объединенный
институт
ядерных
исследований
дубна

E5-85-398

Vu Xuan Minh

ON STABILITY OF MOTIONS OF BILINEAR DYNAMICAL SYSTEMS

Submitted to "ДАН СССР"

1985

We consider the system of ordinary differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x} + B(\vec{x},\vec{x}), \tag{1}$$

where $x \in R^n$, $A \in R^{n\times n}$, $t \in R$, $B(\vec{x},\vec{x})$ is a vector-column of n quadratic forms

$$B_r(\vec{x},\vec{x}) = \sum_{\substack{i\leq j\\ i,j=1,\dots,n}} B_r^{ij} x_i x_j \qquad (B_r^{ij} \in R\ ,\ \ r=1,\dots,n). \tag{2}$$

The dynamical system generated by the equation (1) is called bilinear. Many nonlinear evolution equations in hydrodynamics (dissipative dynamical systems) [1-3], nonlinear optics [4], biosphere [5], mathematical biophysics [6,7], and smooth control bilinear dynamical system [8] belong to the class of bilinear dynamical systems.

In this paper the criterions of the stability of the motions of system (1), orbits of which are found in a bounded part of the space R^n, are defined. Such motions are called steady by Lagrange [9]. We will mark the operation of transposition in T.

Lemma 1. Let $\phi(t)$ be a motion of system (1) steady by Lagrange. Then it can be analytically continued on all the complex plane, i.e., $\phi(z)$, where $z \in C$, is an entire function.

The proof of lemma 1 uses Cauchy theorem for differential equations with complex variable and Abel theorem about the convergency of power series.

Lemma 2. Let $\phi(t)$ be a motion of system (1) steady by Lagrange. Then all its derivatives satisfy the following recurrent equation

$$\phi^{(m+1)} = A\phi^{(m)} + \beta(\phi^{(k)},\phi^{(m-k)}),\quad k=1,\dots,m,\quad \phi^{(m)} = \frac{d^m\phi}{dt^m}.$$

where

$$\beta(\phi^{(k)},\phi^{(m-k)}) = [\beta_1(\phi^{(k)},\phi^{(m-k)}),\dots,\ \beta_r(\phi^{(k)},\phi^{(m-k)}),\dots,$$

$$\beta_n(\phi^{(k)},\phi^{(m-k)})]^T,$$

$$\beta_r(\phi^{(k)},\phi^{(m-k)}) = \sum_{\substack{i\le j\\ i,j=1,\dots,n}} B_r^{ij}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\phi_i^{(k)}\phi_j^{(m-k)}.$$

Lemma 2 is proved by induction on m.

For arbitrary point of the phase space $\phi_0 \in R^n$ we construct the vector of series

$$S(t) = [\sum_{m=0}^{\infty} C_m^1 t^m, \dots, \sum_{m=0}^{\infty} C_m^r t^m, \dots, \sum_{m=0}^{\infty} C_m^n t^m]^T, \quad (3)$$

where the coefficients are determined by the following recurrent equation

$$C_{m+1} = \frac{1}{m+1}(AC_m + \beta(C_k, C_{m-k})), \quad k=1,\dots,m \quad (4)$$

$$C = \phi_0, \quad C_m = [C_m^1, \dots, C_m^r, \dots, C_m^n]^T$$

$$\beta(C_k, C_{m-k}) = [\beta_1(C_k, C_{m-k}), \dots, \beta_r(C_k, C_{m-k}), \dots, \beta_n(C_k, C_{m-k})]^T$$

$$\beta_r(C_k, C_{m-k}) = \sum_{\substack{i\le j\\ i,j=1,\dots,m\\ k=0,1,\dots,m}} B_r^{ij} C_k^i C_{m-k}^j. \quad (5)$$

With the help of lemmas 1 and 2 we have the following statement.

Theorem 1. Let $\phi(t)$ be a motion of systems (1) passing through the point $\phi_0 \in R^n$ at the moment $t=0$. If it is steady by Lagrange, then for any $t \in R$ vector of series (3) absolutely converges. Besides $\phi(t) = S(t)$.

Let D_m (m = 0,1,2,...) be a sequence of the $(n\times n)$-matrices defined in the following

$$D_0 = A + A_0, \quad (A_0)_{rk} = \sum_{i=1}^{n} d_{rk}^i C_0^i, \quad (6)$$

$$(D_m)_{rk} = \sum_{i=1}^{n} d_{rk}^i C_m^i, \quad (7)$$

where $d_{rk}^i = B_r^{ik}$ $(i \le k)$, $d_{rk}^k = 2B_r^{kk}$ $(i=k)$, $d_{rk}^i = B_r^{ki}$ $(i > k)$.

Lemma 3. Let $\phi(t)$ be a motion of system (1), passing through the point $\phi_0 \in R^n$ at the moment $t=0$. Then the first variation of system (1), relative to $\phi(t)$, has the form

$$\frac{d\vec{y}}{dt} = (\sum_{m=0}^{\infty} D_m t^m)\vec{y}, \quad \vec{y} = [y_1, \dots, y_r, \dots, y_n]^T, \quad (8)$$

where the matrices of coefficients D_m (m = 0,1,2,...) are determined by (6) and (7). Besides the matrix $(\sum_{m=0}^{\infty} D_m t^m)$ is bounded.

Lemma 3 is proved by the direct calculation of the first variation of system (1).

By induction on k it is easily to receive the following

$$\vec{y}^{(k)} = \sum_{s=0}^{k-1}\frac{(k-1)!}{s!(k-1-s)!}(\sum_{m=0}^{\infty}\frac{m!}{(m-s)!}D_m t^{m-s})\vec{y}^{(k-1-s)}, \quad \vec{y}^{(k)} = \frac{d^k\vec{y}}{dt^k}. \quad (9)$$

Let

$$\phi(t) = \sum_{k=0}^{\infty} G_k t^k \quad (10)$$

be a matrix of series the coefficients of which are determined with the help of the recurrent equation

$$G_k = \frac{1}{k}\sum_{s=0}^{k-1} D_s G_{k-1-s}, \quad G_0 = I \quad \text{(identity matrix)}. \quad (11)$$

Following from theorem 1, lemma 3, expression (9) and the integral representation of the solution of system (8) we have

Theorem 2. Let $\phi(t)$ be a motion of system (1) passing through the point $\phi_0 \in R^n$ at the moment $t=0$ is steady by Lagrange. Then we have the following statements:

1) Matrix of power series (10) absolutely converges for any $t \in R$;

2) $\phi(t)$ is steady by Lyapunov if and only if $\phi(t)$ is a bounded function;

3) $\phi(t)$ is asymptotically steady if and only if $\psi(t)$ tends to zero when $t \to \infty$;

4) If vector of series $[S(t) - C_0]$ has although one root, then the orbit of $\phi(t)$ is asymptotic steady if and only if $n-1$ characteristic roots of matrix

$$Q = \frac{1}{t^*}\ln\psi(t^*)$$

have negative real parts, where t^* is any its root.

In conclusion author expresses his sincere gratitude to Prof. N.N.Govorun for permanent attention to the work.

REFERENCES

1. Strange Attractors. "Mir", Moscow, 1981.
2. Richtmyer R.D. Principles of Advanced Mathematical Physics. Springer-Verlag, New York-Heidelberg-Berlin, 1981, vol.2.

3. Rabinovich M.I. Usp.Fiz.Nauk, 1978, vol.125, No.2, p.123-168.
4. Afraimovich V.S., Bykov V.V., Shilnikov L.P. Dokl.Akad. Nauk SSSR, 1977, No.2, p.336-339.
5. Kostitzin V.A. Evolution de l'atmosphere. Herman, Paris, 1935.
6. Murray J.D. Lectures on Nonlinear-Differential-Equation Models in Biology. Clarendon Press, Oxford, 1977.
7. Romanovskij Yu.M., Stepanova N.V., Chernauskij D.S. Math.Biophys., "Nauka", M., 1984.
8. Mathematical Methods in Systems Theory. "Mir", M., 1979.
9. Nemytskij V.V., Stepaniv V.V. Qualitative Theory of Differential Equations. "Gostechizdat". M.-L., 1947.

Received by Publishing Department
on May 29, 1985.

Ву Суан Минь E5-85-398
Об устойчивости движений билинейных динамических систем

Даются критерии устойчивости ограниченных движений динамических систем, порождаемых системой обыкновенных дифференциальных уравнений с билинейной правой частью. Они основаны на разложении таких движений в степенные ряды.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Vu Xuan Minh E5-85-398
On Stability of Motions of Bilinear Dynamical Systems

The criterions of the stability of the bounded motions of the dynamical systems generated by the system of ordinary differential equations with the bilinear right part are defined. They are based on the expansion of such motions into power series.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985