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объединенного
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ядерных
исследований
дубна

E5-85-13

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A NEW CLASS OF REALIZATIONS OF THE LIE ALGEBRA sp(n,R)

1985

1. Introduction

1.1 The real symplectic algebra sp(n,R), which is the algebra of the group of linear canonical transformation in 2n-dimensional phase space, plays an outstanding role in many physical problems [2]. This is why various sorts of representations of this algebra are interesting. In this paper, we are going to concentrate our attention on a purely algebraic method of constructing the representations. The central notion is that of the canonical, or boson realization which means an expression of elements of sp n,R by means of polynomials in quantum canonical variables p_i, q_i, such that the commutation relations are preserved. For physical relevance of such canonical realizations of sp(n,R) we refer to the articles [3] and other references quoted therein.

1.2 In the papers [4,5], explicit forms of realizations for sp(n,R) have been constructed using the method of coherent state representation [6]. These realizations are defined by means of $\frac{1}{2}n(n+1)$ canonical pairs and of generators of a subalgebra gl(n,R). Another class of realizations has been described in papers [7]; here realizations of sp(n,R) in terms of (2n-1) canonical pairs and of generators of sp(n-1,R) are obtained.

1.3 We have formulated recently a method [1], which enables to construct wide families of canonical realizations of a semisimple Lie algebra *g* staring from induced representations of *g* with respect to a suitable subalgebra of *g*. In the present paper, we apply the method of Ref. 1 to the case of $sp(n,R)$. For any $r=1,2,\dots,n$, we construct reccurent formulae which give realizations of $sp(n,R)$ in term of $r\left(2n-\frac{3}{2}r+\frac{1}{2}\right)$ - canonical pairs and of generators of the subalgebra $gl(r,R)\oplus sp(n-r,R)$. Both the above-mentioned types of realizations appear to be particular cases for r=n and r=1, respectively.

2. Preliminaries

2.1 The algebra $g \equiv sp(n,R)$ is the $n(2n+1)$ -dimensional real Lie algebra. We choose a basis consisting of $n(2n+1)$ generators $X_{ij} = -\varepsilon_i \varepsilon_j X_{-j,-i}$, $i,j= -n,\dots,-1,1,\dots,n$ satisfying the commutation rules

$$[X_{ij},X_{kl}] = \delta_{jk}X_{il} - \delta_{il}X_{kj} + \varepsilon_i \varepsilon_j \delta_{j,-l}X_{k,-i} - \varepsilon_i \varepsilon_j \delta_{i,-k}X_{-j,l}, \quad (1)$$

where $\varepsilon_i \equiv \operatorname{sgn} i$.

2.2 For any $r=1,2,\dots,n-1$, we define

$$b_r = \sum_{i=1}^{r} X_{ii}$$

any such b_r gives the following decomposition of algebra $sp(n,R)$.

$$\begin{aligned} g &= n_+^{b_r} \oplus g_0^{b_r} \oplus n_-^{b_r} \\ n_+^{b_r} &\equiv R\{X \in g, [b_r,X] = \alpha_X^r X, \text{ where } \alpha_X^r > 0\} \\ g_0^{b_r} &\equiv \{X \in g, [b_r,X] = 0\} \\ n_-^{b_r} &\equiv R\{X \in g, [b_r,X] = -\alpha_X^r X, \text{ where } \alpha_X^r > 0\}. \end{aligned} \quad (2)$$

This decomposition will be used as a starting point for our construction. More details about properties of such decompositions can be found in Ref. 1, sec. 4.

2.3 The Weyl algebra W_{2N_r} is the complex associative algebra with identity generated by $2N_r$ elements

$$\begin{aligned} &p_{it},\ p_{i,-t},\ p_{i,-j} \quad (p_{i,-j} = \varepsilon_i \varepsilon_j p_{j,-i}) \\ &q_{it},\ q_{i,-t},\ q_{i,-j} \quad (q_{i,-j} = \varepsilon_i \varepsilon_j q_{j,-i}), \end{aligned} \quad (3)$$

where $i,j=1,2,\dots,r$, $t=r+1,r+2,\dots,n$. They satisfy the commutation relations

$$\begin{aligned} [p_{it}, q_{js}] &= \delta_{ij}\delta_{st} 1 \\ [p_{i,-t}, q_{j,-s}] &= \delta_{ij}\delta_{st} 1 \\ [p_{i,-j}, q_{k,-l}] &= \frac{(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}{(1+\delta_{ij})} 1. \end{aligned} \quad (4)$$

2.4 Let g_0 be a real Lie algebra. By $\tilde{g}_0$ we denote its complexification; furthemore, $U(\tilde{g}_0)$ is the enveloping algebra of this complexification.

<u>Definition:</u> A realization of g is a homomorphism τ

$$\tau:\ g \to W_{2N_r} \otimes U(\tilde{g}_0). \quad (5)$$

2.5 The homomorphism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ) of the enveloping algebra $U(\tilde{g})$ into $W_{2N_r} \otimes U(\tilde{g}_0)$.

<u>Definition:</u> Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realization τ is called Schurean or Schur-realization if all central elements $C \in Z(\tilde{g})$ are realised by $1 \otimes C_0$ where C_0's are central elements of the enveloping algebra $U(\tilde{g}_0)$.

2.6 In view of possible applications to the representation theory, we introduce the involution "+" in W_{2N} by means of the following relations

$$\begin{aligned} q_{\alpha\beta}^+ &= -q_{\alpha\beta} \\ p_{\alpha\beta}^+ &= p_{\alpha\beta}, \end{aligned} \quad (6a)$$

where α,β run through all the allowed values of the indices. Similarly, the involution "+" on $U(\tilde{g}_0)$ is defined by

$$Y^+ = -Y \quad \text{for } Y \in g_0. \quad (6b)$$

These involutions define naturally an involution on $W_{2N_r} \otimes U(\tilde{g}_0)$:

$$\left(\sum_j \alpha_j \Pi_j \otimes g_j\right)^+ = \sum_j \bar{\alpha}_j \Pi_j^+ \otimes g_j^+, \quad (6c)$$

where $\Pi_j \in W_{2N_r}$ and $g_j \in U(\tilde{g}_0)$.

<u>Definition:</u> Let g be a real Lie algebra and let "+" be the involution

on $W_{2N_r}\otimes U(\tilde{g}_o)$ described above. A realization τ of g on $W_{2N_r}\otimes U(\tilde{g}_o)$ is called skew-Hermitean, if for all elements $X\in g$, the following relations hold

$$(\tau(X))^{+} = -\tau(X). \tag{7}$$

3. Construction of realizations

3.1 Using the commutation relations (1) we can bring the decomposition (2) into the form

$$\begin{aligned} \overset{b_r}{n}_{+} &= R\{X_{it},\ X_{i,-t},\ X_{i,-j}\}\\ \overset{b_r}{g}_{o} &= R\{X_{ij},\ X_{st},\ X_{s,-t},\ X_{-t,s}\}\sim gl(r,R)\oplus sp(n-r,R)\\ \overset{b_r}{n}_{-} &= R\{X_{ti},\ X_{-t,i},\ X_{-j,i}\}, \end{aligned}\tag{8}$$

where again $i,j=1,2,\dots,r$ and $s,t=r+1,r+2,\dots,n$.

Evidently, the set $\{L_{it}, L_{i,-t}, L_{i,-j}:\ i\le j; i,j=1,2,\dots,r; t=r+1,\dots,n\}$ is a basis in $\overset{b_r}{n}_{+}$. We write the elements of this basis as the matrix

$$\left|\begin{array}{ccccc} L_{1,r+1} & L_{1,r+2} & \cdots & L_{1,n} & \\ \vdots & & & & \\ L_{r,r+1} & L_{r,r+2} & \cdots & L_{r,n} & \\ L_{1,-1} & L_{1,-2} & \cdots & L_{1,-n} & \\ & \ddots & L_{r,-r} & \cdots & L_{r,-n} \end{array}\right| . \tag{9}$$

We introduce an ordering in the above basis in which its elements are ordered lexicographically. The monomials of $U(\overset{b_r}{n}_{+})$ can be then written as the matrices

$$\left|\begin{array}{ccc} n_{1,r+1} & \cdots & n_{1,n}\\ \vdots & & \\ n_{r,r+1} & \cdots & n_{r,n}\\ n_{1,-1} & \cdots & n_{1,-n}\\ \ddots & n_{r,-r}\cdots & n_{r,-n} \end{array}\right| \equiv \left(X_{1,r+1}^{n_{1,r+1}}\dots X_{1,n}^{n_{1,n}}\right)\dots\left(X_{r,r+1}^{n_{r,r+1}}\dots X_{r,n}^{n_{r,n}}\right)\times$$
$$\times\left(X_{1,-1}^{n_{1,-1}}\dots X_{1,-n}^{n_{1,-n}}\right)\dots\left(X_{r,-r}^{n_{r,-r}}\dots X_{r,-n}^{n_{r,-n}}\right), \tag{10}$$

where, of course n_{it}, $n_{i,-t}$, $n_{i,-j}$ belongs to N_o, the set of all

non-negative integers, for any $i,j=1,2,\dots,r$ and $t=r+1,r+2,\dots,n$.

3.2 Now we are able to apply the general construction described in Ref. 1. Let σ_r be an auxiliary representation of the algebra $\overset{b_r}{g}_{o}\oplus\overset{b_r}{n}_{-}$ on a vector space V such that

$$\begin{aligned} &\sigma_r(\overset{b_r}{n}_{-}) = 0\\ &\sigma_r|\overset{b_r}{g}_{o}\ \text{is faithful.} \end{aligned}\tag{11}$$

We denote by W the carrier space of the induced representation $\rho_r = ind(g,\sigma_r)$. If $\{v_1,\dots,v_d\}$ is a basis in the space V, then the vectors

$$\left|\begin{array}{ccc} n_{1,r+1} & \cdots & n_{1,n}\\ \vdots & & \\ n_{r,r+1} & \cdots & n_{r,n}\\ n_{1,-1} & \cdots & n_{1,-n}\\ \ddots & n_{r,-r}\cdots & n_{r,-n} \end{array}\right| \otimes v_i \tag{12}$$

form a basis in W.

3.3 We define the creation and annihilation operators $\bar{a}_{it}$, a_{js} on the space W in the following way:

$$\bar{a}_{it}\left|\begin{array}{ccc} n_{1,r+1} & \cdots & n_{1,n}\\ \cdots & n_{it} & \cdots\\ n_{r,r+1} & \cdots & n_{r,n}\\ n_{1,-1} & \cdots & n_{1,-n}\\ \ddots & n_{r,-r}\cdots & n_{r,-n} \end{array}\right| \otimes v_i \equiv \left|\begin{array}{ccc} n_{1,r+1} & \cdots & n_{1,n}\\ \cdots & n_{it}+1 & \cdots\\ n_{r,r+1} & \cdots & n_{r,n}\\ n_{1,-1} & \cdots & n_{1,-n}\\ \ddots & n_{r,-r}\cdots & n_{r,-n} \end{array}\right| \otimes v_i$$

$$a_{js}\left|\begin{array}{ccc} n_{1,r+1} & \cdots & n_{1,n}\\ \cdots & n_{js} & \cdots\\ n_{r,r+1} & \cdots & n_{r,n}\\ n_{1,-1} & \cdots & n_{1,-n}\\ \ddots & n_{r,-r}\cdots & n_{r,-n} \end{array}\right| \otimes v_i \equiv n_{js}\left|\begin{array}{ccc} n_{1,r+1} & \cdots & n_{1,n}\\ \cdots & n_{js}-1 & \cdots\\ n_{r,r+1} & \cdots & n_{r,n}\\ n_{1,-1} & \cdots & n_{1,-n}\\ \ddots & n_{r,-r}\cdots & n_{r,-n} \end{array}\right| \otimes v_i \tag{13a}$$

and similarly we define the operators $\bar{a}_{i,-t}, a_{i,-t}, \bar{a}_{i,-j}, a_{i,-j}$ i j. Furthermore, we define the operators $\tilde{X}_{ik}$ $i,k=1,2,\dots,r$; $\tilde{X}_{st}, \tilde{X}_{s,-t}, \tilde{X}_{-s,t}$, $s,t=r+1,r+2,\dots,n$, by the relations

$$\tilde{X}_{ik} = 1 \otimes \sigma_r(X_{ik}), \qquad \tilde{X}_{st} = 1 \otimes \sigma_r(X_{st}),$$
$$\tilde{X}_{s,-t} = 1 \otimes \sigma_r(X_{s,-t}), \qquad \tilde{X}_{-s,t} = 1 \otimes \sigma_r(X_{-s,t}). \tag{13b}$$

3.4 According to theorem 3.6 of Ref. 1, the induced representation $\rho_r = \mathrm{ind}(\rho, \sigma_r)$ can be rewritten using the above defined operators (13a-b). We get the formulae

$$\rho_r(X_{ij}) = \sum_{s=r+1}^{n} (\bar{a}_{is}\, a_{js} + \bar{a}_{i,-s}\, a_{j,-s}) + \bar{a}_{i,-j}\, a_{j,-j} + \sum_{k=1}^{r} \bar{a}_{k,-i}\, a_{k,-j} + \tilde{X}_{ij}$$

$$\rho_r(X_{st}) = \sum_{k=1}^{r} (\bar{a}_{k,-s}\, a_{k,-t} - \bar{a}_{kt}\, a_{ks}) + \tilde{X}_{st}$$

$$\rho_r(X_{s,-t}) = \sum_{j=1}^{r} \sum_{k=j}^{r} (\bar{a}_{j,-k}\, a_{kt} - \bar{a}_{j,-t}) a_{js} + \sum_{j=1}^{r} \sum_{k=j+1}^{r} (\bar{a}_{j,-k}\, a_{ks} - a_{j,-s}) a_{jt} + \tilde{X}_{s,-t} \tag{14}$$

$$\rho_r(X_{-s,t}) = \sum_{j=1}^{r} \sum_{k=1}^{j-1} (\bar{a}_{k,-j}\, a_{k,-t} - \bar{a}_{jt})\, a_{j,-s} + \sum_{j=1}^{r} \sum_{k=1}^{j} (\bar{a}_{k,-j}\, a_{k,-s} - a_{js})\, a_{j,-t} + \tilde{X}_{-s,t}$$

$$\rho_r(X_{r,r+1}) = \bar{a}_{r,r+1}$$

$$\rho_r X_{r+1,r} = \sum_{s=r+1}^{n} \rho_r^{*}(X_{r+1,s})\, a_{rs} + a_{r,-s}\, \tilde{X}_{r+1,s} - \sum_{j=1}^{r} \rho_r(X_{jr})\, a_{j,r+1} + 2\bar{a}_{r,-(r+1)}\, a_{r,-r},$$

where $\rho_r^{*}(X_{r+1,s}) = \sum_{k=1}^{r} \bar{a}_{k,-(r+1)}\, a_{k,-s} + \tilde{X}_{r+1,s}$.

3.5 Now the sought skew-Hermitean realizations are obtained easily by replacing the operators in the above expressions by suitable algebraic objects. For details,(see Ref. 1., sections 3.7-3.9). They are given by the formulae:

$$\tau_r'(X_{ij}) = \sum_{s=r+1}^{n} (q_{is}\, p_{js} + q_{i,-s}\, p_{j,-s}) + q_{i,-j}\, p_{j,-j} + \sum_{k=1}^{r} q_{k,-i}\, p_{k,-j} + X_{ij} + \left(n - \frac{r}{2} + \frac{1}{2}\right) \delta_{ij}$$

$$\tau_r'(X_{st}) = \sum_{k=1}^{r} (q_{k,-s}\, p_{k,-t} - q_{kt}\, p_{ks}) + X_{st}$$

$$\tau_r'(X_{s,-t}) = \sum_{j=1}^{r} \sum_{k=1}^{r} (q_{j,-k}\, p_{kt} - q_{j,-t})\, p_{js} + \sum_{j=1}^{r} \sum_{k=j+1}^{r} (q_{j,-k}\, p_{ks} - q_{j,-s})\, p_{jt} + X_{s,-t} \tag{15}$$

$$\tau_r'(X_{-s,t}) = \sum_{j=1}^{r} \sum_{k=1}^{r} (q_{k,-j}\, p_{k,-t} - q_{jt})\, p_{j,-s} + \sum_{j=1}^{r} \sum_{k=1}^{j} (q_{k,-j}\, p_{k,-s} - q_{js})\, p_{j,-t} + X_{-s,t}$$

$$\tau'_r(X_{r,r+1}) = q_{r,r+1}$$

$$\tau'_r(X_{r+1,r}) = \sum_{s=r+1}^{n} \tau^*_r(X_{r+1,s})\, p_{rs} + p_{r,-s} X_{r+1,-s} - \sum_{j=1}^{r} \left(\tau'_r(X_{jr}) + 1\right) p_{j,r+1} + 2q_{r,-(r+1)}\, p_{r,-r},$$

where $\tau^*_r(X_{r+1,s}) = \sum_{k=1}^{r} q_{k,-(r+1)}\, p_{k,-s} + X_{r+1,s} + \frac{r}{2}\delta_{r+1,s}.$

3.8 For any $r=1,2,\dots,n$, the element b_r has the same meaning as the element b from sec. 4 of Ref. 1. Therefore, we can apply theorem 4.3 of Ref. 1 to the realizations (15) obtaining in this way:

Proposition: τ'_r are Schur-realizations of $sp(n,R)$ in the $W_{2N_r} \otimes U(gl(r,R) \oplus sp(n-r,R))$ for any $r=1,2,\dots,n$.

Acknowledgement

The author is grateful to Drs. P. Exner, M. Havlíček and O. Navrátil for useful discussions.

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Received by Publishing Department
on January 4, 1985.

Бурдик Ч. E5-85-13

Новый класс реализаций алгебр Ли sp(n,R)

В данной работе применяется метод работы /1/ для построения бозонных реализаций алгебр Ли sp(n,R). Эти реализации описываются рекуррентными формулами, содержащими $r(2n-\frac{3}{2}r+\frac{1}{2})$ канонических пар и генераторов подалгебры $gl(r,R) \oplus sp(n-r,R)$, где $r = 1,2,\dots,n$. Они антиэрмитовы и шуровские.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1985

Burdík Č. E5-85-13

A New Class of Realizations of the Lie Algebra sp(n,R)

The method of Ref. /1/ is applied to the construction of boson realizations for Lie algebras sp(n,R). These realizations are espressed by means of certain recurrent formulae in terms of $r(2n-\frac{3}{2}r+\frac{1}{2})$ canonical pairs and generators of the subalgebra $gl(r,R) \oplus sp(n-r,R)$, where $r = 1,2,\dots,n$. They are skew-Hermitean and Schurean.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985