объединенный
институт
ядерных
исследований
дубна

E5-85-49

K.Mühle, Š.Šujan

LINEAR FUNCTIONAL RELATIONSHIP: SOME APPROXIMATE ESTIMATES

Submitted to "Inverse Problems"

1985

1. INTRODUCTION AND PROBLEM SETTING

In many problems of experimental physics we meet the question of estimating an unknown functional relationship between two quantities X and Y, both of which are observed and thus subject to random measurement errors. Furthermore, often it is possible to perform mutually independent observations of pairs (X, Y), however, X and Y themselves are correlated.

In a wider sense, the problem falls within the category of inverse problems, that is, we seek for a model (from a certain given class of models) that exhibits the best fit to experimental data.

This paper is devoted to the following problem. We are given $n \geq 2$ mutually independent groups of measured values

$$\{(x_{ij}, y_{ij}) : j = 1, 2, \ldots, p_i\};\ i = 1, 2, \ldots, n, \qquad (1)$$

where $p_i \geq 2$ for all i. In the i-th group there are the results of p_i independent observations of a two-dimensional Gaussian vector

$$(X^i, Y^i)' \sim N_2[(\mu_i, \eta_i)', \Sigma];\ i = 1, 2, \ldots, n. \qquad (2)$$

It is supposed that the covariance matrix Σ is the same for all n groups (thus, all measurements are performed with the same precision), and the parameters μ_i and η_i are related linearly:

$$\eta_i = \alpha + \beta\mu_i,\ i = 1, 2, \ldots, n. \qquad (3)$$

Thus, we come to the model of a linear functional relationship /1/. Our intention is to discuss the possibilities of employing certain approximate estimates of the unknown parameters (α, β). In all methods to be described below an estimate of α will be obtainable from the formula

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}, \qquad (4)$$

where $\hat{\beta}$ is an estimate of β, and

$$\left.\begin{aligned} &\bar{x} = \Big(\sum_{i=1}^{n} p_i\Big)^{-1} \sum_{i=1}^{n} p_i \bar{x}_i, \quad \bar{y} = \Big(\sum_{i=1}^{n} p_i\Big)^{-1} \sum_{i=1}^{n} p_i \bar{y}_i, \\ &\bar{x}_i = \frac{1}{p_i} \sum_{j=1}^{p_i} x_{ij}, \quad \bar{y}_i = \frac{1}{p_i} \sum_{j=1}^{p_i} y_{ij}, \quad 1 \le i \le n. \end{aligned}\right\} \qquad (5)$$

Consequently, we shall concentrate ourselves upon estimating the parameter β under various assumptions concerning the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}. \qquad (6)$$

Sometimes it will be more convenient to write $\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$ so that $\sigma_{xx} = \sigma_x^2$, $\sigma_{yy} = \sigma_y^2$.

2. MAXIMUM LIKELIHOOD ESTIMATES

Recall some known results (more precisely, results which can be obtained by slight modifications of known ones). If $p_i = 1 \, (1 \le i \le n)$ and if Σ contains more than one unknown parameter, then the estimates are to be more or less arbitrary. In particular, if $\rho_{xy} = 0$ and $\sigma_x^2 \ne \sigma_y^2$ are unknown, then the likelihood function does not possess a maximum [1].

If $\sigma_x^2 = \sigma_y^2$ and $\rho_{xy} = 0$ then we get the model (1)-(3) with $\Sigma = \sigma^2 I$. In this case (cf. Chpt.29 of [2]) the maximum likelihood estimate (MLE) of β can be expressed as follows:

- if $SPD^*_{xy} \ne 0$, then

$$\hat{\beta}_M = (2\,SPD^*_{xy})^{-1}\Big[SSD^*_y - SSD^*_x + \big[(SSD^*_y - SSD^*_x)^2 + 4\,SPD^{*2}_{xy}\big]^{1/2}\Big], \qquad (7)$$

- if $SPD^*_{xy} = 0$ and $SSD^*_x > SSD^*_y$, then $\hat{\beta}_M = 0$,
- if $SPD^*_{xy} = 0$ and $SSD^*_x < SSD^*_y$, then $\hat{\beta}_M = \infty$

(the reader is encouraged to imagine all this geometrically). Here

$$\left.\begin{aligned} &SSD^*_x = \sum_{i=1}^{n} p_i(\bar{x}_i - \bar{x})^2, \quad SSD^*_y = \sum_{i=1}^{n} p_i(\bar{y}_i - \bar{y})^2, \\ &SPD^*_{xy} = \sum_{i=1}^{n} p_i(\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y}) \end{aligned}\right\} \qquad (8)$$

(cf. (5)). The symbols SSD_x, SSD_y, and SPD_{xy} will denote the usual (unweighted) sums of differences. In case $\Sigma = \sigma^2 I$ and under the assumption that there are no errors when observing X^i (i.e., $\sigma_x^2 = 0$ and $\sigma_{xy} = 0$) we get the usual estimate following the weighted least squares method (LSE; in our situation it coincides with MLE):

$$\hat{\beta}_L = SPD^*_{xy} / SSD^*_x. \qquad (9)$$

Let

$$\hat{\beta}_R = SSD^*_y / SSD^*_x \qquad (10)$$

denote the regression estimate. Then we get (within the general form (1)-(3) of the model considered) the inequalities

$$\left.\begin{aligned} &0 < \hat{\beta}_L < \hat{\beta}_M < \hat{\beta}_R \qquad SPD^*_{xy} > 0, \\ &\hat{\beta}_R < \hat{\beta}_M < \hat{\beta}_L < 0 \qquad SPD^*_{xy} < 0. \end{aligned}\right\} \qquad (11)$$

Since $\hat{\beta}_M$ is a consistent estimate, (11) shows that the estimates $\hat{\beta}_L$ and $\hat{\beta}_R$ are in the general case biased. However, this does not necessarily mean that $\hat{\beta}_M$ is always better than $\hat{\beta}_L$. In fact, the answer to this question depends on the difference between β and the regression of errors in Y^i on the errors in X^i, σ_{xy}/σ_{xx}. If

$$\Big(\beta - \frac{\sigma_{xy}}{\sigma_{xx}}\Big)\Big(\frac{\sigma_{xx}}{\sigma_{yy} - \sigma_{xy}^2/\sigma_{xx}}\Big)^{1/2} \approx 0, \qquad (12)$$

then $\hat{\beta}_L$ is better than $\hat{\beta}_M$ [3]. The reason is that when (12) is true, then the bias of $\hat{\beta}_L$ will be vanishingly small and, at the same time, the confidence interval for $\hat{\beta}_L$ will be narrower than that one for $\hat{\beta}_M$.

If the general model (1)-(3) is considered with $p_1 = \ldots = p_n = k \ge 2$ (for different p_i's one can employ identical considerations but the algebra will become rather cumbersome), then the MLE $\hat{\beta}_M$ can be obtained as follows. Let

$$\mathbf{S} = \begin{pmatrix} SSD_x & SPD_{xy} \\ SPD_{xy} & SSD_y \end{pmatrix} \qquad (13)$$

and

$$\mathbf{W} = \frac{1}{n(k-1)} \sum_{i=1}^{n} \sum_{j=1}^{k} \begin{pmatrix} x_{ij} - \bar{x}_i \\ y_{ij} - \bar{y}_i \end{pmatrix} \begin{pmatrix} x_{ij} - \bar{x}_i \\ y_{ij} - \bar{y}_i \end{pmatrix}'. \qquad (14)$$

Suppose $\beta \neq 0$. Then the conditions (3) can be expressed in the form of a linear constraint

$$(\delta_1, \delta_2)'(\mu_i, \eta_i) = \gamma, \; 1 \le i \le n. \tag{15}$$

Indeed, if $\alpha = \gamma/\delta_2$ and $\beta = -\delta_1/\delta_2$, then (15) yields (3). The model (1), (2), (15) is a special case of the model studied by Anderson /3,4/. Let λ_M denote the smaller solution to

$$\det(\mathbf{S} - \lambda \mathbf{W}) = 0. \tag{16}$$

Then MLE $\hat{\delta}_M$ of $\delta = (\delta_1, \delta_2)$ can be obtained as the solution d of the equation

$$\mathbf{S}\vec{d} = \lambda_M \mathbf{W}\vec{d}. \tag{17}$$

In the particular case $\Sigma = \sigma^2 I$, if $W = (W_{ij})(i, j = x, y)$ this method yields the estimate

$$\hat{\beta}_M = \frac{SSD_y - \lambda_M W_{yy}}{SPD_{xy} - \lambda_M W_{xy}} = \frac{SPD_{xy} - \lambda_M W_{xy}}{SSD_x - \lambda_M W_{xx}}, \tag{18}$$

3. LIKELIHOOD EQUATIONS WHEN Σ IS KNOWN

The likelihood function of the ensemble (1) can be written in the form

$$\left.\begin{aligned} &\left(2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}\right)^{-\sum_{i=1}^{n} p_i} \exp\Big\{-\frac{1}{2(1-\rho_{xy}^2)} \sum_{i=1}^{n}\sum_{j=1}^{p_i}\Big[\frac{1}{\sigma_x^2}\times \\ &\times (x_{ij}-\mu_i)^2 + \frac{1}{\sigma_y^2}(y_{ij}-\eta_i)^2 - \frac{2\rho_{xy}}{\sigma_x\sigma_y}(x_{ij}-\mu_i)(y_{ij}-\eta_i). \end{aligned}\right\} \tag{19}$$

If Σ is a known matrix then using (3) we get likelihood equations

$$\left.\begin{aligned} &\partial S/\partial\mu_i = 0, \quad 1 \le i \le n, \\ &\partial S/\partial\alpha = 0, \\ &\partial S/\partial\beta = 0, \end{aligned}\right\} \tag{20}$$

where

$$\left.\begin{aligned} S = &\sum_{i=1}^{n}\sum_{j=1}^{p_i}\big[\sigma_x^{-2}(x_{ij}-\mu_i)^2 + \sigma_y^{-2}(y_{ij}-\alpha-\beta\mu_i)^2 - \\ &- 2\rho_{xy}\sigma_x^{-1}\sigma_y^{-1}(x_{ij}-\mu_i)(y_{ij}-\alpha-\beta\mu_i)\big]. \end{aligned}\right\} \tag{21}$$

Use (5) together with the notation

$$\bar{\mu} = \left(\sum_{i=1}^{n} p_i\right)^{-1} \sum_{i=1}^{n} p_i\mu_i. \tag{22}$$

After straightforward calculations we get the following form of (20):

$$\begin{aligned} &(\mu_i - \bar{x}_i)(\sigma_y^2 - \rho_{xy}\sigma_x\sigma_y) + \\ &+ (\bar{y}_i - \alpha - \beta\mu_i)(\rho_{xy}\sigma_x\sigma_y - \beta\sigma_x^2) = 0, \quad 1 \le i \le n, \end{aligned} \tag{23a}$$

$$\rho_{xy}\sigma_x\sigma_y\left(\sum_{i=1}^{n} p_i\bar{x}_i - \sum_{i=1}^{n} p_i\mu_i\right) - \sigma_x^2\left(\sum_{i=1}^{n} p_i\bar{y}_i - \alpha\sum_{i=1}^{n} p_i - \beta\sum_{i=1}^{n} p_i\mu_i\right) = 0, \tag{23b}$$

$$\rho_{xy}\sigma_x\sigma_y\left(\sum_{i=1}^{n} p_i\mu_i\bar{x}_i - \sum_{i=1}^{n} p_i\mu_i^2\right) - \sigma_x^2\left(\sum_{i=1}^{n} p_i\mu_i\bar{y}_i - \alpha\sum_{i=1}^{n} p_i\mu_i - \beta\sum_{i=1}^{n} p_i\mu_i^2\right) = 0. \tag{23c}$$

Remark 1. Observe the following: if we consider only the i-th group of results then we shall get instead of (23) only the i-th equation in (23a) and (23b), (23c) both reduce to

$$\rho_{xy}\sigma_x\sigma_y(\bar{x}_i - \mu_i) - \sigma_x^2(\bar{y}_i - \alpha - \beta\mu_i) = 0.$$

Let $S^{(i)}$ denote the exponent of the likelihood function of the i-th group. Then the latter assertion says that

$$\partial S^{(i)}/\partial\alpha = \partial S^{(i)}/\partial\beta = 0.$$

Hence, the likelihood function of the i-th group does not possess a maximum. Consequently, it is not possible to use the usual approach of "pasting together" the estimates, obtained from separate groups the common approach in multiple regression /5/.

4. APPROXIMATE MAXIMUM LIKELIHOOD METHOD

The above Remark 1 together with the well-known fact that finite sample properties of MLE of a linear functional relationship are rather poor /4/ motivated the search upon analytically simpler estimates. Their statistical properties were investigated using computer experiments.

4.1. The essence of the first (and of the second) approach consists of employing the observations of random variables X^i, $1 \le i \le n$, in order to get estimates of μ_i. Formally, we express α from (23b) and substitute it into (23a). Then we get the following system of quadratic equations for the unknown β:

$$\beta^2\sigma_x^2(\mu_i - \bar{\mu}) + \beta[\rho_{xy}\sigma_x\sigma_y(2\bar{\mu} - 2\mu_i - (\bar{x} - \bar{x}_i)) + \sigma_x^2(\bar{y} - \bar{y}_i)] + \sigma_y^2(\mu_i - \bar{x}_i - \rho_{xy}^2\bar{\mu} + \rho_{xy}^2 x) + \rho_{xy}\sigma_x\sigma_y(\bar{y}_i - \bar{y}) = 0, \quad 1 \le i \le n. \tag{24}$$

Since $x^i \sim N_1[\mu_i, \sigma_x^2]$, the MLE of μ_i is simply the sample mean[5], i.e.,

$$(\mu_i)^{\wedge}_M = \bar{x}_i, \quad 1 \le i \le n. \tag{25}$$

Using (25), independence of groups, and the well-known fact that a MLE commutes with an arbitrary one-to-one function of estimated parameters[5], we have (cf. (22))

$$(\bar{\mu})^{\wedge}_M = \bar{x}. \tag{26}$$

Now we make the simplifying assumption that the substitution of estimates (25), (26) does not influence upon validity of likelihood equations (strictly speaking, this is so only when $\rho_{xy} = 0$ so that the likelihood function factorizes into two sums of quadratic terms, one for the X^i's and the second one for the Y^i's). Then the equations (24) take on the form

$$\left.\begin{aligned} &\beta^2\sigma_x^2(\bar{x}_i - \bar{x}) + \beta[\rho_{xy}\sigma_x\sigma_y(\bar{x} - \bar{x}_i) + \sigma_x^2(\bar{y} - \bar{y}_i)] + \\ &+ \rho_{xy}\sigma_x\sigma_y(\bar{y}_i - \bar{y}) = 0, \quad 1 \le i \le n. \end{aligned}\right\} \tag{27}$$

By solving (27) we get for any i the root

$$\hat{\beta}^{(0)} = \rho_{xy}\sigma_y/\sigma_x = \sigma_{xy}/\sigma_{xx}. \tag{28}$$

Second solutions result in the estimates

$$\hat{\beta}^{(i)} = (\bar{y} - \bar{y}_i)/(\bar{x} - \bar{x}_i), \quad i = 1, 2, \dots, n. \tag{29}$$

Remark 2. When Σ is unknown we can take the estimates of σ_{xy} and σ_{xx} in (28) and thereby get $\hat{\beta}^{(0)} = SPD^*_{xy}/SSD^*_x = \hat{\beta}_L$ (cf. (9)). This is in complete agreement with our simplifying assumption. In fact, our approach is fully justified when $\rho_{xy} = 0$. But in this case we have to obtain a MLE which coincides with weighted LSE $\hat{\beta}_L$.

4.2. Using the same approach, but substituting the expression for α obtained from (23b) into (23a), making again the substitutions $\mu_i \to \bar{x}_i$, $\bar{\mu} \to \bar{x}$, we get the estimate

$$\hat{\beta} = \frac{(\Sigma p_i)(\Sigma p_i \bar{x}_i \bar{y}_i) - (\Sigma p_i \bar{x}_i)(\Sigma p_i \bar{y}_i)}{(\Sigma p_i)(\Sigma p_i \bar{x}_i^2) - (\Sigma p_i \bar{x}_i)^2}, \tag{30}$$

where all sums are over $i = 1, 2, \dots, n$. The same result is obtained when first substituting $\mu_i \to \bar{x}_i$, $\bar{\mu} \to \bar{x}$ into (21), expressing $\hat{\alpha}\,(= \bar{y} - \beta\bar{x})$ from the equation corresponding to (23b), and finally calculating $\hat{\beta}$ from the equation corresponding to (23c). It is easy to see that such an approach leads to the problem of weighted least squares for group averages:

$$\sum_{i=1}^{n} p_i(\bar{y}_i - \beta\bar{x}_i - \alpha)^2 = \text{MIN}. \tag{31}$$

Its solution leads precisely to (30).

4.3. Express μ_i from (23a) $(1 \le i \le n)$, and denote this by $\hat{\mu}_i$. Then

$$\left.\begin{aligned} &\bar{x}_i - \hat{\mu}_i = (\bar{y}_i - \alpha - \beta\bar{x}_i)(\rho\sigma_x\sigma_y - \beta\sigma_x^2)/K(\beta), \\ &\bar{y}_i - \alpha - \beta\hat{\mu}_i = (\bar{y}_i - \alpha - \beta\bar{x}_i)(\sigma_y^2 - \rho\sigma_x\sigma_y\beta)/K(\beta), \end{aligned}\right\} \tag{32}$$

where $\rho = \rho_{xy}$ and $K(\beta) = \sigma_y^2 - 2\rho\sigma_x\sigma_y\beta + \beta^2\sigma_x^2$. Next observe that

$$(\bar{x}_i, \bar{y}_i)' \sim N_2[(\mu_i, \alpha + \beta\mu_i)', p_i^{-1}\Sigma], \tag{33}$$

hence we can write down the likelihood function for the vector of group averages $((\bar{x}_1, \bar{y}_1), \dots, (\bar{x}_n, \bar{y}_n))$. Substituting (32) into its exponent we get the expression

$$R(\alpha, \beta) = \frac{1 - \rho^2}{K(\beta)^2} \sum_{i=1}^{n} p_i(\bar{y}_i - \alpha - \beta\bar{x}_i)^2. \tag{34}$$

By minimizing over α we get $\hat{\alpha} = \bar{y} - \beta\bar{x}$. After substituting this into (34) we get

$$Q(\beta) = \frac{1-\rho^2}{K(\beta)}(SSD_y^* - 2\beta SPD_{xy}^* + \beta^2 SSD_x^*). \tag{35}$$

Since $Q(\beta)$ is a ratio of two polynomials of second order in β, analytic calculation of an explicit form of the estimate $\hat{\beta} = \text{Arg}\min Q(\beta)$ is practically impossible (indeed, the estimates will depend on relations among three parameters ρ, σ_x, σ_y). Therefore it is necessary to employ numerical methods. However, in certain special cases we can obtain explicit solutions.

4.3.1. If $\sigma_x^2 \to 0$ or $\sigma_y^2 \to 0$ (in both cases $\sigma_{xy} \to 0$) we get, as expected, regression estimates

$$\left.\begin{aligned} \hat{\beta} &= SPD_{xy}^*/SSD_x^* \quad (\sigma_x^2 \to 0), \\ \hat{\beta} &= SSD_y^*/SPD_{xy}^* \quad (\sigma_y^2 \to 0). \end{aligned}\right\} \tag{36}$$

4.3.2. If $\sigma_x^2 = \sigma_y^2 = \sigma^2$, then the problem $Q(\beta) = \text{MIN}$ has a unique solution

$$\left.\begin{aligned} \hat{\beta} = {} & [2(SPD_{xy}^* - SSD_x^*)]^{-1}\{SSD_y^* - SSD_x^* + [SSD_y^* + \\ & + 2(2\rho^2 - 1)SSD_x^* SSD_y^* + SSD_x^{*2} + 4SPD_{xy}^* \times \\ & \times (SPD_{xy}^* - \rho SSD_y^* - \rho SSD_x^*)]^{1/2}\}. \end{aligned}\right\} \tag{37}$$

In case $\rho = 0$ we come again to the model (1)-(3) with $\Sigma = \sigma^2 I$. In this case the estimate (37) coincides with the MLE $\hat{\beta}_M$ (cf. (7)). The estimate $\hat{\beta}$ is a MLE also when the precisions of measurements of X^i and Y^i are the same, and hence the model is given by (1)-(3) with $\Sigma = \sigma^2\begin{pmatrix}1 & \rho \\ \rho & 1\end{pmatrix}$. If σ^2 is unknown, we may estimate it by

$$\left.\begin{aligned} \hat{\sigma}^2 &= \tilde{Q}(\hat{\beta})/2n, \\ \tilde{Q}(\beta) &= \frac{1-\rho^2}{1-2\rho\beta+\beta^2}(SSD_y^* - 2\beta SPD_{xy}^* + \beta^2 SSD_x^*), \end{aligned}\right\} \tag{38}$$

where $\hat{\beta}$ is the estimate (37). If ρ is also unknown, we can take in (37), (38) the sample correlation coefficient [6]. The formula

(38) can be obtained also in a different way. Suppose $\rho \neq 0$ and $\sigma_x^2 = \sigma_y^2 = \sigma^2$. Then we get (see (32))

$$\bar{x}_i - \hat{\mu}_i = (\bar{y}_i - a - \beta\bar{x}_i)\frac{\rho - \beta}{1 - 2\rho\beta + \beta^2}, \quad \bar{y}_i - a - \beta\hat{\mu}_i = (\bar{y}_i - a - \beta\bar{x}_i)\frac{1 - \rho\beta}{1 - 2\rho\beta + \beta^2}.$$

Then by repeating the considerations leading above to (35) we get instead of (35) the expression $\tilde{Q}(\beta)$ from (38).

5. CONCLUDING REMARKS

Using numerical experiments we studied the estimates (29) and observed a large variety of behaviour. Subsequent considerations serve the purpose of explaining the observed phenomena and thereby giving some practical recommendations.

Suppose i and $p_1, \dots, p_n$ are chosen so that $\mu_i = \bar{\mu}$. By normalizing the scale parameter we can always assume that

$$(\bar{x}_i - \bar{x}, \bar{y}_i - \bar{y})' \sim N_2[(0,0)', \begin{pmatrix}1 & \rho \\ \rho & 1\end{pmatrix}], \quad |\rho| < 1.$$

As is well-known, this entails

$$\hat{\beta}^{(i)} = (\bar{y}_i - \bar{y})/(\bar{x}_i - \bar{x}) \sim C(\rho, (1 - \rho^2)^{1/2}),$$

where $C(\rho, \sigma)$ is the Cauchy distribution with parameters ρ and σ. This is in complete agreement with numerical experiments, which are showing in the situation just described an almost arbitrary scattering of estimates even when the sample sizes were very large.

On the other hand, if the group sizes p_i are large enough, then $\bar{x}_i - \bar{x} \simeq \mu_i - \bar{\mu}$, $\bar{y}_i - \bar{y} \simeq \beta(\mu_i - \bar{\mu})$, since MLE's are consistent (rigorously speaking, such a conclusion is valid only when $\rho = 0$; however, see (33) according to which $\rho/p_i \approx 0$ for large p_i). Consequently,

$$\frac{y_i - \bar{y}}{x_i - \bar{x}} \simeq \frac{\mu_i - \bar{\mu}}{\mu_i - \bar{\mu}}\beta = \beta \quad \text{if} \quad \mu_i - \bar{\mu} \neq 0.$$

Numerical experiments have shown that for large sample sizes the estimates $\hat{\beta}^{(i)}$ (cf. (29)) exhibit low sensitivity to changes in the parameter ρ (even at moderate sample sizes ~50 the difrefences were quite small). In experiments we considered the situation $\mu_1 < \mu_2 < \dots < \mu_n$. Consequently, for large sample sizes practically always $SPD_{xy}^* > 0$. Since the estimates $\hat{\beta}^{(i)}$ are, from the point of view of their structure, closer to type $\hat{\beta}_L$-estimates than to type $\hat{\beta}_M$ ones, a bias towards smaller values was observed, in agreement with (11). However, if $\bar{x}_i - \bar{x} \doteq \mu_i - \bar{\mu}$ (large samples) and $|\mu_i - \bar{\mu}| >> \sigma_x$, the bias approached zero very fastly.

In conclusion suppose the basic model is of the form

$$(X^i, X^i)' = N_2[(\mu_i, \eta_i)', \begin{pmatrix} \sigma_i^2 & \rho_i\sigma_i\tau_i \\ \rho_i\sigma_i\tau_i & \tau_i^2 \end{pmatrix}], \quad 1 \le i \le n. \tag{39}$$

By repeating the considerations from 4.1. and 4.2. we get the estimates

$$\hat{\beta}^{(0)} = \rho_i\tau_i/\sigma_i, \tag{40}$$

$$\hat{\beta}^{(i)} = (\bar{y}_i - \bar{y})/(\bar{x}_i - \bar{x}), \quad 1 \le i \le n, \tag{41}$$

$$\hat{\beta} = \frac{(\Sigma w_i)(\Sigma w_i\bar{x}_i\bar{y}_i) - (\Sigma w_i\bar{x}_i)(\Sigma w_i\bar{y}_i)}{(\Sigma w_i)(\Sigma w_i\bar{x}_i^2) - (\Sigma w_i\bar{x}_i)^2}, \tag{42}$$

where

$$w_i = p_i/\tau_i^2, \quad 1 \le i \le n. \tag{43}$$

The estimate (42) can be considered again as a solution to the problem of weighted least squares

$$\sum_{i=1}^{n} w_i(\bar{y}_i - \alpha - \beta\bar{x}_i)^2 = \text{MIN}. \tag{44}$$

Remark 3. The estimates (30) and (42) can be significantly simplified, if the original linear functional relationship is shifted to the "centre of gravity" of the pooled sample, i.e., when the model is of the form

$$E(Y^i | X^i = x) = \alpha + \beta(x - \mu), \quad 1 \le i \le n. \tag{45}$$

Indeed, then we may take $\hat{\alpha} = \bar{y}$ and

$\hat{\beta} = (\Sigma p_i\bar{x}_i\bar{y}_i)/(\Sigma p_i\bar{x}_i^2)$ (cf.(30)),

$\hat{\beta} = (\Sigma w_i\bar{x}_i\bar{y}_i)/(\Sigma w_i\bar{x}_i^2)$ (cf.(42)),

respectively.

REFERENCES

1. Anderson T.W., Rubin H. Ann.Math.Statist., 1949, 20, p.46.
2. Kendall M.G., Stuart A. The Advanced Theory of Statistics II, 3rd ed. Griffin, London, 1973.
3. Anderson T.W. Ann.Math.Statist., 1951, 22, p.327.
4. Anderson T.W. J.Roy.Stat.Soc., 1976, B38, p.1.
5. Rao C.R. Linear Statistical Inference and Its Applications. J.Wiley, New York, 1973.
6. Cramer H. Mathematical Methods of Statistics. Princeton Univ.Press, Princeton, N.J., 1949.

Received by Publishing Department
on January 23,1985.

В Объединенном институте ядерных исследований начал выходить сборник *"Краткие сообщения ОИЯИ"*. В нем будут помещаться статьи, содержащие оригинальные научные, научно-технические, методические и прикладные результаты, требующие срочной публикации. Будучи частью "Сообщений ОИЯИ", статьи, вошедшие в сборник, имеют, как и другие издания ОИЯИ, статус официальных публикаций.

Сборник "Краткие сообщения ОИЯИ" будет выходить регулярно.

The Joint Institute for Nuclear Research begins publishing a collection of papers entitled *JINR Rapid Communications* which is a section of the JINR Communications and is intended for the accelerated publication of important results on the following subjects:

Physics of elementary particles and atomic nuclei.
Theoretical physics.
Experimental techniques and methods.
Accelerators.
Cryogenics.
Computing mathematics and methods.
Solid state physics. Liquids.
Theory of condenced matter.
Applied researches.

Being a part of the JINR Communications, the articles of new collection like all other publications of the Joint Institute for Nuclear Research have the status of official publications.

JINR Rapid Communications will be issued regularly.

Мюле К., Шуян Ш. E5-85-49

Линейная функциональная зависимость: некоторые приближенные оценки

Предложен класс приближенных оценок линейной функциональной зависимости, полученных при помощи приближенного метода максимального правдоподобия. Подробно обсуждены свойства оценок, подсказанные в итоге численных экспериментов. Установлены связи с оценками по методу максимального правдоподобия, методу наименьших квадратов и регресиионными оценками.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Mühle K., Šujan Š. E5-85-49

Linear Functional Relationship: Some Approximate Estimates

A class of approximate estimates of a linear functional relationship is proposed based on an approximate maximum likelihood method. The properties of estimates suggested by computer simulations are discussed in detail, and relations to maximum likelihood, least squares, and regression estimates are described.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985