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A DISCRETE MODIFIED COUNTER WITH PROLONGING DEAD TIME

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1. INTRODUCTION

The dead time, B, of the counter with prolonging dead time is defined as the time period during that at least one impulse of particle is present. It is produced after registration of all impulses of emitted particles. The dead time distribution is known only in special cases. Takács [1,2] has derived it for the case of a Poisson homogeneous input process of emitted particles. Some asymptotic properties, in a language of the M/GI/ queue, may be found in paper by Afanas'eva, Mikhailova [3].

The idle period, I, of the counter with prolonging dead time is defined as the time period during that the counter is idle. The sum, C, of the dead time and the following by him idle period is said to be the cycle. That is, the cycle is the time period between two successive registered particles. The distribution of the cycle or its Laplace transform were obtained only in the form of complicated contour integrals [4] or in the form of integral equations [5,6]. The Laplace transform in the explicit form has been derived in [7].

The automated blob-length measurements in track chambers in high-energy physics [8] lead to a counter with prolonging dead time where the input process and the length of particles are discrete random variables with the same step h. For the case of the geometric input, the distribution law of the length of the dead time (in the queueing theory language) has been obtained in [8]. The general case of a discrete GI/GI/∞ has been described in [9].

In the present note we derive the joint distribution of the dead time and the idle period for a discrete modified counter. In the particular case of the geometric input of emitted particles we derive it in a different way to the general case. For this aim we use the independence of the dead time from the idle period.

We suppose that the counter is idle before the registration process. The discrete modified counter with prolonging dead time (in the sequel we shall write only discrete modified counter) is a counter for which the first dead time is produced by sequences of impulses, $\{\chi_k\}_{k=1}^{\infty}$, and interarrival times, $\{T_k\}_{k=1}^{\infty}$. $\{\chi_k\}_{k=1}^{\infty}$ are assumed to be independent positive discrete random variables with the distribution laws

$$h_k(n) = P(\chi_k = nh), \quad n \geq 1, \quad k \geq 1, \tag{1.1}$$

where $\sum_{n=1}^{\infty} h_k(n) = 1$, $k \geq 1$. The sequence $\{\chi_k\}_{k=1}^{\infty}$ is independent of the interarrival times $\{T_k\}_{k=1}^{\infty}$, where $T_k = \tau_k - \tau_{k+1}$, $k \geq 1$, and $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of the arrival moments of particles with $0 = \tau_1 < \tau_2 < \dots$. We assume that $\{T_k\}_{k=1}^{\infty}$ is a sequence of positive discrete random variables with the same step h and with the distribution laws

$$f_k(n) = P(T_k = nh), \quad n \geq 1, \quad k \geq 1, \tag{1.2}$$

where $\sum_{n=1}^{\infty} f_k(n) = 1$, $k \geq 1$. Moreover, we assume that the successive dead times are produced by the analogous way independently of previous ones. This counter will be denoted by $\mathcal{C} = (f_1, f_2, \dots; h_1, h_2, \dots)$. The basic properties of the modified counter (not necessarily discrete) are described in /10/.

2. DEAD TIME AND IDLE PERIOD

In the present section we derive the joint distribution law, $W(n,m)$ of the dead time and the following by him idle period, that is, $W(n,m) = P(B = nh, I = mh)$, $n, m \geq 1$. For the simplicity we put $h = 1$.

For a given counter $\mathcal{C} = (f_1, f_2, \dots; h_1, h_2, \dots)$, it is convenient to consider a sequence of discrete modified counters, $\{\mathcal{C}_k\}_{k=1}^{\infty}$, where $\mathcal{C}_k = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$. Also we define the dead time, B_k, the idle period, I_k, and the cycle, C_k, of the counter $\mathcal{C}_k$, for any $k \geq 1$.

Let us put

$$W_k(n,m) = P(B_k = n, I_k = m), \quad n, m \geq 1, \quad k \geq 1. \tag{2.1}$$

It is clear that $W_1(n,m) = W(n,m)$. Suppose that the first dead time of any counter $\mathcal{C}_k$ is produced by sequences $\{\chi_n^k\}_{n=1}^{\infty}$ and $\{T_n^k\}_{n=1}$, where $\chi_n^k = \chi_{k+n-1}$, $T^k = T_{k+n-1}$.

We define

$$W_k(n,j,m) = P(B_k = n, \chi_1^k = j, I_k = m). \tag{2.2}$$

It is evident that

$$W_k(n,m) = \sum_{j=1}^{n} W_k(n,j,m). \tag{2.3}$$

Using the independence of $\{\chi_k\}_{k=1}^{\infty}$ from $\{T_k\}_{k=1}^{\infty}$ we have, for any $k \geq 1$ and $m \geq 1$,

$$W_k(1,1,m) = h_k(1) f_k(m+1), \quad \overline{W}_k(1,m) = W_k(1,1,m). \tag{2.4}$$

$$W_k(2,1,m) = h_k(1) f_k(1) \overline{W}_{k+1}(1,m),$$
$$W_k(2,2,m) = h_k(2)(f(m+2) + f_k(1) \overline{W}_{k+1}(1,m)). \tag{2.5}$$

For the general case $n \geq 3$ we obtain

$$W_k(n,1,m) = h_k(1) f_k(1) W_{k+1}(n-1, m). \tag{2.6}$$

Define by recurrence, for $2 \leq j \leq n-1$,

$$W_k(n,j,m) = h_k(j)(f_k(j) \overline{W}_{k+1}(n-j,m) + \sum_{j=1}^{j-1} f_k(i) A_k(n,j,m,i)), \tag{2.7}$$

where

$$A_k(n,j,m,i) = \sum_{r=1}^{[\frac{j-i+2}{2}]} B_k(n,j,m,i,r). \tag{2.8}$$

Here $B_k(n,j,m,i,r)$ is the probability that $B_k = n, \chi_1^k = j$, $T_1^k = i$ and from the impulses of the second and following it impulses, r cycles, with the total length of the first $r-1$ cycles and the r-th dead time equal to $n-i$, without of the impulse of the first particle, are creared. The event $\{B_k = n, I_k = m\}$ is dependent only on $\{\chi_k, \dots, \chi_{k+n-1}, T_k, \dots, T_{k+n-1}\}$. Therefore

$$B_k(n,j,m,i,r) = \sum \overline{W}_{s_1}(j_1,k_1) \dots \overline{W}_{s_{r-1}}(j_{r-1},k_{r-1}) \cdot W_{s_r}(j_r, m), \tag{2.9}$$

where $s_1 = k+1$, $s_2 = s_1 + j_1$, ..., $s_r = s_{r-1} + j_{r-1}$, and the summation is taken over integers $j_s, k_s \geq 1$, for $s = 1, \dots, r-1$, $j_r \geq n-j$ with $j_1 + k_1 + \dots + j_{r-1} + k_{r-1} + j_r = n-i$.

By analogous manner we have, for $j = n$,

$$W_k(n,n,m) = h_k(n)(f_k(n+m) + \sum_{i=1}^{n-1} f_k(i) A_k(n,n,m,i)), \tag{2.10}$$

where

$$A_k(n,n,m,i) = \sum_{r=1}^{[\frac{n-i+1}{2}]} B_k(n,n,m,i,r) \tag{2.11}$$

and

$$B_k(n,n,m,i,r) = \sum W_{s_1}(j_1,k_1) \dots \overline{W}_{s_{r-1}}(j_{r-1},k_{r-1}) \cdot W_{s_r}(j_r, k_r + m),$$

where s_i are same as above and the summation runs over integers $j_s, k_s \ge 1$ for $s = 1, \dots, r-1$, $j_r \ge 1$, $k_r \ge 0$ with $j_1 + k_1 + \dots + j_r + k_r = n - i$.

This proves the following theorem:

Theorem 1. The joint distribution of the dead time and the idle period of the discrete modified counter $\mathcal{C}_k = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $k \ge 1$, is given by the formula (2.3), where $W_k(n,m)$ is calculated from (2.4) through (2.12).

Remark. For the distribution law, $P_k^C(i) = P(C_k = i)$, of the cycle of the discrete modified counter $\mathcal{C}_k = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$ we have

$$P_k^C(i) = \sum_{\substack{n+m=i \\ n, m \ge 1}} W_k(n,m), \quad i \ge 2, \quad k \ge 1. \qquad (2.13)$$

3. DEAD TIME

The use of the formula (2.3), for the dead time distribution law only, leads to the following recurrent formulae:

Define

$$P_k(n) = P(B_k = n), \qquad (3.1)$$

and

$$W_k(n, j, .) = \sum_{m=1}^{\infty} W_k(n, j). \qquad (3.2)$$

Then $P_k(n) = \sum_{j=1}^{n} W_k(n, j, .)$. Putting $Q_k(n) = \sum_{i=n}^{\infty} f_k(i)$, $n \ge 1$, we have $W_k(1,1,.) = h_k(1)\, Q_k(2) = P_k(1)$, $W_k(2,1,.) = h_k(1)\, f_k(1)\, P_{k+1}(1)$, $W_k(2,2,.) = h_k(2)(Q_k(3) + f_k(1)\, P_{k+1}(1))$. For $n \ge 3$ we obtain $W_k(n,1,.) = h_k(1)\, f_k(1)\, P_{k+1}(n-1)$. If $2 \le j \le n-1$, then

$$W_k(n, j, .) = h_k(j)(f_k(j)\, P_{k+1}(n-j) + \sum_{i=1}^{j-1} f_k(i)\, A_k(n, j, ., i)),$$

where

$$A_k(n, j, ., i) = \sum_{r=1}^{\left[\frac{j-i+2}{2}\right]} B_k(n, j, ., i, r)$$

and $B_k(n, j, ., r, .) = \sum W_{s_1}(j_1, k_1) \dots W_{s_{r-1}}(j_{r-1}, k_{r-1})\, P_{s_r}(j_r)$.

Here the summation is the same as in (2.9).

Similarly, for $j = n$, we have

$$W_k(n, n, .) = h_k(n)(Q_k(n+1) + \sum_{i=1}^{n-1} f_k(i)\, A_k(n, n, ., i)),$$

where

$$A_k(n, n, ., i) = \sum_{r=1}^{\left[\frac{n-i+1}{2}\right]} B_k(n, n, ., i, r),$$

and

$$B_k(n, n, ., i, r) = \sum W_{s_1}(j_1, k_1) \dots W_{s_{r-1}}(j_{r-1}, k_{r-1})\, P(B_{s_r} = j_r,\ I_{s_r} \ge k_r + 1).$$

In the last formula the summation is the same as in (2.12).

4. GEOMETRIC INPUT

It is evident that, for a modified counter, the dead time and the idle period are, in general, dependent random variables. In particular case of the counter $\mathcal{C} = (f_1, f_2, \dots; h_1, h_2, \dots)$, where

$$f_k(n) = (1-p)\, p^{n-1}, \quad n \ge 1, \quad k \ge 1, \qquad (4.1)$$

and $0 < p < 1$, it is simple to show that the idle period distribution law, $P_k^I(n) = P(I_k = n)$, obeys the geometric law (4.1) with the same parameter p, for any $k \ge 1$. Using the known properties of the geometric input of emitted particle, or directly the formula (2.3) we may see that in this particular case the dead time, B_k, and the idle period, I_k, are independent random variables.

In this section we derive simpler formulae, for the case of (4.1), than (2.3) by a different way from using the independence of the dead time from the idle period. The used method is analogous to that from [8].

Thus, we suppose that the counter at the moment $t = 0$ is idle and the first particle arrives at the time τ_1 distributed by

$$P(\tau_1 = n) = (1-p)\, p^n, \quad n = 0, 1, \dots. \qquad (4.2)$$

Denote by A an event that the dead time begins from $t = 0$. Due to (4.2), we have $P(A) = 1 - p$. We define $p_k(j) = P(\chi_1^k = j, A)$. Therefore

$$p_k(j) = h_k(j)(1-p), \qquad (4.3)$$

and for $j = 0$ we put

$$p_k(0) = p. \tag{4.4}$$

We denote the conditional probability in question, $P(B_k = n|A)$, by $P(n)$, and the joint probability, $P(B_k = n, A)$, by $PP_k(n)$. Clearly

$$P_k(n) = PP_k(n)/(1-p), \quad n \geq 1, \quad k \geq 1. \tag{4.5}$$

Let $\bar{W}_k(n,j) = P(B_k = n, \chi_1^k = j, A)$, $n \geq 1$, $j = 1,\dots,n$, $k \geq 1$. Then

$$PP_k(n) = \sum_{j=1}^{n} \bar{W}_k(n,j), \quad n \geq 1, \quad k \geq 1. \tag{4.6}$$

It is evident that $\bar{W}_k(n,j) = W_k(n,j,.)$, where $W_k(n,j,.)$ is the expression (3.2) from Section 3.

Using the properties of $W_k(n,j)$ and the independence of the dead time from the idle period we may prove the following relationships for any $k \geq 1$:

$$\bar{W}_k(1,1) = p_k(1)\, p_k(0), \qquad PP_k(1) = \bar{W}_k(1,1). \tag{4.7}$$

Let $n \geq 2$ and put $B_k(1,1) = PP_k(1)$,

$$B_k(n,1) = PP_k(n), \tag{4.8}$$

then

$$\bar{W}_k(n,1) = p_k(1)\, B_{k+1}(n-1,1). \tag{4.9}$$

Define by recurrence, for $2 \leq j \leq n-1$,

$$B_k(n,j) = pB_k(n-1,j-1) + B_{k+1}(n-1,j-1) \sum_{i=1}^{j-1} p_k(i) + \sum_{i=j}^{n-1} \bar{W}_k(n-1,i). \tag{4.10}$$

then

$$\bar{W}_k(n,j) = p_k(j)\, B_{k+1}(n-1,j). \tag{4.11}$$

For the following it is useful to introduce a function D_k, $k \geq 1$, via

$D_k(0) = p$,

$$D_k(m) = pD_k(m-1) + D_{k+1}(m-1) \sum_{i=1}^{m} p_k(i), \quad m \geq 1, \tag{4.12}$$

then

$$\bar{W}_k(n,n) = p_k(n)\, D_{k+1}(n-1), \tag{4.13}$$

and this proves the following theorem.

Theorem 2. The probability laws of the dead time of the discrete modified counter $\mathcal{C}_k = (f_k, f_{k+1}, \dots; h_k, h_{k+1}, \dots)$, $k \geq 1$, with (4.1) is given by the formula (4.5), where $PP_k(n)$ is calculated from (4.3) through (4.13).

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Двуреченский А., Ососков Г.А. E5-84-828

Дискретный модифицированный счетчик с мертвым временем продлевающегося типа

Автоматические измерения длин сгустков в трековых камерах физики высоких энергий приводят к дискретному модифицированному счетчику с мертвым временем продлевающегося типа. Настоящая статья рассматривает вопрос определения совместного распределения мертвого времени и последующего за ним свободного периода. Частный случай счетчика с геометрическим входным потоком изучается более подробно. С использованием независимости мертвого времени от последующего за ним свободного периода определяется распределение мертвого времени путем, отличным от общего случая.

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A Discrete Modified Counter with Prolonging Dead Time

The automated blob-length measurements in track chambers in high-energy physics lead to a discrete modified counter with prolonging dead time with the geometric input of emitted particles. The present paper deals with the joint distribution determination of the dead time and following it idle period. The particnlar case of the counter with the geometric input is studied in more detail. Using the independence of the dead time from the idle period we determine the dead time distribution in a different way as in general.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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