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ON THE D-L-R EQUATIONS IN THE EUCLIDEAN FIELD THEORY AND STATISTICAL MECHANICS.

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B.O. INTRODUCTION

Let us consider a system consisting of particles located at the space R^d and interacting through two-particle potential V. Particles may have some internal degrees of freedom labelled by points q of measurable space $\{\Sigma, d\lambda\}$. Then the two-particle potential will be a function on the set $(R^d \otimes \Sigma)^{\otimes 2}$.

Configuration space of our system is then the space of locally finite configurations $\hat{\Omega} \subset (R^d \otimes \Sigma)^{\infty}$. Let us denote by $\hat{\omega}(\Lambda)$ restriction of the given configuration $\hat{\omega} \in \hat{\Omega}$ to a Borel subset $\Lambda \subset R^d$ and by $|\hat{\omega}(\Lambda)|$ a number of particles from the configuration $\hat{\omega}$ which are located at the set Λ. For a given Borel $\Lambda \in R^d$ let us denote by $\mathfrak{F}(\Lambda)$ the smallest σ-algebra generated by the sets of the form $\{\hat{\Omega}_{\Lambda',k}\}$, where Λ' runs over bounded Borel subsets of Λ, $k \in N$ }, where the set $\hat{\Omega}_{\Lambda,k} = \{\hat{\omega} \in \hat{\Omega} \,|\, |\hat{\omega}(\Lambda)| = k\}$. Under mild restriction made on $\{\Sigma, d\lambda\}$ it can be proved that there exists a Polish topology τ on the space $\hat{\Omega}$ whose Borel σ-algebra coincides with $\mathfrak{F}(R^d)$. One important property of these σ-algebras is the property $\mathfrak{F}(R^d) = \mathfrak{F}(\Lambda) \otimes \mathfrak{F}(\Lambda^c)$ valid for any Borel subset $\Lambda \subset R^d$.

On the measure space $\{\hat{\Omega}, \mathfrak{F}(R^d)\}$ there exists a unique measure $\pi_0(z)$ such that for any family $\Lambda_1, \ldots, \Lambda_n$ of Borel subsets of R^d which are pairwise disjoint, the random elements $|\hat{\omega}(\Lambda_1)|, \ldots, |\hat{\omega}(\Lambda_n)|$ are independent and have expectation values equal to $z\,|\hat{\omega}(\Lambda_1)| \cdot \ldots \cdot |\hat{\omega}(\Lambda_n)|$. Here $z > 0$ is the chemical activity. The measure π_0 is a well-known Poisson process on the space $\{\hat{\Omega}_0, \mathfrak{F}(R^d)\}$. By the grand canonical Gibbs ensemble for our system we understand any probabilistic measure ν_∞ on the measure space $\{\hat{\Omega}, \mathfrak{F}(R^d)\}$ which have the properties:

gcG1) ν_∞ is supported on the set

$$\hat{\Omega}(V) = \{\hat{\omega} \in \hat{\Omega} \;\cdot\; \forall \Lambda \subset R^d, \quad \Lambda \quad \infty:$$

$$\operatorname*{ess\,sup}_{(q,x) \in \hat{x} \in \Lambda \otimes \Sigma} \exp \sum_{\hat{y} \in \hat{\omega}(\Lambda^c)} V(q, x; q_y, y) < \infty \}$$

gcG2) $\nu_\infty \cdot E_\Lambda = \nu_\infty$, where the local specification E_Λ is given for every Borel bounded $\Lambda \subset R^d$

$$E_\Lambda\{F \,|\, \mathfrak{F}(\Lambda^c)\}(\hat{\omega}) = \int_{\hat{\Omega}} F(\hat{\omega}(\Lambda) \cup \hat{\omega}(\Lambda^c))\,\nu_\Lambda(z, \hat{\omega}(\Lambda^c) | d\hat{\omega}(\Lambda)), \qquad (B.1)$$

where the conditioned by $\hat{\omega}(\Lambda^c)$ measure $\nu_\Lambda(z, \hat{\omega}(\Lambda^c)|d\hat{\eta})$ is given by its Radon Nikodym derivative:

$$\frac{d\nu_{\Lambda \upharpoonright \mathcal{F}(\Lambda)}(z, \hat{\omega} | d\hat{\eta})}{d\pi_{0 \upharpoonright \mathcal{F}(\Lambda)}} = Z_\Lambda^{-1}(z, \hat{\omega}) \exp(\mathcal{E}(\hat{\eta}|\hat{\omega}(\Lambda^c))) , \tag{B.2}$$

$$Z_\Lambda(z, \hat{\omega}) = \int_{\hat{\Omega}} \pi_{0 \upharpoonright \mathcal{F}(\Lambda)}(d\hat{\eta}) \exp(\mathcal{E}(\hat{\eta}|\hat{\omega}(\Lambda^c))) , \tag{B.3}$$

$$\mathcal{E}(\hat{\eta}|\hat{\omega}) = \sum_{\hat{x} \in \hat{\eta}} \sum_{\hat{y} \in \hat{\varphi}} V(q_x, x; q_y, y) . \tag{B.4}$$

The sets of all Gibbs measures with fixed $\{\Sigma, \lambda\}, z$ and V are denoted by $\mathcal{G}(\{\Sigma,\lambda\}, V, z)$. Similarly as in the field theory this set seems to be too large to exclude some pathological from the physical point of view situations. We restrict ourselves to the tempered g.c. Gibbs measures. They are defined via gcG 2) and

gcG 1´) ν_∞ is supported on the set $\hat{\Omega}^f_\infty$ of configurations with a finite density of particles at infinity.

As has been demonstrated by Ruelle [B2] the set $\hat{\Omega}^f_\infty$ naturally arises as a carrier set of g.c.G. measures corresponding to the class of superstable interactions. A general existence and uniqueness theorems can be found in papers /3/.

B.1. NEUTRAL SYSTEMS WITH POSITIVE DEFINED PAIR POTENTIAL /5/

Let us assume that $\Sigma \subset R^1$ and $d\lambda$ in some positive, even and bounded measure on Σ. We assume also that

$$V(q, x; q', x') = q \cdot q' V(x - x'), \tag{B.5}$$

where

$$0 \leq \hat{V}(k) = (2\pi)^{-d/2} \int e^{ikx} V(x)\, dx . \tag{B.6}$$

Then from (B.5) and (B.6) it follows that $V(\cdot,\cdot)$ is a positive definite function on $(R^d \otimes \Sigma)^{\otimes 2}$. The following new uniqueness criterion has been proved in paper /4/. Conditioned by $\hat{\omega} \in \hat{\Omega}(V)$ the grand canonical partition function $Z_\Lambda(z, \hat{\omega})$ defined by (B.3) may be written also as

$$Z_\Lambda(z, \hat{\omega}) = \sum_{n=0}^\infty \frac{z^n}{n!} \int_{\Lambda^{\otimes n}} d(\hat{x})_n\, e^{\mathcal{E}((\hat{x})_n|(\hat{x})_n)}\, e^{\mathcal{E}((\hat{x})_n|\hat{\omega}(\Lambda^c))} \tag{B.7}$$

and the conditioned correlation functions

$$\rho^n_\Lambda(z, \hat{\omega}|(\hat{x})_n) = Z_\Lambda^{-1}(z, \hat{\omega}) \cdot \sum_{m=0}^\infty \frac{z^{m+n}}{m!} \int_{\Lambda^{\otimes T}} d(\hat{y})_m - \times \exp(\mathcal{E}((\hat{x})_n \cup (\hat{y})_m | (\hat{x})_n \cup (\hat{y})_m)) \cdot \exp(\mathcal{E}((\hat{x})_n \cup (\hat{y})_m | \hat{\omega}(\Lambda^c))) . \tag{B.8}$$

can be analytically continued in the complex parameter ζ to some neighbourhoods of $\mathrm{Re}\zeta$ and $\mathrm{Im}\zeta$ axes. These analytic continuations are given by the formulas

$$Z_\Lambda(z, \hat{\omega}, \zeta) = \int_\Omega \pi_{0 \upharpoonright \mathcal{F}(\Lambda)} d(\hat{\omega}(\Lambda)) \cdot \exp(\mathcal{E}(\hat{\omega}(\Lambda)|\hat{\omega}(\Lambda))) \cdot \exp(-i\zeta\, \mathcal{E}(\hat{\omega}(\Lambda)|\hat{\omega}(\Lambda^c))). \tag{B.9}$$

and similarly for $\rho^n_\Lambda(z, \hat{\omega}, \zeta)$. Define the infinite volume conditioned (by $\hat{\omega}$) pressure (whenever exists):

$$p^{\hat{\omega}}_\infty(z, \zeta) = - \lim_{\Lambda \uparrow R^d} \frac{1}{|\Lambda|} \ln Z_\Lambda(z, \hat{\omega}, \zeta). \tag{B.10}$$

In paper [4] we have proved the following perspectively looking new criterion for uniqueness.

Theorem B.1.1.

Let us assume that V fulfills (B.5), (B.6) and additionally is Γu-regular function and $V(0) < \infty$, $V \in L^1(R^d)$. Assume that $\hat{\omega} \in \hat{\Omega}(V)$, $z_0 > 0$ are such that the following assumptions are fulfilled.

1. z_0 is the regular point of $p^{(\phi)}_\infty(z)$.
2. $p^{(\hat{\omega})}_\infty(z, \zeta) = p^{(\phi)}_\infty(z_0)$ for $\zeta \in [(+1-\epsilon)i, (1+\epsilon)i]$ with some $\epsilon > 0$.
3. One-particle infinite-volume correlation function $\rho^1_\infty(z_0, \hat{\omega})$ is translationally invariant.
4. There exists convex regions $O_n(0, i)$ – in the complex plane C^1, which contain the points $\zeta = i$ and $\zeta = 0$ to which one can analytically continue the conditioned correlation functions $\rho^n_\Lambda(z_0, \hat{\omega}, \zeta)$. Moreover in the sets $O_n(0, i)$ there exist uniform in Λ estimates:

$$\forall_{(\hat{x})_n} \sup_{\zeta \in O_n(0,i)} |\rho^n(z_0, \hat{\omega}, \zeta, |(\hat{x})_n)| < \xi_n((\hat{x})_n) < \infty . \tag{B.11}$$

Then the thermodynamic limits (in the sense of a weak convergence) of the conditioned measure $\nu_\Lambda(z_0, \hat{\omega})$ exist and are equal to $\nu_\infty(z, \phi)$.

Sketch of the Proof:

Let $\mu^{\circ}_{V}(d\phi)$ be a Gaussian measure on the space $S'(R^d)$ with mean equal to zero and the covariance given by V. In terms of this measure conditioned partition function (B.3) and conditioned correlation functions can be rewritten as:

$$Z_{\Lambda}(z,\hat{\omega},\zeta) = \frac{\int_{S'(R^d)} \mu^{\circ}_{V}(d\phi) \exp(z \int_{\Lambda} d(\hat{x}) : \cos(a\phi(x) + \zeta\,\mathcal{E}((\hat{x})|\hat{\omega}(\Lambda^c)):)}{\int_{S'(R^d)} \mu^{\circ}_{V}(d\phi) \prod_{y\in\hat{\omega}(\Lambda^c)} : e^{ia_y \phi(y)} :} \quad (B.12)$$

and

$$\rho^{n}_{\Lambda}(z,\hat{\omega},\zeta)(\hat{x})_n) = z^n \int_{S'(R^d)} \mu^{\circ}_{\Lambda}(d\phi) z,\hat{\omega},\zeta) \prod_{i=1}^{n} : e^{ia_i \phi(x_i)} : \quad (B.13)$$

respectively. Here

$$\mu_{\Lambda}(d\phi|z,\hat{\omega},\zeta) = Z^{-1}_{\Lambda}(z,\hat{\omega},\zeta) \exp(z \int_{\Lambda} d(\hat{x}) : \cos(a\phi(x) + \zeta\,\mathcal{E}((\hat{x})_n|\hat{\omega}(\Lambda)):) \times$$
$$\times \mu^{\circ}_{V}(d\phi). \quad (B.14)$$

From these formulas it follows that on the line Reζ the correlation inequalities very similar to those used for the sine-Gordon theories hold with the same consequences. The conditioning in the space $\hat{\Omega}$ corresponds here to the value $\zeta = i$. Using the analysis based on the notion of the $*$-weak topology in the standard Banach space (similar to those described in the Ruelle book /8/) and the Mayer-Montrolle equations, it is possible to prove locally uniform convergence of the whole family of the correlation functions in the regions $O_n(0,i)$ under the assumptions that the symmetrized one-particle correlation functions

$$C^{1}_{\Lambda}(z,\hat{\omega},t)(q,x)) = \frac{1}{2}[\rho^{1}_{\Lambda}(z,\hat{\omega},t|(-q,x)) + \rho^{1}_{\Lambda}(z,\hat{\omega},t|(q,x))] \quad (B.15)$$

converge in thermodynamic limit to those corresponding to the empty external configuration. Then arguments very similar to those used in the proof of the Theorem A.2.3 supported with the Vitali Theorem lead to the proof.

In the region of small z, where the contraction map principle can be applied to the Kirkwood-Salsburg equations, it is possible to verify all the hypotheses of the Theorem B.1.1.

Theorem B.1.2.

Let V be as in the Theorem B.1.1. Then for all $|z|$ suff. small and $\hat{\omega} \in \hat{\Omega}^{f}_{\infty}$ all the assumptions of the Theorem B.1.1. hold.

Corollary B.1.

In the situation described in the Theorem B.1.2. there exists at least one tempered g.c. G measure.

But the most fascinating feature of the Theorem B.1.1. not explored fully, is its weak restriction on z!. One-dimensional systems are expected to check the usefulness of this Theorem.

B.2. LEE-YANG-LIKE THEOREM FOR A SYSTEM WITH REPULSIVE PAIR INTERACTION

In the theory of lattice systems the Lee-Yang theorem is one of the fundamental tools to establish the analyticity in the large domains of coupling, thus excluding the possibility of phase transitions there. See review in /6/ for the present status of this tool. In the case of Ising model, it says that in the presence of (no matter how small) external magnetic field the appearance of a phase transition is impossible.

For the class of continual systems we prove a similar statement. Let us consider at temperature $T = 1/\beta$ one-component system of particles interacting through pair potential V. We assume that an external field Ψ is switched on.

Theorem B.2.

Assume that $V(x) \geq 0$; $V(x) \in L^1(R^d)$, V is lower regular, $e^{-\beta\Psi} \in L_1(R^d)$. Then there exists for any $z \geq 0$, $0 < \beta < \infty$ at least one tempered grand canonical Gibbs measure.

Sketch of the Proof

For a sequence of parameters $t_1, \ldots, t_n \in [0,1]$ and external configurations $\hat{\omega} \in \hat{\Omega}^{f}_{\infty}$ let us define the following - particle functions:

$$\rho^{n}_{\Lambda}(z,\beta,\hat{\omega}|(t,x)_n) =$$
$$= \exp(\beta\mathcal{E}((t,x)_n|(t,x)_n)\cdot\exp(\beta\mathcal{E}((t,x)_n|\hat{\omega}(\Lambda^c)))\cdot\bar{\rho}^{n}_{\Lambda}(z,\beta,\hat{\omega}(\Lambda^c)|(t,x)_n), \quad (B.16)$$

where:

$$\bar{\rho}^{n}_{\Lambda}(z,\beta,\hat{\omega}(\Lambda^c)|(t,x)_n) =$$

$$= \sum_{m=0}^{\infty} \frac{z^m}{m!} \int \prod_{i=1}^{m} e^{-\beta\Psi(y_i)} dy_i \exp(-\beta\mathcal{E}((t,x)_n \cup (1,y)_m \mid (t,x)_n \cup (q,y)_m) \times$$

$$\times \exp(-\beta\,\mathcal{E}((1,y)_m \mid \hat{\omega}(\Lambda^c))) \exp(-\beta\,\mathcal{E}((t,x)_n \mid \hat{\omega}(\Lambda^c))\,. \tag{B.17}$$

are connected to standard correlation functions $\rho_\Lambda^n(z,\beta,\hat{\omega}\mid(x)_n)$ by the formula:

$$\rho_\Lambda^n(z,\beta,\hat{\omega}\mid(x)_n) = z^n \tilde{\rho}_\Lambda^n(z,\beta,\hat{\omega}\mid(t,x)_n)\mid t_i = 1\,. \tag{B.18}$$

It is possible to derive the following system of linear integral equations for the functions $\hat{\rho}_\Lambda^n$

$$\rho_\Lambda^n(z,\beta,\hat{\omega}(\Lambda^c)\mid(t,x)_n) = \Pi_\Lambda(z,1,\beta,\hat{\omega}(\Lambda^c)\mid(t,x)_n) +$$

$$+\beta z^2\, \Pi_\Lambda(z,1,\beta,\hat{\omega}(\Lambda^c)\mid(t,x)_n) \times$$

$$\times \int_0^1 d\alpha \int_0^1 d\tau \int dy_1\, e^{-\beta\Psi(y_1)} \int dy\, e^{-\beta\Psi(y)}\, \mathcal{E}((t,x)_n \mid (1,x)) \times$$

$$\times\, \mathcal{E}((q,y)\mid(1,y_1))\cdot \Pi_\Lambda(-z,\alpha,\beta,\hat{\omega}(\Lambda^c)\mid(t,x)_n) \times \tag{B.19}$$

$$\times \exp-\beta\alpha\,\mathcal{E}((t,x)_n\mid(1,y)_1)\exp-\beta\tau\,\mathcal{E}((1,y)\mid(1,y)) \times$$

$$\times \exp-\beta\mathcal{E}((\hat{y})_1\mid\hat{\omega}(\Lambda^c))\exp(-\beta\,\mathcal{E}((\hat{y})_2\mid\hat{\omega}(\Lambda^c)) \times$$

$$\times\, \hat{\rho}_\Lambda^{m+2}(z,\beta,\hat{\omega}(\Lambda^c)\mid(\alpha,t)_n,(\tau,y_1),(1,y_2))\,,$$

where

$$\Pi_\Lambda(z,t,\beta,\hat{\omega}(\Lambda^c)\mid(t,x)_n) =$$

$$= \exp - z\int_\Lambda [1-\exp(-\beta t\sum_{i=1}^{m} V(x_i-y)\,t_i\,]\, e^{-\beta\mathcal{E}(y)\,\omega(\Lambda^c)}\, d\mu_\psi(y) \tag{B.20}$$

and

$$d\mu_\psi(y) = e^{-\beta\Psi(y)}\, dy\,. \tag{B.21}$$

The proof relies on the analysis of these systems of equations. Let us denote by $\hat{\mathbf{G}}_\Lambda(z,\beta,\hat{\omega}(\Lambda^c))$ a generating linear operator for equations (B.19). It is possible to define a special Banach space in which the norm of $\|_\Lambda \mathbf{G}_\Lambda \|_\Lambda$ can be bounded uniformly from above by 1, for any $z \geq 0$! Applying some additional topological arguments one can prove then that the thermodynamic limits of $\rho_\Lambda^n(z,\beta,\hat{\omega}(\Lambda^c)\mid(\hat{x})_n)$ are equal to those with $\hat{\omega}=\phi$.

Using Wiener integrals [8] a similar theorem can be proved at least for the Maxwell-Boltzmann statistic case.

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Гелерак Р. E5-85-109

Уравнения ДЛР в евклидовой квантовой теории поля и статистической механике. Статистическая механика

Рассматриваются два класса систем классических частиц с точки зрения гиббсовского подхода. В случае нейтральных систем частиц, взаимодействующих с помощью парного, положительно определенного потенциала, доказан новый критерий единственности предельного гиббсовского состояния. Для систем частиц, взаимодействующих с помощью отталкивающего парного потенциала и в присутствии внешнего поля, доказана единственность предельной меры Гиббса умеренного роста.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Gielerak R. E5-85-109

On the D-L-R Equations in the Euclidean Field Theory and Statistical Mechanics. Statistical Mechanics

Two classes of systems of classical point particles are considered from the Gibbsian point of view. New criterion for uniqueness of the grand canonical Gibbs measure for the case of neutral systems of particles interacting via two-body potential of positive type is proved. In the case of systems with repulsive two-particle interactions, enclosed in an external field, the uniqueness of the tempered, grand canonical Gibbs measure is proven.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1985