k has the meaning as above. Put $k_0 = 0$ and define a sequence (φ_r) as follows:

$$\varphi_r = \varrho_{k_n} \quad \text{for } k_{n-1} < r \leq k_n .$$

Clearly (φ_r) is t-bounded. Next form

$$\psi_m = \varphi_r \quad \text{for all } m \in U_r \setminus U_{r+1}$$

Then $\| P_{k_n}(\varphi_r) \|$ is equal to one for $r \leq k_n$ and equal to zero for $r > k_n$, and moreover $S_{k_n,p}(\psi_m) = \{ m : \| P_{k_n}(\psi_m) \| < 1/p \} =$

$$= \{ m : m \in U_r \setminus U_{r+1},\ r > k_n \} = U_{k_n+1} .$$

The inclusion $S_{k,p} \supset S_{l,p}$ for $l > k$ implies $\psi_m \xrightarrow{\sigma_{\mathcal{U}}} 0$ because this is true if and only if $S_{k_n,p} \in \mathcal{U}$ for all $n,p \in \mathbb{N}$ and an arbitrary sequence (k_n) converging to infinity. By the assumption $\hat{\mathcal{D}}_{\mathcal{U}} = \check{\mathcal{D}}_{\mathcal{U}}$ the sequence (ψ_m) can be represented as $(\psi_m) = (\alpha_m) + (\beta_m)$; (α_m), (β_m) are t-bounded, $\alpha_m \xrightarrow{\sigma} 0$ and $\lim_{\mathcal{U}} \|\beta_m\|^2 = 0$. Lemma 3.4 implies the existence of a $U_p = U \in \mathcal{U}$ almost contained in $S_{k_n,p}(\psi_m) = U_{k_n+1}$, so U is almost contained in U_k for every k. But that means that $\mathcal{U}$ is δ-stable.

q.e.d.

Remark 3.6

From the proof it can be seen that the implication if $\mathcal{U}$ is δ-stable, then $\hat{\mathcal{D}}_{\mathcal{U}} = \check{\mathcal{D}}_{\mathcal{U}}$ holds without the assumption that $\mathcal{D}[t]$ is not a Montel space. On the other hand, if $\mathcal{D}[t]$ is a Montel space, then $\hat{\mathcal{D}}_{\mathcal{U}} = \check{\mathcal{D}}_{\mathcal{U}} = (0)$ for any ultrafilter $\mathcal{U}$.

In what follows let $\mathcal{D}[t]$ be not a Montel space. Our next aim is to decide when π_1 or π is an irreducible representation of $\mathcal{L}^+(\mathcal{D})$ (hence of $\mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$). For C^*-algebras the notion "irreducible representation" is unambiguous, because "all" possible definitions of irreducibility coincide.

In the unbounded case the situation is much more complicated and not completely clarified (for some results see /2/,/11/). We will return to this problem in a forthcoming paper and use here the following notion.

Definition 3.7

A $*$-representation π of a $*$-algebra R (i.e., π is a $*$-homomorphism from R into some $\mathcal{L}^+(\mathcal{D})$) is said to be topologically irreducible if $\{\pi(x)\varphi : x \in R\}$ is t_π-dense in $\mathcal{D}$ for any non-zero $\varphi \in \mathcal{D}$. (here t_π is the graph topology induced by $\pi(R)$).

Theorem 3.8

Let π, π_1 resp. be the representations of $\mathcal{L}^+(\mathcal{D})$ on $\hat{\mathcal{D}}_{\mathcal{U}}$, $\check{\mathcal{D}}_{\mathcal{U}}$ resp. defined in the beginning of the section. Then

i) π is irreducible if and only if $\mathcal{U}$ is rare.

ii) π_1 is irreducible if and only if $\mathcal{U}$ is absolute.

The proof is somewhat lengthy and so we divide it into several Lemmata. Let $f: \mathbb{N} \longrightarrow \mathbb{N}$ be an arbitrary map, $\mathcal{U}$ a free ultrafilter and $\mathcal{V} = f(\mathcal{U})$.

Lemma 3.9

1) $\check{\mathcal{D}}_{\mathcal{V}}$ can be isometrically embedded into $\check{\mathcal{D}}_{\mathcal{U}}$.

ii) $\check{\mathcal{D}}_{\mathcal{V}} = \check{\mathcal{D}}_{\mathcal{U}}$ if and only if f is a $\mathcal{U}$-equivalence.

Proof:

i) Let $\check{g} = (\psi_n) \in \check{\check{\mathfrak{D}}}_{\mathcal{V}}$. Then the map $F = F_f$ with $F([\check{g}]_{\mathcal{V}}) = [\check{h}]_{\mathcal{U}}$, $\check{h} = (\chi_n) = (\psi_{f(n)}) \in \check{\check{\mathfrak{D}}}_{\mathcal{U}}$ gives the embedding. The isometry is easily established.

ii) Let $\check{f} = (\varrho_n)$, where (ϱ_n) is an orthonormal basis of an infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathfrak{D}$. Then $\check{f} \in \check{\mathfrak{D}}_{\mathcal{U}}$ and the equality $\check{\mathfrak{D}}_{\mathcal{V}} = \check{\mathfrak{D}}_{\mathcal{U}}$ imply the existence of a sequence $\check{g} = (\varphi_n) \in \check{\check{\mathfrak{D}}}_{\mathcal{V}}$ such that $F([\check{g}]_{\mathcal{V}}) = [(\varphi_{f(n)})]_{\mathcal{U}} = [\check{f}]_{\mathcal{U}} = [(\varrho_n)]_{\mathcal{U}}$. This means $0 = [(\varphi_{f(n)} - \varrho_n)]_{\mathcal{U}} = \lim_{\mathcal{U}} \|\varphi_{f(n)} - \varrho_n\|$, i.e., there is a $U \in \mathcal{U}$ so that $\|\varrho_n - \varphi_{f(n)}\| < 1/2$ for all $n \in U$.
Consider all $n, m \in U$ with $f(n) = f(m)$. Then the estimation $\|\varrho_n - \varrho_m\| \le \|\varrho_n - \varphi_{f(n)}\| + \|\varrho_m - \varphi_{f(m)}\| < 1$ implies $n = m$. Thus, f is injective on $U \in \mathcal{U}$ and f is a $\mathcal{U}$-equivalence.
On the other hand, if f is a $\mathcal{U}$-equivalence then from Lemma 2.5 i) there follows the existence of a bijection $g: \mathbb{N} \longrightarrow \mathbb{N}$ so that $\mathcal{U}$ and $\mathcal{V} = g(\mathcal{U})$ are equivalent. Then $\check{\mathfrak{D}}_{\mathcal{V}} \subset \check{\mathfrak{D}}_{\mathcal{U}}$ by i). If $\check{h} = (\psi_n) \in \check{\check{\mathfrak{D}}}_{\mathcal{U}}$ then $\check{g} = (\psi_{g^{-1}(n)}) = (\chi_n) \in \check{\check{\mathfrak{D}}}_{\mathcal{V}}$ and obviously $\lim_{\mathcal{U}} \|\psi_n - \chi_{g(n)}\| = 0$, i.e., $\check{\mathfrak{D}}_{\mathcal{U}} \subset \check{\mathfrak{D}}_{\mathcal{V}}$.

q.e.d.

Remark 3.10

i) $\check{\mathfrak{D}}_{\mathcal{V}}$ is a t_{π_1}-closed, π_1-invariant subspace of $\check{\mathfrak{D}}_{\mathcal{U}}$.

ii) $\check{\mathfrak{D}}_{\mathcal{V}} = (0)$ if and only if $\mathcal{V} = f(\mathcal{U})$ is a fixed ultrafilter.

The situation for $\hat{\mathfrak{D}}_{\mathcal{U}}$ and $\hat{\mathfrak{D}}_{\mathcal{V}}$ is a little bit more complicated as the next lemma shows.

Lemma 3.11

i) $\hat{\mathfrak{D}}_{\mathcal{V}} \subset \hat{\mathfrak{D}}_{\mathcal{U}}$ if and only if f is $\mathcal{U}$-finite.

ii) Let f be $\mathcal{U}$-finite, then $\hat{\mathfrak{D}}_{\mathcal{V}} = \hat{\mathfrak{D}}_{\mathcal{U}}$ if and only if f is a $\mathcal{U}$-equivalence.

Proof:

i) Let (ϱ_n), (ϱ_{n_k}), P_k have the same meaning as in the proof of Theorem 3.5. Put $\hat{f} = (\varphi_k) = (\varrho_{n_k}) \in \hat{\hat{\mathfrak{D}}}_{\mathcal{V}}$. The embedding $F_f: \hat{\mathfrak{D}}_{\mathcal{V}} \longrightarrow \hat{\mathfrak{D}}_{\mathcal{U}}$ implies the existence of a sequence $\hat{g} = (\psi_k) \in \hat{\hat{\mathfrak{D}}}_{\mathcal{U}}$ such that $0 = [(\varphi_{f(k)} - \psi_k)]_{\mathcal{U}} = \lim_{\mathcal{U}} \|\varphi_{f(k)} - \psi_k\|$. Hence, there is a $U \in \mathcal{U}$ with $\|\varphi_{f(k)} - \psi_k\| < 1/2$ for all $k \in U$. Moreover, $\hat{g} \in \hat{\hat{\mathfrak{D}}}_{\mathcal{U}}$ means that for all $1 \in \mathbb{N}$ there is an $j_1 \in \mathbb{N}$ such that $\|P_{n_1} \psi_k\| < 1/2$ for all $k \ge j_1$.

Therefore $\|P_{n_1}(\varphi_{f(k)})\| \le \|\varphi_{f(k)} - \psi_k\| + \|P_{n_1}(\psi_k)\| < 1$ for all $k \ge j_1$ and $k \in U$. Consequently for $\varphi_{f(k)} = \varrho_{n_{f(k)}}$ we get $\|P_{n_1} \varrho_{n_{f(k)}}\| < 1$, i.e., $n_{f(k)} > n_1$ and so $f(k) > 1$ for all $k > j_1$, $k \in U$. Thus, for arbitrary $k \in \mathbb{N}$: $f^{-1}(k) \cap U \subset [1, j_1 - 1]$ and so f is $\mathcal{U}$-finite.
On the other hand, let $\hat{f} = (\psi_n) \in \hat{\hat{\mathfrak{D}}}_{\mathcal{V}}$. We show that if f is $\mathcal{U}$-finite, then there is a $\hat{g} = (\chi_n) \in \hat{\hat{\mathfrak{D}}}_{\mathcal{U}}$ with $0 = [\psi_{f(n)} - \chi_n]_{\mathcal{U}} = \lim_{\mathcal{U}} \|\psi_{f(n)} - \psi_n\|$. Because f is $\mathcal{U}$-finite, there is a $U \in \mathcal{U}$ so that $f^{-1}(n) \cap U = \{m \in U: f(m) = n\}$ is finite for all n. Put

$$\chi_n = \begin{cases} \psi_{f(n)} & \text{for } n \in U \\ 0 & \text{for } n \notin U \end{cases}$$

Clearly, $\lim_{\mathcal{U}} \|\psi_{f(n)} - \chi_n\| = 0$, so it remains to prove that $\hat{g} = (\chi_n) \in \hat{\hat{\mathfrak{D}}}_{\mathcal{U}}$.
Since $\hat{f} \in \hat{\hat{\mathfrak{D}}}_{\mathcal{V}}$, for every $k \in \mathbb{N}$, $\varepsilon > 0$ there is a $j(k, \varepsilon) > 0$ with $\|P_k \psi_n\| \le \varepsilon$ for all $n \ge j(k, \varepsilon)$. Thus

$$\|P_k \chi_n\| = \begin{cases} \|P_k(\psi_{f(n)})\| & \text{for } n \in U \\ 0 & \text{for } n \notin U \end{cases}$$

By assumption the set $\{n \in U: f(n) < j(k, \varepsilon)\}$ is finite, consequently there is a $q(k, \varepsilon) \in \mathbb{N}$ so that for all $n \in U$, $n > q(k, \varepsilon)$ it follows that $f(n) \ge j(k, \varepsilon)$. Therefore $\|P_k \chi_n\| \le \varepsilon$ for all $n > q(k, \varepsilon)$, i.e., $(\chi_n) \in \hat{\hat{\mathfrak{D}}}_{\mathcal{U}}$.

ii) The proof is analogous to that of Lemma 3.9.ii).

q.e.d.

Remark 3.12

For a free ultrafilter $\mathcal{U}$ and $\mathcal{U}$-finite f the space $\hat{\mathfrak{D}}_{\mathcal{V}}$ is a π-invariant t_π-closed subspace of $\hat{\mathfrak{D}}_{\mathcal{U}}$ and $\hat{\mathfrak{D}}_{\mathcal{V}} \neq (0)$.
This is quite analogously to Remark 3.10, while the last assertion follows from Lemma 2.5.iv).

Corollary 3.13

i) If π_1 is irreducible (on $\check{\mathfrak{D}}_{\mathcal{U}}$), then $\mathcal{U}$ is absolute.

ii) If π is irreducible (on $\hat{\mathfrak{D}}_{\mathcal{U}}$), then $\mathcal{U}$ is rare.

Proof:

Suppose $\mathcal{U}$ to be not absolute, then according to Lemma 2.5. ii) there is a map $f: \mathbb{N} \longrightarrow \mathbb{N}$ so that $f(\mathcal{U})$ is neither fixed nor equivalent to $\mathcal{U}$. So the assertion follows from Lemma 3.9. and Remark 3.10.

ii) Suppose $\mathcal{U}$ to be not rare, then again there is an f which is $\mathcal{U}$-finite but not a $\mathcal{U}$-equivalence (Lemma 2.5.iii). So the assertion follows from Lemma 3.11 and Remark 3.12.

q.e.d.

Lemma 3.14

Let $\mathfrak{U}$ be rare. Then for all $[\hat{f}] = [(\chi_n)] \in \hat{\mathfrak{D}}_{\mathfrak{U}}$, $[\hat{f}] \neq 0$, the set $\{ \pi(A)[\hat{f}] : A \in \mathcal{L}^+(\mathfrak{D}) \}$ is t_π-dense in $\hat{\mathfrak{D}}_{\mathfrak{U}}$. (Here and in the proof $[\cdot]$ means $[\cdot]_{\mathfrak{U}}$.)

Proof:

Let $[\hat{g}] = [(\psi_n)] \in \hat{\mathfrak{D}}_{\mathfrak{U}}$ be arbitrary. According to section 2 there is a $B \in \mathfrak{B}(\mathfrak{D})$, $B \geq 0$ with $\psi_n = B\varphi_n$, $\|\varphi_n\| = 1$, $\varphi_n \in \mathcal{H}$ (having in mind that the set $\{\psi_n\}$ is t-bounded).

Let $B = \int_0^b \lambda \, dE_\lambda$. For $\varepsilon > 0$, $k \in \mathbb{N}$ by Lemma 2.1 there exists an $a > 0$, so that for $P_a = \int_a^b dE_\lambda$: $\|A_k(I-P_a)B\| \leq \varepsilon$. Remark that $\mathcal{H}^\circ = P_a\mathcal{H} \subset \mathfrak{D}$. The elements $[(P_a\psi_n)]$, $[(P_a\chi_n)]$ are contained in $\hat{\mathcal{H}}^\circ_{\mathfrak{U}}$. Then by /12/ there is a certain operator $S_0 \in \mathfrak{B}(\mathcal{H}^\circ)$ so that $S = S_0 \oplus 0 \in \mathcal{L}^+(\mathfrak{D})$ fulfills $\pi(S)([(P_a\chi_n)]) = [(P_a\psi_n)]$. Therefore

$$|||\pi(A_k)(\pi(S)\pi(P_a)[(\chi_n)] - [(\psi_n)])||| = |||\pi(A_k)([(P_a\psi_n)] - [(\psi_n)])||| = |||\pi(A_k)\pi(I-P_a)([(\psi_n)])||| = \lim_{\mathfrak{U}}\|A_k(I-P_a)\psi_n\| =$$

$$= \lim_{\mathfrak{U}}\|A_k(I-P_a)B\varphi_n\| \leq \|A_k(I-P_a)B\| < \varepsilon .$$

q.e.d.

Corollary 3.15

i) If $\mathfrak{U}$ is rare, then π is irreducible on $\hat{\mathfrak{D}}_{\mathfrak{U}}$.

ii) If $\mathfrak{U}$ is absolute, then π_1 is irreducible on $\check{\mathfrak{D}}_{\mathfrak{U}}$.

Proof:

i) is the content of Lemma 3.14.

ii) $\mathfrak{U}$ absolute implies $\hat{\mathfrak{D}}_{\mathfrak{U}} = \check{\mathfrak{D}}_{\mathfrak{U}}$ (Theorem 3.5) and the irreducibility is a consequence of i).

q.e.d.

Thus, Theorem 3.7. is proved by Corollaries 3.13 and 3.15.

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О неприводимости обобщенных представлений Калкина

Для двух классов обобщенных представлений Калкина для алгебры неограниченных операторов $L^{+}(D)$, D-пространства Фреше исследуется вопрос о неприводимости представлений. Показано, что верны такие же критерии неприводимости, как для ограниченного случая.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Löffler F., Timmermann W. E5-85-522
On the Irreducibility of Generalized Calkin Representations

For two classes of generalized Calkin representations of the $*$-algebra $L^{+}(d)$ of unbounded operators on an (F)-domain D there is investigated the problem of irreducibility of these representations. It is proved that the same criteria as in the bounded case are valid.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985