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ON EXPANSION OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS INTO POWER SERIES

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1. Introduction

We consider the system of ordinary differential equations with complex variables

$$\omega' = f(z,\omega) , \quad \omega' = \frac{d\omega}{dz} . \tag{1.1}$$

where $z \in C$, $\omega \in C^n$, $f(z,\omega)$ is a holomorphic vector-function in a field W of the complex (z,ω)-space.

The classical Cauchy theorem says that for any initial value $(z_0,\omega^0) \in W$ there exists a unique Cauchy problem solution which is holomorphic at the point z_0 /1,2/ . On the theorem's base one have developed the analytical theory of differential equations which has achieved considerable successes. However for applications of the analytical theory it is necessary to elaborate effective methods for calculation of the coefficients of Taylor series of Cauchy problem solution. On the other hand, the idea of straight calculation of coefficients of power series is the effective method of approximate integration of ordinary differential equations and nowadays causes excite interest among specialists /3-10/ . The reasons of the new ascent of the classical idea are economy of computer time, possibility to control the step and the approximate order at every step and possibility to get analytical expressions for qualitative investigation.

In this paper the universal method of expansion of Cauchy problem solutions of system (1.1) is elaborated. The method has no the known difficulty connected with calculation of partial derivatives of high orders of function f . The method is used also for expansion into power series of Cauchy problem solution of the scalar differential equation of the type

$$\omega^{(n)} = g(z,\omega^{(n-1)},\dots,\omega^{(1)},\omega) , \quad \omega^{(k)} = \frac{d^k\omega}{dz^k} , \tag{1.2}$$

where g is an analytical equation for z and every derivative $\omega^{(k)}$ ($k = 0,1, \dots , n-1$) in a field $Q \subset C^{1+n}$.

[illegible] ИНСТИТУТ
[illegible] ИССЛЕДОВАНИЙ

The method of expansion into power series is particularly effective in the cases when the right part of a differential equation is an entire function. In particular we shall consider the system of ordinary differential equations with real variables

$$x' = f(t,x)\ , \quad x' = \frac{dx}{dt}\ , \qquad (1.3)$$

where $f(t,x)$ is a real vector-function being defined for all $t \in R$, $x \in R^n$ and such that there is its analytical continuation which is an entire function in the complex (z, ω) -space. The expansion algorithm of the derivatives of flows generated by system (1.3) (or Jacobi matrices of global solutions) into power series is elaborated. This shows the other advantage of the expansion method into power series, because the flow's derivative is the most important character in the modern theory of qualitative investigation of differential equations /11,12/ .

For simplification of the text we propose that $z_0 = 0$ $(t_0 = 0)$. All the results are easily transferred for the case of an arbitrary value z_0 (t_0) . The operation of matrices transposition will be denoted by T .

2. Expansion of Cauchy Problem Solution into Power Series

In general the vector-function $f(z,\omega)$ being holomorphic in the field W can be presented in the neighbourhood of every point of the field by the vector of convergent power series

$$f_r(z,\omega) = f_r^0(z) + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = 1,\dots,n}} f_r^{\alpha 1,\dots,\alpha K}(z)\, \omega_{\alpha 1} \cdots \omega_{\alpha K} \quad (r = 1,\dots,n) \qquad (2.1)$$

where

$$f_r^0(z) = \sum_{m=0}^{\infty} f_{rm}^0 z^m \ , \quad f_r^{\alpha 1,\dots,\alpha K}(z) = \sum_{m=0}^{\infty} f_{rm}^{\alpha 1,\dots,\alpha K} z^m .$$

For the initial condition $\varphi(0) = \omega^0$ we find the solution in the form

$$\varphi(z) = \left(\sum_{m=0}^{\infty} c_m^1 z^m, \dots, \sum_{m=0}^{\infty} c_m^r z^m, \dots, \sum_{m=0}^{\infty} c_m^n z^m \right)^T . \qquad (2.2)$$

Theorem: Suppose the vector-function $f(z,\omega)$ is holomorphic in the neighbourhood of point $(0,\omega^0) \in W$ and presented

by the form (2.1). Then the coefficients of expansion of Cauchy problem solution into series (2.2) satisfy the following recurrent equation

$$c_{m+1}^r = \frac{1}{m+1} \left(f_{rm}^0 + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = 1,\dots,n \\ S0 + S1 + \cdots + SK = m}} f_{rS0}^{\alpha 1,\dots,\alpha K} c_{S1}^{\alpha 1} \cdots c_{SK}^{\alpha K} \right) \qquad (2.3)$$

$S0, S1, \dots, SK \in N = \{0,1,\dots\}$, $c_0^r = \omega_r^0$, $r = 1,\dots,n$, $m = 0,1,2,\dots$.

Proof. By induction on m it is easy to show that m-derivatives of $\varphi(z)$ satisfy the following recurrent equation

$$\varphi_r^{(m+1)} = \left(f_r^0(z) \right)^{(m)} + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = 1,\dots,n \\ S0 + S1 + \cdots + SK = m}} \frac{m!}{S0!\,S1!\dots SK!} \left(f_r^{\alpha 1,\dots,\alpha K}(z) \right)^{(S0)} \varphi_{\alpha 1}^{(S1)} \cdots \varphi_{\alpha K}^{(SK)} .$$

It follows (2.3). Theorem 1 is proved.

In the particular case when the vector-function f is independent of z , i.e., $\omega' = f(\omega)$, it can be presented by the form

$$f_r(\omega) = a_r + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = 1,\dots,n}} a_r^{\alpha 1,\dots,\alpha K} \omega_{\alpha 1} \cdots \omega_{\alpha K} \quad (r = 1,\dots,n)\ , \qquad (2.4)$$

where a_r , $a_r^{\alpha 1,\dots,\alpha K} \in C$.
Then we have

Corollary 1. Suppose the vector-function f is independent of z and is presented by the form (2.4). Then the coefficients of expansion of Cauchy problem solution into series (2.2) satisfy the following recurrent equation

$$c_{m+1}^r = \frac{1}{m+1} \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = 1,\dots,n \\ S1 + \cdots + SK = m}} a_r^{\alpha 1,\dots,\alpha K} c_{S1}^{\alpha 1} \cdots c_{SK}^{\alpha K}$$

$S1, \dots, SK \in N$, $c_0^r = \omega_r^0$, $c_1^r = f_r(\omega^0)$, $m = 1,2,\dots$.

The right part of equation (1.2) being analytical in the neighbourhood of an initial condition can be presented in the neighbourhood by the convergent power series

$$g(z,\omega^{(1)},\ldots,\omega^{(1)},\omega) = q(z) + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 0,1,\ldots,n-1 \\ \cdots\cdots\cdots \\ \alpha K = 0,1,\ldots,n-1}} a^{\alpha 1,\ldots,\alpha K}(z)\,\omega^{(\alpha 1)}\ldots\omega^{(\alpha K)}, \tag{2.5}$$

where

$$q(z) = \sum_{m=0}^{\infty} q_m z^m, \quad a^{\alpha 1,\ldots,\alpha K}(z) = \sum_{m=0}^{\infty} a_m^{\alpha 1,\ldots,\alpha K} z^m .$$

The Cauchy problem solution of equation (1.2) is found in the form

$$\varphi(z) = \sum_{m=0}^{\infty} c_m z^m . \tag{2.6}$$

Theorem 2. Suppose the function $g(z,\omega^{(n-1)},\ldots,\omega^{(1)},\omega)$ is holomorphic in the neighbourhood of initial condition $(0,\omega_0^{(n-1)},\ldots,\omega_0^{(1)},\omega_0) \in Q$ and presented by the form (2.5). Then the coefficients of expansion of Cauchy problem solution into series (2.6) can be defined by the following recurrent equation

$$C_{m+n} = \frac{1}{(m+1)\ldots(m+n)}\Big(q_m + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 0,1,\ldots,n-1 \\ \cdots\cdots\cdots \\ \alpha K = 0,1,\ldots,n-1 \\ S0+S1+\cdots+SK=m}} a_{S0}^{\alpha 1,\ldots,\alpha K} \prod_{\substack{i=1 \\ \alpha i \neq 0}}^{K} \prod_{j=1}^{\alpha i} (Si+j) C_{\alpha 1+S1}\cdots C_{\alpha K+SK}\Big) \tag{2.7}$$

$$C_0 = \omega_0,\ C_1 = \frac{1}{1!}\omega_0^{(1)},\ldots, C_{n-1} = \frac{1}{(n-1)!}\omega_0^{(n-1)}, \quad m = 0,1,2,\ldots .$$

Proof. By induction on m it is easy to prove that (m+n)-derivatives of solution $\varphi(z)$ can be found by the following recurrent equation

$$\varphi^{(m+n)} = q^{(m)}(z) + \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 0,1,\ldots,n-1 \\ \cdots\cdots\cdots \\ \alpha K = 0,1,\ldots,n-1 \\ S0+S1+\cdots+SK=m}} \frac{m!}{S0!\,S1!\ldots SK!}\left(a^{\alpha 1,\ldots,\alpha K}(z)\right)^{(S0)} \varphi^{(\alpha 1+S1)}\ldots\varphi^{(\alpha K+SK)} .$$

It follows (2.7). Theorem 2 is proved.

Lower we consider some typical examples.

Example 1. Kostitzin equation /13/

$$\omega_1' = -\lambda\omega_1 + \beta\omega_1\omega_2$$

$$\omega_2' = \mu\omega_2 - \beta\omega_1\omega_2 .$$

The coefficients of Taylor series (2.2) of its Cauchy problem solution are determined by the following

$$C_0^1 = \omega_1(0), \quad C_0^2 = \omega_2(0)$$

$$C_{m+1}^1 = \frac{1}{m+1}\Big(-\lambda C_m^1 + \beta \sum_{S=0}^{m} C_S^1 C_{m-S}^2\Big)$$

$$C_{m+1}^2 = \frac{1}{m+1}\Big(\mu C_m^2 - \beta \sum_{S=0}^{m} C_S^1 C_{m-S}^2\Big) .$$

Example 2. The mathematical pendulum

$$\omega^{(2)} = \sin\omega = -\sum_{K=1}^{\infty} (-1)^K \frac{\omega^{2K+1}}{(2K+1)!} .$$

The coefficients of Taylor series (2.6) are defined so

$$C_0 = \omega(0), \quad C_1 = \omega^{(1)}(0)$$

$$C_{m+2} = -\frac{1}{(m+1)(m+2)} \sum_{K=1}^{\infty} \sum_{S1+\cdots+S(2K+1)=m} \frac{(-1)^K}{(2K+1)!} C_{S1}\cdots C_{S(2K+1)}$$

$$m = 0,1,2,\ldots .$$

Example 3. Second Painleve equation

$$\omega^{(2)} = 2\omega^3 + z\omega + \alpha$$

$$C_0 = \omega(0), \quad C_1 = \omega^{(1)}(0), \quad C_2 = 2\omega^3(0) + \alpha ,$$

$$C_{m+2} = \frac{1}{(m+1)(m+2)}\Big(C_{m-1} + 2 \sum_{S1+S2+S3=m} C_{S1}C_{S2}C_{S3}\Big)$$

$$m = 1,2,\ldots .$$

Example 4. Van der Pol equation

$$\omega^{(2)} = (\alpha - \gamma^2\omega^2)\omega^{(1)} - \mu^2\omega + \varepsilon\beta\cos z$$

$$C_{m+2} = \frac{1}{(m+1)(m+2)}\Big(-\mu^2 C_m + \alpha C_{m+1} - \sum_{S1+S2+S3=m} \gamma^2 C_{S1}C_{S2}C_{S3} + q(m)\alpha\beta\frac{(-1)^{m/2}}{m!}\Big),$$

$$q(m) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

$$C_0 = \omega(0), \quad C_1 = \omega^{(1)}(0), \quad m = 0,1,2,\ldots .$$

So the given above algorithms are highly simple and on principle they have no any difficulty in their realization. However, from the practical point of view they are effective on the whole in the cases when the right part of differential equation is the polynomial of independent variables (examples 1,3 and 4) or enough quickly covergent series (example 2). In the last case the infinite series can be replaced by a polynomial. The coefficients of polynomial can be infinite series and they do not lead to the infinite sum in calculation of coefficients of the solution.

Practical calculation has shown that the method of expansion into power series has the great advantage for the rate of calculation as compared with the others at least in the following cases: a) the linear systems with polynomials coefficients, b) the system with a bilinear (quadratic) right part and c) polynomial systems with small parameters of the nonlinear part. For such system calculation of solutions is not more complicated then calculation of the exponential.

3. Expansion of Flow's Derivative into Power Series

The right part of equation (1.3) can be analytically continued on all the complex (z, ω) - space and its analytic continuation is an entire function. Therefore the function $f(t,x)$ can be also presented by the form (2.1) with the unique difference such that the complex variable z will be replaced by the real variable t when the question is a real solution.

For every point x on compact manifold $M \subset R^n$ its tangent space will be denoted by $T_x(M)$, the derivative of a flow $\varphi_t : M \longrightarrow M$ by $D\varphi_t$ /11,12/.

Theorem 3. Suppose $f(t,x)$ satisfies the above condition and $\varphi_t : M \longrightarrow M$ is a flow on compact manifold $M \subset R^n$ generated by system (1.3). Then for every $x \in M$ we have the following statements:

1) The motion $\varphi_t(x)$ passing through point x can be presented by the vector of series

$$\varphi_t(x) = \Big(\sum_{m=0}^{\infty} c^1_m(x) t^m, \dots, \sum_{m=0}^{\infty} c^r_m(x) t^m, \dots, \sum_{m=0}^{\infty} c^n_m(x) t^m \Big)^T \tag{3.1}$$

which absolutely converges for all $t \in R$, where the coefficients are found by recurrent equation (2.3) with the initial values $c^r_0 = x_r \quad (r=1,\dots,n)$;

2) For every $t \in R$ the derivative $D\varphi_t(x) : T_x(M) \longrightarrow T_y(M)$, $y = \varphi_t(x)$ can be presented by the matrix of series

$$D\varphi_t(x) = \sum_{m=0}^{\infty} d^m t^m ,$$

where matrices of coefficients d^m (m=0,1,2, ...) are determined by the following recurrent equation

$$d^{m+1}_{jr} = \frac{1}{m+1} \sum_{K=1}^{\infty} \sum_{\substack{\alpha 1 = 1,\dots,n \\ \cdots\cdots \\ \alpha K = 1,\dots,n \\ S0+S1+\dots+SK=m}} f^{\alpha 1,\dots,\alpha K}_{rS0} \sum_{h=1,\dots,K} d^{sh}_{jsh} \left(c^{\alpha h}_{sh}\right)^{-1} c^{\alpha 1}_{s1} \dots c^{\alpha K}_{sK} \tag{3.2}$$

$d^0 = I$ (identity matrix).

Proof. The right part of system (1.3) is an entire function in the complex (z, ω) space. Therefore by theorem 1 motion $\varphi_t(x)$ can be presented by the vector of series (3.1) which absolutely converges in a some interval $(-\delta, \delta)$, $\delta > 0$. Then vector of series (3.1) also converges at the points $\pm\delta$, because the motion $\varphi_t(x)$ is bounded. Further the condition of continuation of motion $\varphi_t(x)$ follows that (3.1) also absolutely converges in the interval $(-\delta-\varepsilon, \delta+\varepsilon)$ for some $\varepsilon > 0$, consequently in all the real axis. Finally, direct calculation of Jacobi matrix of vector of series (3.1) at point x gives equation (3.2). Theorem 3 is proved.

For example we consider the bilinear system

$$x' = Ax + B(x,x)$$

$$A \in R^{n \times n}, \quad B_r(x,x) = \sum_{\nu,\mu=1,\dots,n} B^{\nu\mu}_r x_\nu x_\mu , \; (B^{\nu\mu}_r \in R) , \quad r = 1,\dots,n .$$

The Taylor coefficients of the motion $\varphi(t)$ are defined by the formula

$$c_{m+1} = \frac{1}{m+1} \left(A c_m + \beta(c_s, c_{m-s}) \right), \quad \beta_r(c_s, c_{m-s}) = \sum_{\substack{\nu,\mu=1,\dots,n \\ s=0,1,\dots,m}} B^{\nu\mu}_r c^\nu_s c^\mu_{m-s} .$$

Then the Taylor coefficients of derivative $D\varphi_t(x)$ is found by the following

$$d^{m+1}_{jr} = \frac{1}{m+1} \Big(\sum_{\alpha=1}^{n} A_{r\alpha} d^m_{j\alpha} + \sum_{\substack{\nu,\mu=1,\dots,n \\ s=0,1,\dots,m}} B^{\nu\mu}_r \left(d^s_{j\nu} c^\mu_{m-s} + d^{m-s}_{j\mu} c^\nu_s \right) \Big) .$$

Thus the algorithm of calculation of the coefficients of flow's derivative is also simple. For example, for calculation of 200 first coefficients of the solution and the derivative of the corresponding flow of Kostitzin equation (example 1) by EC 1060 only one minute has been spended.

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Ву Суан Минь E5-85-456
О разложении решений обыкновенных
дифференциальных уравнений в степенные ряды

Даются рекуррентные формулы последовательного вычисления коэффициентов Тейлора решений задачи Коши для обыкновенных дифференциальных уравнений с аналитической правой частью. Дается также рекуррентная формула разложения в степенные ряды производных потоков, порождаемых этими уравнениями.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Vu Xuan Minh E5-85-456
On Expansion of Solutions of Ordinary
Differental Equations into Power Series

The recurrent formulae of successive calculation of Taylor coefficients of Cauchy problem solutions for ordinary differental equations with analytical right part are obtained. For the flows generated by the eguations the recurrent formula of expansion of their derivatives into power series is elaborated too.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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