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# A NOTE ON A MODIFIED COUNTER WITH PROLONGING DEAD TIME

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## 1. INTRODUCTION

The dead time, B, of the counter with prolonging dead time is defined as the time period during that at least one impulse of the particle is present. It is produced after registration of all impulses of emitted particles. The idle period, I, of the counter is defined as the time period during that the counter is idle. The cycle, C, is the time period between two successive registered particles. It is clear that the cycle is the sum of the dead time and the successive idle period.

The dead time distribution is known only in special cases. Takács[12,13] has derived it for the case of a Poisson homogeneous input process of emitted particles. Some limit properties, in language of $M/G/\infty$ queue, may be found in[1]. The distribution of the cycle or its Laplace transform were obtained for non-modified counter in[6,10,11,14]. The joint Laplace transform of the cycle and the number, $\nu$, of particles arrived at the counter during the dead time, for the modified counter, is in[8]. The joint distribution of the dead time and the successive idle period is known only for a discrete modified counter[5,7,9].

In the present note we derive an integral equation for the joint distribution of the dead time and the successive idle period for the modified counter with prolonging dead time. The asymptotic exponential law of the dead time of the counter of order m with the Poisson homogeneous input process is proved. We introduce classes of t-recurrent and t-semirecurrent events and apply them to the counter theory.

## 2. PRELIMINARIES

We suppose that the counter is idle before the registration process. The modified counter with prolonging dead time is a counter for which the first dead time is produced by sequences of impulses, $\{\chi_n\}_{n=1}^{\infty}$, and interarrival times, $\{T_n\}_{n=1}^{\infty}$. $\{\chi_n\}_{n=1}^{\infty}$ are assumed to be independent positive random variables with the distribution functions

$$H_n(t) = P(\chi_n < t), \; n \geq 1 . \qquad (2.1)$$

The sequence $\{\chi_n\}_{n=1}^{\infty}$ is independent of $\{T_n\}_{n=1}^{\infty}$, where $T_n = \tau_{n+1} - \tau_n$, $n \ge 1$ and $\{\tau_n\}_{n=0}^{\infty}$ is a sequence of the arrival moments with $0 = \tau_0 < \tau_1 < \dots$. We assume that $\{T_n\}_{n=1}^{\infty}$ is a sequence of positive random variables with the distribution functions

$$F_n(t) = P(T_n < t), \quad n \ge 1. \tag{2.2}$$

Moreover, we assume that all successive dead time are resumed with the initial conditions, independently of the previous dead times. This modified counter with prolonging dead time will be denoted by $\mathfrak{C} = (F_1, F_2, \dots; H_1, H_2, \dots)$. Some of its basic properties were established in /8/.

For a given modified counter with prolonging dead time (a modified counter for short), $\mathfrak{C} = (F_1, F_2, \dots; H_1, H_2, \dots)$, it is convenient to consider a sequence of modified counters, $\{\mathfrak{C}_k\}_{k=1}^{\infty}$, where $\mathfrak{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, $k \ge 1$. Suppose that the first dead time of any counter $\mathfrak{C}_k$, $k \ge 1$, is produced by sequences $\{\chi_n^k\}_{n=1}^{\infty}$ and $\{T_n^k\}_{n=1}^{\infty}$, where $\chi_n^k = \chi_{k+n-1}$, $T_n^k = T_{k+n-1}$. Also we define the dead time, $B_k$, the idle period, $I_k$, the cycle, $C_k$, and the number, $\nu_k$, of the emitted particles arrived at the counter during the dead time, for any $k \ge 1$.

We obtain an important class of modified counters, named modified counters of order $m$, $m \ge 1$, if there is an integer $m$ such that $F_m = F_{m+1} = \dots$, $H_m = H_{m+1} = \dots$, and we write simply $\mathfrak{C} = (F_1, \dots, F_m; H_1, \dots, H_m)$. It is clear that the modified counter of order 1 is the (non-modified) counter with prolonging dead time.

Let us put

$$A_n^k = \{\chi_k < T_k + \dots + T_{k+n-1},\ \chi_{k+1} < T_{k+1} + \dots + T_{k+n-1},\ \dots,\ \chi_{k+n-1} < T_{k+n-1}\},\quad n \ge 1,\ k \ge 1. \tag{2.3}$$

Then $\{A_n^k : n \ge 1, k \ge 1\}$ are semirecurrent events /12/ that is, for any $k$ and integers $i_j$ with

$$1 \le i_0 < i_1 < \dots < i_n, \quad n \ge 1, \tag{2.4}$$

we have

$$P(A_{i_1}^k \dots A_{i_n}^k \mid A_{i_0}^k) = P(A_{i_1 - i_0}^{k+i_0} \dots A_{i_n - i_0}^{k+i_0}). \tag{2.5}$$

As has been proved in /8/, for $P_n^k = P(\nu_k = n)$ we have

$$\left.\begin{aligned} P_1^k &= P(A_1^k), \\ P_n^k &= P(A_n^k) - \sum_{j=1}^{n-1} P(A_j^k) P_{n-j}^{k+j}, \quad n \ge 2. \end{aligned}\right\} \tag{2.6}$$



For every $\nu_k$ we denote by $f_k(z)$ its generating function, that is, $f_k(z) = \sum_{n=1}^{\infty} P_n^k z^n$, $|z| \le 1$, $k \ge 1$.

In order to determine the joint Laplace transform, $\Phi_k(s,z) = M(e^{-sC_k} z^{\nu_k})$ of the modified counter $\mathfrak{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, $k \ge 1$, let us put $a_k(s) = \int_0^{\infty} e^{-sx} dF_k(x)$, $s \ge 0$. The new counter $\mathfrak{C}_k^s = (F_k^s, F_{k+1}^s, \dots; H_k, H_{k+1}, \dots)$, $s \ge 0$, $k \ge 1$, where $F_k^s(x) = a_k(s)^{-1} \int_0^x e^{-st} dF_k(t)$, determines the integer-valued random variable $\nu_k^s$ defined as the number of emitted particles arrived at the modified counter $\mathfrak{C}_k^s$ during its dead time. Let us put $P_n^k(s) = P(\nu_k^s = n)$, $n \ge 1$, $k \ge 1$, $s \ge 0$.

Theorem 2.1. The joint Laplace transform of the modified counter $\mathfrak{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, $k \ge 1$, is given by

$$\Phi_k(s,z) = \sum_{n=1}^{\infty} \Big(\prod_{i=0}^{n-1} a_{k+i}(s)\Big) z^n P_n^k(s), \quad s \ge 0,\ |z| \le 1,\ k \ge 1. \tag{2.7}$$

In particular, if $F = F_1 = F_2 = \dots$, then

$$\Phi_k(s,z) = f_k^s(a(s) z), \quad s \ge 0,\ |z| \le 1,\ k \ge 1, \tag{2.8}$$

where $f_k^s$ is the generating function of $\nu_k^s$ and $a(s) = a_1(s) = a_2(s) = \dots$.

If $M(T_1) < \infty$, then

$$M(C_k) = M(T_1) M(\nu_k). \tag{2.9}$$

Proof. Since $C_k = T_1^k + \dots + T_{\nu_k}^k$, we have

$$\Phi_k(s,z) = \sum_{n=1}^{\infty} \int_{A_n} e^{-s(t_1 + \dots + t_n)} z^n dF_k(t_1) \dots dF_{k+n-1}(t_n) \times dH_k(x_1) \dots dH_{k+n-1}(x_n),$$

where the integration area $A_n$ has the following form

$$(x_1 < t_1)^c, \quad \begin{pmatrix} x_1 < t_1 + t_2 \\ x_2 < t_2 \end{pmatrix}^c, \quad \dots, \quad \begin{pmatrix} x_1 < t_1 + \dots + t_{n-1} \\ \vdots \\ x_{n-1} < t_{n-1} \end{pmatrix}^c,$$

$$\left(\begin{array}{l} x_1 < t_1 + \dots + t_n \\ \vdots \\ x_n < t_n \end{array}\right)$$

(here the sign "c" denotes the complement of the set mentioned in the parentheses).

The Wald identity inplies the mean value of $C_k$. Q.E.D.

Using the one-to-one correspondence between the distribution functions and their Laplace transform, we may assume, due to (2.7) and (2.8), that

$$V_k^i(t) = P(C_k < t, \ \nu_k = i), \ i \ge 1, \ k \ge 1, \qquad (2.10)$$

is known.

## 3. DEAD TIME AND IDLE PERIOD

The dead time and the idle period are, in general, stochastically dependent random variables. The exceptions are, for example, in the case of the modified counters with the Poisson homogeneous input process of emitted particles, and the counters of order 1 with constant impulse lengths. The joint distribution of the dead time and the successive idle period of the modified counter is known only in the case of discrete modified counter [7]. In the present part we derive the integral equation to the joint distribution, $W_k(z,u) = P(B_k < z, I_k < u), k \ge 1$, and the distribution function $Z_k(z) = P(B_k < z)$, $k \ge 1$, respectively, for the modified counter $\mathbb{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$.

We shall suppose $F_k(0+) = 0$, $k \ge 1$.

<u>Theorem 3.1.</u> (i) For all $z \ge 0$, $u \ge 0$, $k \ge 1$

$$W_k(z,u) = \int_0^z (F_k(y+u) - F_k(y+))\,dH_k(y) + \int_0^z\int_x^z (W_{k+1}(z-x,u) - W_{k+1}(y-x,u))\,dH_k(y)\,dF_k(x) + \sum_{r=1}^{\infty}\ \sum_{i_1,\dots,i_r=1}^{\infty} \times$$

$$\times \int_0^z\int_x^z\int_0^{y-x} (W_{m_{r+1}}(z-x-t,u) - W_{m_{r+1}}(y-x-t,u))\,dV_{m_1}^{i_1} * \dots * V_{m_r}^{i_r}(t) \times dH_k(y)\,dF_k(x), \qquad (3.1)$$

where $m_1 = k+1$, $m_2 = m_1 + i_1, \dots, m_{r+1} = m_r + i_r$ and the sign "$*$" denotes the convolution.



(ii) For all $z \ge 0$ and $k \ge 1$

$$Z_k(z) = \int_0^z (1 - F_k(y+))\,dH_k(y) + \int_0^z\int_x^z (Z_{k+1}(z-x) - Z_{k+1}(y-x))\,dH_k(y)\,dF_k(x) + \sum_{r=1}^{\infty}\ \sum_{i_1,\dots,i_r=1}^{\infty} \int_0^z\int_x^z\int_0^{y-x} \times \qquad (3.2)$$

$$\times (Z_{m_{r+1}}(z-x-t) - Z_{m_{r+1}}(y-x-t))\,dV_{m_1}^{i_1} * \dots * V_{m_r}^{i_r}(t)\,dH_k(y)\,dF_k(x),$$

where $m_1 = k+1$, $m_2 = m_1 + i_1, \dots, m_{r+1} = m_r + i_r$.

<u>Proof.</u> Since the proof of (3.1) is analogous to that of (3.2), we prove only (3.2). The event $\{B_k < z\}$ is the union of two disjoint events $A_1$ and $A_2$, where $A_1 = \{B_k < z, \chi_1^k < T_1^k\}$ and $A_2 = \{B_k < z, \chi_1^k \ge T_1^k\}$.

Clearly $P(A_1) = \int_0^z (1 - F_k(y+))\,dH_k(y)$;

Under the condition $\{0 < x = T_1^k \le \chi_1^k = y < z\} = B$ say,

$$P(A_2 | B) = P(y - x \le B_{k+1} < z - x) + \sum_{r=1}^{\infty} P(C_{\mu_1} + C_{\mu_2} + \dots + C_{\mu_r} < y - x \le$$

$$\le C_{\mu_1} + \dots + C_{\mu_r} + B_{\mu_{r+1}} < z - x),$$

where $\mu_1 = k+1, \mu_2 = \mu_1 + \nu_{\mu_1}, \dots, \mu_{r+1} = \mu_r + \nu_{\mu_r}$, $r \ge 1$.

Using the Markov property of the random variables $\mu_1, \dots, \mu_{r+1}$ and (2.10), we have

$$P(A_2 | B) = P(y - x \le B_{k+1} < z - x) + \sum_{r=1}^{\infty}\ \sum_{i_1,\dots,i_r=1}^{\infty} P(C_{m_1} + \dots + C_{m_r} \le$$

$$< y - x \le C_{m_1} + \dots + C_{m_r} + B_{m_{r+1}} < z - x, \ \nu_{m_1} = i_1, \dots, \nu_{m_r} = i_r)$$

which implies (3.2). Q.E.D.

<u>Corollary 3.1.1.</u> For the modified counter of order 1, $\mathbb{C} = (F;H)$, we have

$$W(z,u) = \int_0^z (F(y+u) - F(y+))\,dH(y) + \int_0^z\int_x^z\int_0^{y-x} (W(z-x-t,u) - W(y-x-t,u))\,dN(t)\,dH(y)\,dF(x), \quad z \ge 0, \ u \ge 0, \qquad (3.3)$$

where $N(t)$ is the renewal function of the cycle, that is, $N(t) = \sum_{n=0}^{\infty} G^{*n}(t)$, and $G^{*n}(t)$ is the n-th convolution of the distribution function, G, of the cycle with itself.

Corollary 3.1.2. For the modified counter of order 2, $\mathfrak{C} = (F, F^*; H, H^*)$, we have

$$W(z, u) = \int_0^z (F(y+u) - F(y+))\,dH(y) + \int_0^z \int_x^z \int_0^{y-x} (W^*(z-x-t, u) - W^*(y-x-t, u))\,dN^*(t)\,dH(y)\,dF(x), \quad z \geq 0, \quad u \geq 0, \tag{3.4}$$

where $W^*$ is the joint distribution of the dead time and the successive idle period, and $N^*$ is the renewal function of the cycle for the modified counter $\mathfrak{C}^* = (F^*; H^*)$.

## 4. LIMIT PROPERTIES

In this part we shall deal with the modified counter of order m, $\mathfrak{C} = (F, \ldots, F; H_1, \ldots, H_m)$, with the homogeneous Poisson input process of emitted particles with the rate $\lambda$. In this case the dead time and the idle period are independent random variables and, moreover, the idle period distribution is exponential with the same parameter $\lambda$.

Theorem 4.1. Let $\mathfrak{C} = (F, \ldots, F; H_1, \ldots, H_m)$ be a modified counter of order m with (i) $H_1(t) \geq \ldots \geq H_m(t)$, $t \geq 0$; (ii) $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$; (iii) $\int_0^\infty t^2 dH_m(t) < \infty$; (iv) $H_m(0+) = 0$. Then

$$\lim_{\lambda \to \infty} P(B_k / M(B_k) > t) = e^{-t}, \quad t > 0, \quad k = 1, \ldots, m. \tag{4.1}$$

Proof. According to the above note, we conclude that $M(B_k) = M(C_k) - 1/\lambda$, $M(e^{-sB_k}) = M(e^{-sC_k})(\lambda + s)/\lambda$, $k = 1, \ldots, m$. Due to either (2.8) and (2.9) or /4/ we have

$$M(e^{-sC_k/M(B_k)}) = f_k^{\bar{s}}(a(\bar{s})),$$

where $\bar{s} = s/(M(C_k) - 1/\lambda)$.

Therefore

$$M(e^{-sB_k/M(B_k)}) = a(\bar{s})\,[\,\phi_k^{\bar{s}}(a(\bar{s})) + \sum_{j=1}^{m-k-1} P_j^k(\bar{s})\,a(\bar{s})^j$$



$$(\phi_m^{\bar{s}}(a(\bar{s})) - \phi_{k+j}^{\bar{s}}(a(\bar{s})))]/[1 - a(\bar{s}) + a(\bar{s})\,\phi_m^{\bar{s}}(a(\bar{s}))]\,/$$
$$/a(\bar{s}) = [1 + \sum_{j=1}^{m-k-1} P_j^k(\bar{s})\,a(\bar{s})^j\,(\phi_m^{\bar{s}}(a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s})) - \phi_{k+j}^{\bar{s}}(a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s})))]/[(1 - a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s})) + a(\bar{s})\,\phi_m^{\bar{s}}(a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s}))], \tag{4.2}$$

where

$$\phi_k^s(z) = 1 - \sum_{n=0}^{\infty} (P(A_n^k(s)) - P(A_{n+1}^k(s)))\,z^n,$$

$P(A_0^k(s)) \equiv 1$, $k \geq 1$, and $P(A_n^k(s))$ corresponds to that of the associated counter $\mathfrak{C}_k^s$, $s \geq 0$.

The proof will be divided into four steps.

(I) First of all we prove that, under our conditions, for any $\lambda > 0$, $|\phi_k^{\bar{s}\prime}(1)| < \infty$ and $\lim_{\lambda \to \infty} \phi_k^{\bar{s}\prime}(1) = 0$, $k = 1, \ldots, m$. We note that

$$-\phi_k^{s\prime}(1) = \sum_{n=1}^{\infty} (P(A_n^k(s)) - e^{-(\lambda+s)D}), \quad s \geq 0, \tag{4.3}$$

where $D = \int_0^\infty t\,dH_m(t)$. Due to that and (i), we have that

$|\phi_1^{\bar{s}\prime}(1)| < \infty$ iff $|\phi_2^{\bar{s}\prime}(1)| < \infty$ iff, etc., iff $|\phi_m^{\bar{s}\prime}(1)| < \infty$.

In order to prove (I) it is sufficient to show the same for $\phi_m'(1) = \phi_m^{s\prime}(1)$ when $s = 0$. Indeed, $\phi_m^{\bar{s}}$ (corresponds to $\phi_m$ when $F(t) = 1 - e^{-\Lambda t}$, $t > 0$, where $\Lambda = \lambda(1 + s/(e^{\lambda D} - 1))$, and if $\lambda \to \infty$, then $\Lambda \to \infty$.

It is clear that

$$P(A_1^1) = \lambda \int_0^\infty H_1(t) e^{-\lambda t}\,dt,$$
$$P(A_n^m) = \lambda^{n+1}/n! \int_0^\infty \left( \int_0^t H_m(x)\,dx \right)^n e^{-\lambda t}\,dt, \quad n \geq 1, \tag{4.4}$$

$$\phi_m(z) = 1 - \lambda \int_0^\infty \exp\left(-\lambda \int_0^y (1 - zH_m(u))\,du\right)(1 - H_m(y))\,dy. \tag{4.5}$$

Since $D_2 = \int_0^\infty y(1 - H_m(y))\,dy < \infty$, the integral

$$\int_0^\infty (1 - H_m(y)) \exp\left(-\lambda \int_0^y (1 - zH_m(u))\,du\right) \int_0^y H_m(u)\,du\,dy$$

converges uniformly in $z \in [0,1]$. Therefore we may take the derivative of (4.5) with respect to $z=1$ and obtain

$$\phi'_m(z) = -\lambda^2 \int_0^\infty (1 - H_m(y)) \exp(-\lambda \int_0^y (1 - zH_m(u))du \int_0^y H_m(u)du\,dy .$$

Let $A>0$ be arbitrary. Denote

$$I_1(\lambda, A) = \lambda^2 \int_0^A (1 - H_m(y)) \exp(-\lambda \int_0^y (1 - H_m(u))du) \int_0^y H_m(u)\,du\,dy ,$$

$$I_2(\lambda, A) = \phi'_m(1) - I_1(\lambda, A) .$$

It is obvious that $I_2(\lambda, A) \le \lambda^2 \exp(-\lambda \int_0^A (1 - H_m(u))du D_2$. Hence, for an arbitrary $\epsilon > 0$, there is $\Lambda_2(\epsilon, A) > 0$ so that $I_2(\lambda, A) < \epsilon/4$ whenever $\lambda > \Lambda_2(\epsilon, A)$.

Using the per-partes integration method we conclude

$$I_1(\lambda, A) = I_3(\lambda, A) + I_4(\lambda, A) + I_5(\lambda, A),$$

where

$$I_3(\lambda, A) = -\lambda \exp(-\lambda \int_0^A (1 - H_m(u))\,du) \int_0^A H_m(u)\,du ,$$

$$I_4(\lambda, A) = \exp(-\lambda \int_0^A (1 - H_m(u))du) ,$$

$$I_5(\lambda, A) = \lambda \int_0^A \exp(-\lambda \int_0^y (1 - H_m(u))du)\,dy - 1 .$$

It is clear that

$$|I_3(\lambda, A)| \le \lambda A \exp(-\lambda \int_0^A (1 - H_m(u))du) .$$

Hence there is $\Lambda_3(\epsilon, A) > 0$ so that $|I_3(\lambda, A)| < \epsilon/4$ whenever $\lambda > \Lambda_3(\epsilon, A)$. Similarly there is $\Lambda_4(\epsilon, A) > 0$ so that $|I_4(\lambda, A)| < \epsilon/4$ when $\lambda > \Lambda_4(\epsilon, A)$.

The condition $H_m(0+) = 0$ entails that, for any $\epsilon_1 > 0$, there is $A(\epsilon_1)$ with $1 - \epsilon_1 \le 1 - H_m(u) < 1$ whenever $0 < u < A(\epsilon_1)$. Therefore

$$\lambda \int_0^{A(\epsilon_1)} e^{-\lambda y}\,dy - 1 \le I_5(\lambda, A) \le \lambda \int_0^{A(\epsilon_1)} e^{-\lambda(1-\epsilon_1)y}\,dy - 1$$



and

$$-e^{-\lambda A(\epsilon_1)} \le I_5(\lambda, A) \le (1 - e^{-\lambda A(\epsilon_1)})/(1 - \epsilon_1) - 1 < \epsilon_1/(1 - \epsilon_1) .$$

Using the inequality $\epsilon_1/(1 - \epsilon_1) < 2\epsilon_1$ which holds for $0 < \epsilon_1 < 1/2$ we get $|I_5(\lambda, A)| \le \max(e^{-A(\epsilon_1)}, 2\epsilon_1)$.

Now, for a given $\epsilon_1 > 0$, we may choose $\Lambda_5(\epsilon_1) > 0$ so that $e^{-\lambda A(\epsilon_1)} < 2\epsilon_1$ whenever $\lambda > \Lambda_5(\epsilon_1)$. From this restriction we may find $\epsilon_1$ and $A$ so that $\epsilon_1 = \epsilon/8$ and $A = A(\epsilon/8)$. Hence, if $\lambda > \max_{2 \le i \le 5} (\Lambda_i(\epsilon, A(\epsilon/8)))$ then $|\phi'_m(1)| < \infty$.

(II) From (4.4) we have that $\lim_{\lambda\to\infty} P(A_1^1) = 0$, and the condition (i) of the Theorem implies $P(A_{n+1}^k) \le P(A_n^{k+1})$, $n \ge 1$, $k = 1, \dots, m-1$, and consequently, $P(A_n^1) \le P(A_{n-m+1}^m)$, $n \ge m$. Therefore, this, (4.3) and $\lim_{\lambda\to\infty} \phi'_m(1) = 0$ yield $\lim_{\lambda\to\infty} \phi'_k(1) = 0$, $k = 1, \dots, m$. Hence $\lim_{\lambda\to\infty} \phi_k^{\bar{s}\prime}(1) = 0$, $k = 1, \dots, m$.

The theory of semirecurrent events [3] entails that $\lim_{n\to\infty} P(A_n^k)$, $k = 1, \dots, m$, exists and equals, in our case, $e^{-\lambda D}$, so that

$$\lim_{n\to\infty} P(A_n^k(s)) = e^{-(\lambda+s)D} \equiv p_s, \quad s \ge 0 .$$

It is evident that $\lim_{\lambda\to\infty} (1 - a(\bar{s}))/p = s$, where $p = p_s$ for $s = 0$.

(III) Here we show that

$$\lim_{\lambda\to\infty} \phi_i^{\bar{s}}(a(\bar{s}))/p_{\bar{s}} = 1, \quad i = 1, \dots, m . \tag{4.6}$$

We note that, for any fixed $\lambda > 0$, $\phi_i^{\bar{s}}(z)$ is non-decreasing function for each $0 \le z \le 1$. Using that and the elementary inequality $e^{-x} \ge 1 - x$, $x \ge 0$, we obtain $p_{\bar{s}} = \phi_i^{\bar{s}}(1) \le \phi_i^{\bar{s}}(a(\bar{s})) \le$ $< \phi_i^{\bar{s}}(1 - \bar{s}/\lambda)$. Due to the inequality $(1-x)^n \ge 1 - nx$, $|x| \le 1$, we have, for sufficiently large $\lambda > 0$,

$$\phi_i^{\bar{s}}(1 - \bar{s}/\lambda) = 1 - \sum_{n=0}^{\infty} (1 - \bar{s}/\lambda)(P(A_n^i(\bar{s})) - P(A_{n+1}^i(\bar{s}))) =$$

$$= p_{\bar{s}} - \bar{s}/\lambda\, \phi_i^{\bar{s}\prime}(1) .$$

Hence

$$1 \le \phi_i^{\bar{s}}(a(\bar{s}))/p_{\bar{s}} \le 1 - \bar{s}/\lambda p_{\bar{s}}^{-1}\, \phi_i^{\bar{s}\prime}(1) \le 1 - \bar{s}/\lambda p^{-1}\, \phi_i^{\bar{s}\prime}(1) .$$

Using (2.9) and

$$M(\nu_k) = (\phi_k'(1) - \phi_m'(1) + 1 - \sum_{j=1}^{m-k-1} P_j^k(\phi_m'(1) - \phi_{k+j}'(1)))/p,$$

we conclude that $\lim_{\lambda\to\infty} \bar{s}/\lambda p^{-1} = s$, so that (4.6) holds.

(IV) Using (4.6) we have

$$\lim_{\lambda\to\infty} \phi_i^{\bar{s}}(a(\bar{s}))/\phi_j^{\bar{s}}(a(\bar{s})) = 1, \quad i, j = 1, \dots, m.$$

Finally, it is easy to show that $\lim_{\lambda\to\infty} p/p_{\bar{s}} = 1$, which completely proves the theorem. Q.E.D.

## 5. t-SEMIRECURRENT EVENTS

The notion of semurecurrent events (see (2-4)-(2.6)) plays an important role in the counter theory in determining some of basic properties of modified counters with prolonging dead time. Here we present the generalization of this notion to t-semirecurrent events and then apply them to the problem of determination of the joint distribution of $I_k$ and $\nu_k$ for the modified counter $\mathcal{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$.

We say that a system of events $\{A_n^k(t): n \geq 1, k \geq 1, t \in [0, \infty]\}$ is t-semirecurrent if, for all $k \geq 1$, integers $i_j$ with $1 \leq i_0 < i_1 < \dots < i_n$, $n \geq 1$, and $t_0, \dots, t_n \in [0, \infty]$, we have

$$P(A_{i_1}^k(t_1) \dots A_{i_n}^k(t_n) \mid A_{i_0}^k(t_0)) = P(A_{i_1-i_0}^{k+i_0}(t_1) \dots A_{i_n-i_1}^{k+i_0}(t_n)). \quad (5.1)$$

If the t-semirecurrent events do not depend of the superscripts k, then they are called t-recurrent events. It is clear, that if $\{A_n^k(t): n \geq 1, k \geq 1, t \in [0,\infty]\}$ are t-semirecurrent events, then $\{A_n^k(t) : n \geq 1, k \geq 1\}$ are semirecurrent events for any $t \in [0, \infty]$.

It is simple to prove that, for an integer-valued random variable, $\nu_k(t)$, $t \in [0,\infty]$, defined via

$$P_n^k(t) = P(\nu_k(t) = n) = P(\bar{A}_1 \dots \bar{A}_{n-1} \bar{A}_n(t)), \quad (5.2)$$

where $A_i = A_i(\infty)$, $i = 1, \dots, n-1$, we have

$$\left.\begin{aligned} P_1^k(t) &= P(A_1^k(t)), \\ P_n^k(t) &= P(A_n^k(t)) - \sum_{j=1}^{n-1} P(A_j^k) P_{n-j}^{k+j}(t), \quad n \geq 2. \end{aligned}\right\} \quad (5.3)$$



If $\{A_n: n \geq 1\}$ are recurrent events, then $\{A_n(t), A_n: n \geq 1\}$ are, for any $t \in [0,\infty]$, recurrent events with delay. For the definition of the recurrent events with delay see, for example,[15].

In order to determine $U_k(t,n) = P(I_k < t, \nu_k = n)$, $n \geq 1$, $t > 0$, $k \geq 1$, for the modified counter $\mathcal{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, we introduce the following system of functions $\{Y_n^k : n \geq 1, k \geq 1\}$ via

$$\left.\begin{aligned} Y_1^k &= 0, \\ Y_n^k &= \max(Y_{n-1}^k, \chi_{n-1}^k) - T_{n-1}^k, \quad n \geq 2. \end{aligned}\right\} \quad (5.4)$$

The interpretation of the functions $\{Y_n^k : n \geq 1, k \geq 1\}$ is the following. Let $\mathcal{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, $k \geq 1$, be a modified counter. Denote by $\mathcal{C}_k^*$ a counter with prolonging dead time for that the n-th emitted particle, $n \geq 1$, has the impulse length $\chi^k$, the interarrival time after the n-th particle is $T_n^k$, and the conditions after the idle period do not resume the initial conditions. Hence, if $Y_n^k \geq 0$, then $Y_n^k$ is the time period starting at the arrival moment of the n-th particle at the counter $\mathcal{C}_k^*$, $k \geq 1$, and needed to finish all impulses of emitted particles arrived before the arrival of the n-th particle. If $Y_n^k < 0$, then the n-th particle begins the dead time and the previous idle period has the length $-Y_n^k$.

The similar functions have been used in[2] to determine the number of customers served during the busy period of a GI/GI/∞ queue.

Let us put

$$\begin{aligned} A_n^k(t) = \{&\chi_k < T_k + \dots + T_{k+n-1}, \chi_{k+1} < T_{k+1} + \dots + T_{k+n-1}, \dots, \\ &\chi_{k+n-1} < T_{k+n-1}, -Y_{n+1}^k < t\}, \quad n \geq 1, k \geq 1, t \in [0,\infty]. \end{aligned} \quad (5.5)$$

Then $\{A_n^k(t) : n \geq 1, k \geq 1, t \in [0,\infty]\}$ are t-semirecurrent events and for $U_k(t,n)$ we have

$$U_k(t,n) = P_n^k(t), \quad n \geq 1, t \geq 0, k \geq 1. \quad (5.6)$$

In order to calculate (5.6) we introduce random variables $q_n^k(x)$, $n \geq 1$, $k \geq 1$, $x \geq 0$, defined as the number of the particles arrived at the counter $\mathcal{C}_k^*$ before the arrival of the (n+1)-th particle whose impulses are present at the counter at the moment x after the (n+1)-th arrival moment.

Let us define, for $x \geq 0$, $-\infty < y < \infty$, $n \geq 1$, $k \geq 1$,

$$f_n^k(x,y) = P(q_n^k(x) = 0, -Y_{n+1}^k < y), \quad (5.7)$$

$$\psi_n^k(x) = P(q_n^k(x) = 0). \tag{5.8}$$

It is evident that

$$P(A_n^k(t)) = f_n^k(0,t) \tag{5.9}$$

and

$$q_n^k(0) = 0 \quad \text{iff} \quad Y_{n+1}^k < 0. \tag{5.10}$$

For the functions $\psi_n^k(x)$ we yield

$$\psi_n^k(x) = \int_0^\infty \dots \int_0^\infty H_k(t_1 + \dots + t_n + x) H_{k+1}(t_2 + \dots + t_n + x) \dots \tag{5.11}$$

$$H_{k+n-1}(t_n + x)\, dF_k(t_1) \dots dF_{k+n-1}(t_n), \quad x \geq 0, n \geq 1, k \geq 1.$$

For $f_n^k(x,y)$ we have the following recurrent equations

$$f_1^k(x,y) = \begin{cases} \int_0^\infty (H_k(t+x) - H_k((t-y)+))\, dF_k(t), & \text{if } y > -x, \\ 0, & \text{if } y \leq -x. \end{cases} \tag{5.12}$$

Let now $n \geq 2$. Then, for any $x \geq 0$, $-\infty < y < \infty$, and $k \geq 1$,

$$f_n^k(x,y) = \begin{cases} \int_0^\infty f_{n-1}^k(t+x, y-t) H_{k+n-1}(t-y)\, dF_{k+n-1}(t) + \\ + \int_0^\infty \psi_{n-1}^k(t+x)(H_{k+n-1}(t+x) - H_{k+n-1}((t-y)+))\, dF_{k+n-1}(t), \\ \text{if } y > -x, \\ \int_0^\infty f_{n-1}^k(t+x, y-t) H_{k+n-1}(t+x)\, dF_{k+n-1}(t), \text{ if } y \leq -x. \end{cases} \tag{5.13}$$

It is easy to show that if $\{H_n\}_{n=1}^\infty$ or $\{F_n\}_{n=1}^\infty$ are continuous from right, then, for $y>0$, we have

$$f_n^k(0,y) = \int_0^\infty \dots \int_0^\infty (H_k(t_1 + \dots + t_n) H_{k+1}(t_2 + \dots + t_n) \dots$$

$$\dots H_{k+n-1}(t_n) - H_k(t_1 + \dots + t_n - y) H_{k+1}(t_2 + \dots + t_n - y) \dots \tag{5.14}$$

$$\dots H_{k+n-1}(t_n - y))\, dF_k(t_1) \dots dF_{k+n-1}(t_n), \quad n \geq 1, \ k \geq 1.$$



Theorem 5.1. The joint distribution, $U_k(t,n)$, of the idle period and the number of emitted particles arrived at the counter during the dead time of the modified counter $\mathcal{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, $k \geq 1$, is given by (5.6), where $P(A_n^k(t))$ is calculated by (5.9) using the formulae from (5.7) through (5.14).

Remark. Let $\mathcal{C} = (F; H)$ be a modified counter of order 1. Put

$$\phi_t(z) = 1 - \sum_{n=0}^\infty (P(A_n^1(t)) - P(A_{n+1}^1(t))) z^n,$$

$$\phi(z) = 1 - \sum_{n=0}^\infty (P(A_n^1) - P(A_{n+1}^1)) z^n, \quad |z| \leq 1, \ t \in [0,\infty],$$

where $P(A_0^1(t)) \equiv P(A_0^1) \equiv 1$. Then, for the generating function $f_t(z) = \sum_{n=1}^\infty P_n^1(t) z^n$, we have

$$f_t(z) = z\phi_t(z)/(1 - z + z\phi(z)). \tag{5.15}$$

Put

$$p_t = \lim_n \int_0^\infty \dots \int_0^\infty H(t_1 - t) \dots H(t_1 + \dots + t_n - t)\, dF(t_1) \dots dF(t_n), \quad t \geq 0.$$

If H or F is continuous, and $p = p_t|_{t=0} > 0$, then, due to (5.15),

$$P(I < t) = 1 - p_t/p, \tag{5.16}$$

$$M(I) = p^{-1} \int_0^\infty p_t\, dt, \quad M(B) = (M(T_1) - \int_0^\infty p_t\, dt) M(\nu). \tag{5.17}$$

Example 5.1. Let $\mathcal{C} = (1 - e^{-\lambda t}; H)$. Then

$$P(A_n^1(t)) = (1 - e^{-\lambda t}) P(A_n^1) = (1 - e^{-\lambda t}) \lambda^{n+1}/n! \int_0^\infty \left( \int_0^\infty H(u)\, du \right)^n e^{-\lambda y}\, dy,$$

and $P(\nu = n, I < t) = P(\nu = n) P(I<t)$, so that we obtain the known result $P(I < t) = 1 - e^{-\lambda t}$, $t \geq 0$.

Example 5.2. Let $\mathcal{C} = (F_1, F_2, \dots; H_1, H_2, \dots)$ be a modified counter, where $H_n$, $n \geq 1$, is the distribution function of the constant $D_n$ which $0 < D_1 \leq D_2 \leq \dots$. Then, for any $n, k \geq 1$ and $t > 0$, we have

$$P(A_n^k(t)) = F_{n+k-1}(t + D_{n+k-1}) - F_{n+k-1}(D_{n+k-1}+),$$

$$P_n^k(t) = F_k(D_k+) \dots F_{k+n-2}(D_{k+n-2}+)(F_{n+k-1}(t + D_{n+k-1}) - F_{n+k-1}(D_{n+k-1}+)).$$

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Двуреченский А. E5-85-322

Заметка о модифицированном счетчике с мертвым временем продлевающегося типа

Для модифицированного счетчика с мертвым временем продлевающегося типа выводится интегральное уравнение для совместного распределения мертвого времени и простоя. Доказывается асимптотический показательный закон для мертвого времени в случае гомогенного пуассоновского процесса испускаемых образцом частиц на счетчик порядка m. Вводится класс t-семирекуррентных событий и применяется к теории светчиков.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Dvurečenskij A. E5-85-322

A Note on a Modified Counter with Prolonging Dead Time

For the modified counter with prolonging dead time the integral equation is derived for the joint distribution of the dead time and the successive idle period. The asymptotic exponential law of the dead time is proved in the case of the homogeneous Poisson input process of emitted particles and the modified counter of order m. The class of t-semirecurrent events is introduced and applied to the counter theory.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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