объединенный
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ядерных
исследований
дубна

E5-85-158

A.Dvurečenskij

THE BUSY AND IDLE PERIODS JOINT DISTRIBUTION OF A DISCRETE MODIFIED GI/GI/c QUEUE

Submitted to "Journal of Applied Probability"

1985

0. INTRODUCTION

In the last years the use in practice of discrete queueing systems with finitely or infinitely many servers has increasing tendency. The discrete systems are used as mathematical models of, for example, mass servicing machines, electronic machines, transport problems, automated filmless blob-length measurements in track chambers in high-energy physics [1], particle counters, etc.

Borovkov [2,3] studied some limit properties for the case of queue with finitely and infinitely many servers.

For a modified queue we suppose that the service times of all customers serving during any busy period are independent random variables with not necessarily identical distribution functions. Modified M/GI/1 queue has been investigated by Yeo [4] and Welch [5] modified GI/M/1 and GI/GI/1 queues by Pakes [6,7] and GI/M/1 by Shanthikumar [8]. Modified GI/GI/ has been studied by Dvurečenskij, Ososkov [9] on the particle counter language.

The joint distribution of the busy and idle periods for the GI/M/1 queue has been derived by Kalashnikov [10]. For the discrete modified counter with prolonging dead time the joint distribution of the dead time and idle period was done by Dvurečenskij, Ososkov [11].

In the present note we derive the busy and idle periods joint distribution for the discrete modified queue GI/GI/c, for $1 \le c < \infty$. First of all we shall concentrate ourselves on the discrete modified GI/GI/1 queue, then we shall continue with the general case of GI/GI/c, for any $2 \le c < \infty$. We note that the presented formulae are computationally convenient for the practical use, and they may be used for the discrete modified GI/GI/∞ queue, too.

1. DISCRETE MODIFIED GI/GI/1 QUEUE

We suppose that the queueing system with c servers, $1 \le c < \infty$, is idle before the service process and the customers arrive at the queue at moments $0 = \tau_1 < \tau_2 < \dots$. Let χ_k, $k \ge 1$, be the service time of the k th customer in the busy period, and let T_k, $k \ge 1$, be the interarrival time between the arrivals of the k th and k + 1st customers. The busy period, B^c, is the time period when at least one customer is served. The idle period, I^c, is the time period when no customer is served. The sum, $C^c = B^c + I^c$, of the busy period and following idle period is said to be a cycle. For the modified queue we suppose that any busy period is produced by the sequence of the service times, $\{\chi_k\}_{k=1}^{\infty}$, and by the interarrival times, $\{T_k\}_{k=1}^{\infty}$. $\{\chi_k\}_{k=1}^{\infty}$ are assumed to be independent positive discrete random variables with the distribution laws

$$h_k(n) = P(\chi_k = nh), \quad n \geq 1, \quad k \geq 1, \tag{1.1}$$

where $\sum_{n=1}^{\infty} h_k(n) = 1$, $k \geq 1$. The sequence $\{\chi_k\}_{k=1}^{\infty}$ is independent of the sequence $\{T_k\}_{k=1}^{\infty}$, where T_k, $k \geq 1$, are assumed to be positive discrete random variables with the same step $h > 0$ and with the distribution laws

$$f_k(n) = P(T_k = nh), \quad n \geq 1, \quad k \geq 1, \tag{1.2}$$

where $\sum_{n=1}^{\infty} f_k(n) = 1$, $k \geq 1$. Any successive busy period is resumed with initial conditions, independently of the previous busy periods. This discrete modified queue will be denoted by $\mathfrak{Q}^c = (f_1, f_2, \dots; h_1, h_2, \dots)$.

For a given queue $\mathfrak{Q}^c = (f_1, f_2, \dots; h_1, h_2, \dots)$ is convenient to consider a sequence of discrete queues, $\{\mathfrak{Q}_k^c\}_{k=1}^{\infty}$, where $\mathfrak{Q}_k^c = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$. Suppose that the first busy period of any queue $\mathfrak{Q}_k^c$ is produced by sequences $\{\chi_n^k\}_{n=1}^{\infty}$ and $\{T_n^k\}_{n=1}^{\infty}$, where $\chi_n^k = \chi_{k+n-1}$, $T_n^k = T_{k+n-1}$, when for $\mathfrak{Q}_k^c$ we define the corresponding busy period, B_k^c, idle period, I_k^c, and the cycle, C_k^c, respectively. For simplicity we put $h = 1$.

Our main aim is to determine the joint distribution, $W_k^c(n,m) = P(B_k^c = n,\ I_k^c = m)$, $n, m \geq 1$, of the busy period, B_k^c, and the idle period, I_k^c, of the discrete modified queue $\mathfrak{Q}_k^c$, $1 \leq c < \infty$, for any $k \geq 1$.

First of all we begin with the discrete modified GI/GI/1 queue $\mathfrak{Q}_k^1$, $k \geq 1$.

It is evident that

$$W_k^1(n, m) = \sum_{j=1}^{n-1} \sum_{t=1}^{j} h_k(j) f_k(t) A_k^1(n, m; j, t) + h_k(n) f_k(n+m), \tag{1.3}$$

where $\qquad n, m \geq 1,$

$$A_k^1(n, m; j, t) = P(B_k^1 = n,\ I_k^1 = m \mid \chi_1^k = j,\ T_1^k = t), \quad 1 \leq t \leq j < n. \tag{1.4}$$

So we have

$$W_k^1(1, m) = h_k(1) f_k(1+m). \tag{1.5}$$

For any $n \geq 2$ and $1 \leq t \leq j \leq n-1$ we have

$$A_k^1(n, m; j, t) = \sum_{r=1}^{j-t+1} B_k^1(n, m; r; j, t), \tag{1.6}$$

where $B_k^1(n, m, r; j, t)$ is the probability that $B_k^1 = n$, $I_k^1 = m$ when $\chi_1^k = j$, $T_1^k = t$, and from the service times of the second and following it service times, r cycles with the sum of all busy periods equal to $n + m - t$, without the service time of the first customer, are created.

The event $\{B_k^1 = n,\ I_k^1 = m\}$ depends only on $\{\chi_k, \dots, \chi_{k+n-1}, T_k, \dots, T_{k+n-1}\}$. Therefore

$$B_k^1(n, m, r; j, t) = \sum W_{s_1}^1(n_1, m_1) \dots W_{s_r}^1(n_r, m_r), \tag{1.7}$$

where $s_1 = k+1$, $s_2 = s_1 + n_1$, ..., $s_r = s_{r-1} + n_{r-1}$ and the summation is taken over all integers $n_1, m_1, \dots, n_r \geq 1$, $m_r \geq m$, with $n_1 + \dots + n_r = n - j$, $m_1 + \dots + m_r = n + j - t$.

This proves the following theorem:

Theorem 1. The joint distribution of the busy and idle periods of the discrete modified GI/GI/1 queue $\mathfrak{Q}_k^1 = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $k \geq 1$, is given by the formula (1.3), where $A_k^1(n, m; j, t)$ is calculated from (1.6) and (1.7).

To determine $W_k^c(n, m)$, $k > 1$, for $c \geq 2$, we need to have the following two distributions: $V_k(n_1, n_2, t)$ and $U_k(n_1, n_2, z, t)$.

Let $V_k(n_1, n_2, t)$, $n_1 \geq 1$, $n_2 \geq 0$, $t \geq 1$ be the probability that at the queue $\mathfrak{Q}_k^1 = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $k \geq 1$, a customer arrives at the moment t and from the service times of all customers, have been arrived before the moment $\min(n_1, t)$, the busy period of the total length $n_1 + n_2$ is created such that the busy period formed without the service time of the last customer participating in the creation of the above busy period would be less than n_1. This probability is evaluated in formulae (1.8)-(1.15).

It is clear that if $n_1 = n$, $n_2 = 0$, $t = n + m$, then $V_k(n, 0, n+m) = W_k^1(n, m)$.

So we have

$$V_k(1, n_2, t) = h_k(1 + n_2)\ f_k(t), \quad t \geq 1, \tag{1.8}$$

$$V_k(2, n_2, 1) = h_k(2 + n_2)\ f_k(t), = V_k(1, 1 + n_2, 1). \tag{1.9}$$

If $1 \leq t \leq n_1 - 1$, $n_2 \geq 2$, then

$$V_k(n_1, n_2, t) = V_k(t, n_1 + n_2 - t, t). \tag{1.10}$$

Now let $t \ge n_1$. Then

$$V_k(n_1, n_2, t) = \sum_{j=1}^{n_1} V_k(n_1, j, n_2, t), \qquad (1.11)$$

where $V_k(n_1, j, n_2, t)$ is the joint probability of the event described in the formulation of $V_k(n_1, n_2, t)$ and $\{$either $\chi_1^k = j$, if $j < n_1$, or $\chi_1^k = n_1 + n_2$, if $j = n_1\}$.

Hence, if $1 \le j \le n_1 - 1$, then

$$V_k(n_1, j, n_2, t) = h_k(j) \sum_{i=1}^{j} f_k(i)\, VA_k(n_1, j, n_2, t, i), \qquad (1.12)$$

where

$$VA_k(n_1, j, n_2, t, i) = \sum_{r=1}^{j-i+1} VB_k(n_1, j, n_2, t, i, r) \qquad (1.13)$$

and $VB_k(n_1, j, n_2, t, i, r)$ has the similar sense as $B_k^1(n_1, m; j, t, r)$.

Therefore

$$VB_k(n_1, j, n_2, t, i, r) = \sum W^1_{s_1}(j_1, k_1) \dots W^1_{s_{r-1}}(j_{r-1}, k_{r-1}) \times \qquad (1.14)$$
$$\times V_{s_r}(j_r, n_2, t - i - j_1 - k_1 - \dots - j_{r-1} - k_{r-1}).$$

Here $s_1 = k + 1$, $s_2 = s_1 + j_1, \dots, s_r = s_{r-1} + j_{r-1}$, and the summation runs over integers $j_1, k_1, \dots, j_{r-1}, k_{r-1}, j_r \ge 1$ with $j_1 + \dots + j_r = n_1 - j$.

For $j = n_1$ we have

$$V_k(n_1, n_1, n_2, t) = h_k(n_1 + n_2) f_k(t). \qquad (1.15)$$

The following expression has great importance in its application to the modified GI/GI/c queue, when $c \ge 2$.

Let $U_k(n_1, n_2, z, t)$, $n_1 \ge 1$, $n_2 \ge 0$, $z \ge 0$, $1 \le t \le z + n_1$, be the probability that at the queue $\mathcal{Q}_k^1 = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $k \ge 1$, a customer arrives at the moment t and from the service times of all customers, having arrived before the moment $\min(z + n_1, t)$ and which are served after the moment z, the busy period of the total length $n_1 + n_2$ is created such that the busy period formed without the service time of the last customer participating in the creation of the above busy period would be less that n_1. This probability is calculated in formulae (1.16)-(1.20).

It is clear that

$$U_k(n_1, n_2, 0, t) = V_k(n_1, n_2, t). \qquad (1.16)$$

The definition of V_k yields

$$U_k(1, n_2, z, t) = h_k(1 + n_2) f_k(t) \qquad (1.17)$$

and

$$U_k(n_1, n_2, z, t) = \sum_{j=1}^{n_1 \wedge t} U_k(n_1, j, n_2, t), \qquad (1.18)$$

where $U_k(n_1, j, n_2, t)$ is the joint probability of the event described in the formulation of $V_k(n_1, n_2, t)$ and $\{$either $\chi_1^k = j$, if $j < n_1 \wedge t$, or $\chi_1^k = n_1 + n_2$, if $j = n_1 \wedge t\}$. Here $x \wedge y = \min(x, y)$.

Hence, if $j < n_1 \wedge t$, then

$$U_k(n_1, j, n_2, z, t) = h_k(j) \sum_{i=1}^{(z+j)\wedge(t-1)} f_k(i)\, U_{k+j}(n_1 - j, n_2, z - i + j, t - i), \qquad (1.19)$$

and if $j = n_1 \wedge t$, then

$$U_k(n_1, n_1 \wedge t, n_2, z, t) = h_k(n_1 + n_2) f_k(t). \qquad (1.20)$$

2. DISCRETE MODIFIED MULTISERVER GI/GI/c QUEUE

In this section we derive the joint distribution of the busy period and idle period, $W_k^c(n, m)$, $n, m \ge 1$, of the discrete modified queue $\mathcal{Q}_k^c = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $k \ge 1$, for any $c \ge 2$, assuming the knowledge of the corresponding formulae for all queues $\mathcal{Q}_k^s$, where $1 \le s < c$, $k \ge 1$.

Let n, $m \ge 1$ be given integers. For any $1 \le i \le n \wedge c$, let integers $t_1, \dots, t_i$ with $t_1 + \dots + t_i < n$ be given. Put

$$J_1 = n, \ J_2 = J_1 - t_1, \dots, J_i = J_{i-1} - t_{i-1}$$

and, for any $1 \le j_1 \le J_1, \dots, 1 \le j_i \le J_i$, define

$$a_1 = j_1, \ a_{s+1} = \max(a_s - t_s, j_{s+1}) \quad \text{for} \quad 1 \le s \le i.$$

To any t_i assign t_i^* via

$$t_i^* = \begin{cases} J_i + m, & \text{if } t_i = J_i = a_i, \\ t_i, & \text{otherwise.} \end{cases}$$

Let us introduce functions $A_k^c(n, m; j_1, t_1, \dots, j_i, t_i)$ for $1 \le j_s \le J_s$, $1 \le t_s \le a_s$, when $1 \le s \le i$, via

$$A_k^c(n, m; j_1, t_1, \dots, j_i, t_i) = P(B_k^c = n, \ I_k^c = m \mid \chi_1^k = j_1, \qquad (2.1)$$
$$T_1^k = t_1, \dots, \ \chi_i^k = j_i, \ T_i^k = t_i^*).$$

It is clear that if $t_i^* = J_i + m$, then

$$A_k^c(n, m; j_1, t_1, \dots, j_i, t_i) = 1 \tag{2.2}$$

and if $t_i^* < J_i + m$ and $i < n \wedge c$, then

$$A_k^c(n, m; j_1, t_1, \dots, j_i, t_i) = \sum_{j_{i+1}=1}^{J_{i+1}} \sum_{t_{i+1}=1}^{a_{i+1}} h_{k+i}(j_{i+1}) f_{k+i}(t_{i+1}^*) \times$$
$$\times A_k^c(n, m; j_1, t_1, \dots, j_{i+1}, t_{i+1}). \tag{2.3}$$

Therefore we have

$$W_k^c(n, m) = \sum_{j_1=1}^{J_1} \sum_{t_1=1}^{a_1} h_k(j_1) f_k(t_1^*) A_k^c(n, m; j_1, t_1). \tag{2.4}$$

We note that the above considerations may be carried for $\mathcal{Q}_k^1$, too. In this case we obtain (1.3) and (1.4).

In order to determine $W_k^c(n,m)$ it is sufficient to determine $A_k^c(n, m; j_1, t_1, \dots, j_i, t_i)$ for any $i \le n \wedge c$. Therefore in the sequel we investigate the properties of (2.1) and calculate them. The investigation will be divided into ten steps.

(i) First of all we assume that $1 \le i \le n \le c$.

It is clear that

$$\left.\begin{aligned} &A_k^c(1, m; 1, 1) = 1 \\ &W_k^c(1, m) = h_k(1) f_k(1 + m). \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} &A_k^c(2, m; 1, 1) = W_{k+1}^c(1, m), \\ &A_k^c(2, m; 2, 1) = W_{k+1}^c(1, m), \\ &A_k^c(2, m; 2, 2) = 1. \end{aligned}\right\} \tag{2.6}$$

By recurrence we have, for $3 \le n \le c$,

$$A_k^c(n, m; 1, 1) = W_{k+1}^c(n - 1, m), \tag{2.7}$$

and for $2 \le j \le n - 1$, $1 \le t \le j$, we obtain

$$A_k^c(n, m; j, t) = \sum_{r=1}^{[\frac{j-t+2}{2}]} B_k^c(n, m, r; j, t). \tag{2.8}$$

Here $B_k^c(n, m; j, t)$ is the probability that $B_k^c = n$, $I_k^c = m$, when $\chi_1^k = j$, $T_1^k = t$, and from the service times of the second and following it customers, r cycles, with the total length of the first $r - 1$ cycles and the r th busy period equal to $n - t$, without the service time of the first customer, are created. Therefore

$$B_k^c(n, m, r; j, t) = \sum W_{s_1}^c(n_1, m_1) \dots W_{s_r}^c(n_r, m_r), \tag{2.9}$$

where $s_1 = k + 1$, $s_2 = s_1 + n_1, \dots, s_r = s_{r-1} + n_{r-1}$, and the summation is taken over integers $n_1, m_1, \dots, n_{r-1}, m_{r-1} \ge 1$, $n_r \ge n - j$, $m_r = m$ with $n_1 + m_1 + \dots + n_r = n - t$.

If $j = n$, then

$$A_k^c(n, m; n, n) = 1. \tag{2.10}$$

For $1 \le t \le n - 1$ we have

$$A_k^c(n, m; n, t) = \sum_{r=1}^{[\frac{n-t+1}{2}]} B_k^c(n, m, r; n, t) \tag{2.11}$$

and

$$B_k^c(n, m, r; n, t) = \sum W_{s_1}^{c-1}(n_1, m_1) \dots W_{s_r}^{c-1}(n_r, m_r), \tag{2.12}$$

where $s_1 = k + 1$, $s_2 = s_1 + n_1, \dots, s_r = s_{r-1} + n_{r-1}$, and the summation runs over integers $n_1, m_1, \dots, m_{r-1}, n_r, m_r \ge 1$, $m_r \ge m$, with $n_1 + m_1 + \dots + n_r + m_r = n - t$.

(ii) Now we suppose that $2 \le i \le n \le c$.

Put $t_0 = 0$ and $\theta_s = t_0 + \dots + t_{s-1}$, $1 \le s \le i + 1$. Denote by s_0 an arbitrary index, $1 \le s_0 \le i$, such that $\theta_{s_0} + j_{s_0} \ge \theta_s + j_s$, $1 \le s \le i$. Then

$$A_k^c(n, m; j_1, t_1, \dots, j_i, t_i) = A_k^c(n, m; \theta_{s_0} + j_{s_0}, \theta_{i+1}). \tag{2.13}$$

In the following we shall suppose that $1 \le i \le c < n$ and $t_i^* < J_i + m$.

(iii) Assume there is $t_q \ge 1$, $2 \le q \le i$, with $j_1, \dots, j_q \ge 2$. Then

$$A_k^c(n, m; j_1, t_1, \dots, j_i, t_i) = A_k^c(n - 1, m; j_1 - 1, t_1, \dots, j_{q-1} - 1,$$
$$t_{q-1}, j_q - 1, t_q - 1, j_{q+1}, t_{q+1}, \dots, j_i, t_i). \tag{2.14}$$

(iv) If, for some u, $2 \le u \le i$, there is v, $1 \le v \le u$, such that $j_v \le t_v + \dots + t_u$, then

$$A_k^c(n, m; j_1, t_1, \ldots, j_i, t_i) = A_{k+1}^c(n, m; j_1, t_1, \ldots, j_{v-1}, t_{v-1} + t_v, j_{v+1}, t_{v+1}, \ldots, j_i, t_i). \tag{2.15}$$

(v) Taking into account the third and fourth steps we see that in the following considerations it suffices to assume $t_1 = t_2 = \ldots = t_i = 1$ and $\theta_s + t_s > \theta_{i+1}$, $1 \le s \le i$. Moreover, from (2.3) we conclude that we may restrict ourselves to the case $i = c$.

Suppose that $\theta_s + j_s < J_s$, $s = 1, \ldots, c$. Denote by s_1 an arbitrary index of such customer whose service is finished fistly, that is, $\theta_{s_1} + j_{s_1} \le \theta_s + j_s$, $1 \le s \le i$, and let s_2 be another index (if there is any) for which the corresponding customer is finished firstly after the moment $\theta_{s_1} + j_{s_1}$.

(vi) Let at least one customer service be finished before another, that is, there is q, $1 \le q \le c$, with $\theta_q + j_q < \theta_s + j_s$ for some s, $1 \le s \le c$. Denote by

$$Z = s_1 - 1 + j_{s_2} - c, \quad N_1 = s_2 - s_1 + j_{s_2} - j_{s_1}, \quad N_2 = n + 1 - s_2 - j_{s_2}.$$

Then

$$A_k^c(n, m; j_1, 1, \ldots, j_c, 1) = \sum_{n_1=1}^{N_1} \sum_{n_2=0}^{N_2} \sum_{z=1}^{Z} \sum_{t=1}^{s+n_1} U_{k+c}(n_1, n_2, z, t) \times A_{k+1}^c(n-1, m; j_1 - 1, 1, \ldots, j_{c-1} - 1, 1, j_c - 1, t). \tag{2.16}$$

(vii) Let now $\theta_1 + j_1 = \ldots = \theta_c + j_c$. Putting $Z = j_1 - c$, $N_1 = n - j_1$, we have

$$A_k^c(n, m; j_1, 1, \ldots, j_c, 1) = \sum_{n=1}^{N_1} \sum_{z=1}^{Z} \sum_{t=1}^{Z} U_{k+c}(n_1, 0, z, t) \times A_{k+1}^c(n-1, m; j_1 - 1, 1, \ldots, j_{c-1} - 1, 1, j_c - 1, t). \tag{2.17}$$

(viii) Let us suppose that $j_1 = n$, $\theta_s + j_s < J_s$, $s = 2, \ldots, c$. Then

$$A_k^c(n, m; n, 1, j_2, 1, \ldots, j_c, 1) = \sum_{r=1}^{[\frac{n+1-J}{2}]} B_k^{c-1}(n, m, r; j_2, 1, \ldots, j_c, 1), \tag{2.18}$$

where $J = \max(j_2, j_3 + 1, \ldots, j_c - c - 1)$, and $B_k^{c-1}(n, m, r; j_2, 1, \ldots, j_c, 1)$ have analogous sense to that explained just after (2.8). Hence

$$B_k^{c-1}(n, m, r; j_2, 1, \ldots, j_c, 1) = \sum A_{s_1}^{c-1}(n_1, m_1; j_2, 1, \ldots, j_c, 1) W_{s_2}^{c-1}(n_2, m_2) \ldots W_{s_r}^{c-1}(n_r, m_r). \tag{2.19}$$

Here $s_1 = k + 1$, $s_2 = s_1 + n_1, \ldots, s_r = s_{r-1} + n_{r-1}$. The summation is taken over integers $n_1 \ge J$, $n_2, \ldots, n_r \ge 1$, $m_1, \ldots, m_{r-1} \ge 1$, $m_r \ge m$.

(ix) Let us consider the case when there is q such that $\theta_s + j_s = J_s$, $1 \le s \le q$, $q \ge 2$, and $\theta_s + j_s < J_s$, $q < s \le c$. Then

$$A_k^c(n, m; n, 1, n-1, 1, \ldots, n-q+1, 1, j_{q+1}, 1, \ldots, j_c, 1) = A_{k+q-1}^{c-q+1}(n-q+1, m; n-q+1, 1, j_{q+1}, 1, \ldots, j_c, 1). \tag{2.20}$$

(x) Finally, we suppose that

$$j_1 = n, \; j_2 < n - 1, \; \ldots, \; j_u < n - u + 1, \; j_{u+1} = n - u,$$

for some $2 \le u < c$. Then

$$A_k^c(n, m; n, 1, j_2, 1, \ldots, j_u, 1, n-u, 1, j_{u+2}, 1, \ldots, j_c, 1) = A_k^c(n, m; n, 1, n-1, j_2 - 1, 1, \ldots, j_u, 1, j_{u+2}, 1, \ldots, j_c, 1). \tag{2.21}$$

Combining all the above ten steps we may obtain all possible cases of (2.1), and this proves the main result:

Theorem 2. The joint distribution of the busy and successive idle periods of the discrete modified multiserver queue, $\mathcal{Q}_k^c = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$, $k \ge 1$, for any $2 \le c < \infty$, is given by (2.2), (2.3) and the formulae from (2.5) through (2.21).

Remark 1. It is evident that for the discrete modified queueing system with infinitely many servers, $\mathcal{Q}_k^\infty = (f_k, f_{k+1}, \ldots; h_k, h_{k+1}, \ldots)$, $k \ge 1$, the joint distribution, $W_k^\infty(n, m)$ of the busy period, B_k^∞, and the successive idle period, I_k^∞, is given by

$$W_k^{\infty}(n,m) = W_k^n(n,m), \quad n,m \geq 1, \quad k \geq 1. \tag{2.22}$$

Remark 2. The distribution law of the cycle C_k^c, $P_k^c(i) = P(C_k^c = i)$, $i \geq 2$, $k \geq 1$, $c = 1,2,\ldots,\infty$, is given by

$$P_k^c(i) = \sum_{\substack{n+m=1 \\ n,m\geq 1}} W_k^c(n,m). \tag{2.23}$$

REFERENCES

1. Dvurečenskij A. et al. J.Appl.Prob., 1984, 21, p.201-206.
2. Borovkov A.A. Teorija veroj. i prim., 1963, 8, p.251-263.
3. Borovkov A.A. Stochastic Processes in Queueing Theory (In Russian). "Nauka", Moscow, 1972.
4. Yeo G.F. J.Austral.Math.Soc., 1962, 3, p.491-502.
5. Welch P.D. Oper.res., 1964, 12, p.736-752.
6. Pakes A.G. Ann.Inst.Statist.Math., 1972, 24, p.589-597.
7. Pakes A.G. J.Appl.Prob., 1973, 10, p.192-197.
8. Shanthikumar J.G. Opsearch, 1982, 19, p.148-159.
9. Dvurečenskij A., Ososkov G.A. J.Appl.Prob., 1985, 22, No.3; JINR, E5-83-882, Dubna, 1983.
10. Kalashnikov V.V. Izv.AN SSSR, Tekh.kiber., 1971, No.6, p.106-109.
11. Dvurečenskij A., Ososkov G.A. JINR, E5-84-823, Dubna, 1984.

Received by Publishing Department
on March 4, 1985.

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Двуреченский А. E5-85-158
Совместное распределение периодов занятости массового обслуживания GI/GI/c

Измерение ионизации треков в стримерных камерах в физике высоких энергий ставит задачу определения длины дискретизованного трека. Эту физическую проблему можно успешно решать в рамках теории дискретных систем массового обслуживания с конечным или бесконечным числом обслуживающих каналов. В настоящей заметке определяется совместное распределение периодов занятости и простоя для дискретной модифицированной системы GI/GI/c для каждого $1 \le c \le \infty$.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Dvurečenskij A. E5-85-158
The Busy and Idle Periods Joint Distribution of a Discrete Modified GI/GI/c Queue

For a discrete modified GI/GI/c queue, $1 \le c < \infty$, where the service times of all customers served during any busy periods are independent random variables with not necessarily identical distribution functions, the joint distribution of the busy period and the successive idle period is derived. The presented formulae are convenient for practical use.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985