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A NEW CLASS OF REALIZATIONS OF THE LIE ALGEBRA gl(n+1,R)

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1. Introduction

1.1 In our preceding paper [1], the method of constructing realizations for an arbitrary real semisimple algebra g was presented. It was shown that any induced representation can be rewritten as the so-called boson representation. The construction starts from a decomposition $g = n_+^b \oplus g_0^b \oplus n_-^b$ of g, which is a simple generalization of the triangle decomposition [2]; it employs substantially induced representations [3] of g with respect to a suitable representation σ of the subalgebra g_0^b n_-^b. It was proved in Ref. 1 that the method gives realizations which posses two properties permitting their application in the representation theory. They are skew-Hermitean and Schurean.

1.2 In the papers [4-5] (see also Ref. 6), extensive families of realizations for the real algebras gl(n+1,R) and sl(n+1,R) were constructed. The method is based on the recurrent formulae derived in Ref. 4 which yield realizations of gl(n+1,R) in terms of n canonical pairs and generators of gl(n,R). Using these formulae, we can obtain pure boson realizations with $\frac{1}{2}d(2n+1-d)$ canonical pairs (see Table 4 of Ref. 6) for d=1,2,...,n.

1.3 In the present paper, we apply the method of Ref. 1 to the case of algebras gl(n+1,R). For any r=1,2,...,n, we construct recurrent formulae which give realizations of gl(n+1,R) in terms of r(n+1-r) - canonical pairs and generators of the subalgebra $gl(r,R)\oplus gl(n+1-r,R)$. It is also shown that these formulae contain the above mentioned realizations of Ref. 4 as a particular case.

1.4 The paper is organised as follows. All the necessary prerequisites are listed in § 2. The following § 3 contains the main results. Here the new wide families of realizations are derived. In the last section the results are discussed, and in particular, a detailed comparison with the realizations, which were derived in Ref. 4 is made.

2. Preliminaries

2.1 The Weyl algebra W_{2N_r} is the associative algebra over $\mathbb{C}$ with identity generated by $2N_r$ elements $\{p_{ti}, q_{ti}\}$, where $N_r=(n-r+1)r$ and $i=1,2,\dots,r$; $t=r+1,r+2,\dots,n+1$, which satisfy the relations

$$\left[p_{ti},q_{sj}\right] = \delta_{ji}\,\delta_{st}\,1 \qquad \begin{matrix} i,j=1,2,\dots,r \\ s,t=r+1,r+2,\dots,n+1. \end{matrix} \tag{1}$$

2.2 Let g be a real Lie algebra. By $\tilde{g}$ we denote its complexification, furthermore, $U(\tilde{g})$ is the enveloping algebra of this complexification.

Definition: A realization of a Lie algebra g is a homomorphism

$$\tau :\ g \to W_{2N_r} \otimes U(\tilde{g}_0). \tag{2}$$

2.3 The homomorphism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ) of the enveloping algebra $U(\tilde{g})$ into $W_{2N_r}\otimes U(\tilde{g}_0)$.

Definition: Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realization τ is called Schurean or Schur-realization if all central elements $C\in Z(\tilde{g})$ are realised by $1\otimes C_0$ where the C_0's are central elements of the enveloping algebra $U(\tilde{g}_0)$.

2.4 In view of possible applications to the representation theory, we introduce the involution "+" in W_{2N_r} by means of the following relations

$$\begin{aligned} (q_{ti})^+ &= -q_{ti} \\ (p_{ti})^+ &= p_{ti} \qquad \text{for } i=1,2,\dots,r, \\ &\qquad\qquad\quad t=t+1,r+2,\dots,n+1. \end{aligned} \tag{3a}$$

Similarly, the involution "+" on $U(\tilde{g}_0)$ is defined by

$$Y^+ = -Y \qquad \text{for } Y\in g_0. \tag{3b}$$

These involutions define naturally an involution on $W_{2N_r}\otimes U(\tilde{g}_0)$:

$$\Big(\sum_j \alpha_j \pi_j \otimes g_j\Big)^+ \equiv \sum_j \bar{\alpha}_j\, \pi_j^+ \otimes g_j^+, \tag{3c}$$

where $\pi_j \in W_{2N_r}$ and $g_j\in U(\tilde{g}_0)$.

Definition: Let g be a real Lie algebra and let "+" be the involution on $W_{2N_r}\otimes U(g_0)$ described above. A realization τ of g on $W_{2N_r}\otimes U(\tilde{g}_0)$ is called skew-Hermitean, if for all elements $X\in g$ the following relations hold

$$(\tau(X))^+ = -\tau(X). \tag{4}$$

2.5 Definition: Two realizations τ and $\tilde{\tau}$ are called to be related if an endomorphism θ of $W_{2N_r}\otimes U(\tilde{g}_0)$ exists such that either

$$\tilde{\tau}(X) = \theta(\tau(X))$$

or

$$\tau(X) = \theta(\tilde{\tau}(X)) \qquad \text{for all } X\in g. \tag{5}$$

2.6 The algebra $gl(n+1,R)$ is the $(n+1)^2$ - dimensional real Lie algebra with the standard basis $\{E_{ij}:\ i,j=1,2,\dots,n+1\}$, the elements of which obey:

$$\left[E_{ij},\,E_{kl}\right] = \delta_{jk}\,E_{il} - \delta_{il}E_{kj}. \tag{6}$$

This algebra is a direct sum of its one dimensional centrum generated by the element $E = \sum_{i=1}^{n+1} E_{ii}$ and the simple subalgebra $sl(n+1,R)$.

2.7 For any $r=1,2,\dots,n$, we define

$$b_r = -\sum_{i=1}^{r} E_{ii}. \tag{7}$$

Any such b_r give a decomposition of algebra $g=gl(n+1,R)$ in this way

$$\begin{aligned} g &= n_+^{b_r} \oplus g_0^{b_r} \oplus n_-^{b_r} \\ n_+^{b_r} &= R\{X\in g, [b_r,X] = \alpha_X^r X\ ,\ \text{where}\ \ \alpha_X^r > 0\} \\ g_0^{b_r} &= \{X\in g, [b_r,X] = 0\} \\ n_-^{b_r} &= R\{X\in g, [b_r,X] = -\alpha_X^r X\ \text{where}\ \ \alpha_X^r>0\}. \end{aligned} \tag{8}$$

This decomposition will be used as a starting point for our construction (see also Ref. 1, sec. 4).

3. Construction of realizations

3.1 Using the commutation relations (6) we can bring the decomposition (8) into the form

$$\begin{aligned} n_+^{b_r} &= R\{E_{ti}\} \\ g_o^{b_r} &= R\{E_{ik}, E_{st}\} \sim gl(r,R) \oplus gl(n+1-r,R) \\ n_-^{b_r} &= R\{E_{it}\}, \end{aligned} \tag{9}$$

where again $i,k=1,2,\ldots,r$ and $s,t=r+1,r+2,\ldots,n+1$. Evidently, the set $\{E_{ti}: i=1,2,\ldots,r;\ t=r+1,r+2,\ldots,r+1\}$ is a basis in $n_+^{b_r}$. We introduce an ordering in the above basis in the following way: $E_{ts} < E_{t'i'}$ holds if $i < i'$ or $i = i'$ and $t < t'$, i.e., if we write the elements of this basis as the matrix

$$\begin{vmatrix} E_{r+1,1} & E_{r+2,1} & \cdots & E_{n+1,1} \\ E_{r+1,2} & E_{r+2,2} & \cdots & E_{n+1,2} \\ \vdots \\ E_{r+1,r} & E_{r+2,r} & \cdots & E_{n+1,r} \end{vmatrix}, \tag{10}$$

then its elements are ordered lexicographically. The monomials of $U(\tilde{n}_+^b)$ can be then written as the matrices

$$\begin{vmatrix} n_{r+1,1} & \cdots & n_{n+1,1} \\ \vdots \\ n_{r+1,r} & \cdots & n_{n+1,r} \end{vmatrix} \equiv \left(E_{r+1,1}^{n_{r+1,1}}, \ldots E_{n+1,1}^{n_{n+1,1}}\right) \ldots \left(E_{r+1,r}^{n_{r+1,r}} \ldots E_{n+1,r}^{n_{n+1,r}}\right), \tag{11}$$

where, of course n_{ti} belongs to N_o the set of all non-negative integers, for any $i=1,2,\ldots,r$ and $t=r+1,r+2,\ldots,n+1$.

3.2 Now we apply the general construction described in Ref. 1 to the case of $gl(n+1,R)$. Let σ_r be an auxiliary representation of the algebra $g_o^{b_r} \oplus n_-^{b_r}$ on the space V such that

$$\begin{aligned} \sigma_r(n_-^{b_r}) &= 0 \\ \sigma_r | g_o^{b_r} & \text{ is faithful}. \end{aligned} \tag{12}$$

We denote by W the space of induced representation $\varrho_r = \mathrm{ind}(g, \sigma_r)$. If $\{v_1,\ldots,v_d\}$ is a basis in the space V, then the vectors

$$\begin{vmatrix} n_{r+1,1} & \cdots & n_{n+1,1} \\ \vdots \\ n_{r+1,r} & \cdots & n_{n+1,r} \end{vmatrix} \otimes v_i, \tag{13}$$

where $n_{tj} \in N_o$ for $j=1,2,\ldots,r$, $t=r+1,r+2,\ldots,n+1$ and $i=1,2,\ldots,d$ form a basis in W.

3.3 We define the creation and annihilation operators $\bar{a}_{tj}$, a_{sk} $j,k=1,2,\ldots,r;\ s,t=r+1,r+2,\ldots,n+1$ on the space W in the following way:

$$\begin{aligned} \bar{a}_{tj} \begin{vmatrix} n_{r+1,1} & \cdots & n_{n+1,1} \\ \cdots & n_{tj} & \cdots \\ n_{r+1,r} & \cdots & n_{n+1,r} \end{vmatrix} \otimes v_i &\equiv \begin{vmatrix} n_{r+1,1} & \cdots & n_{n+1,1} \\ \cdots & n_{tj}+1 & \cdots \\ n_{r+1,r} & \cdots & n_{n+1,r} \end{vmatrix} \otimes v_i \\ a_{sk} \begin{vmatrix} n_{r+1,1} & \cdots & n_{n+1,1} \\ \cdots & n_{sk} & \cdots \\ n_{r+1,r} & \cdots & n_{n+1,r} \end{vmatrix} \otimes v_i &\equiv n_{sk} \begin{vmatrix} n_{r+1,1} & \cdots & n_{n+1,1} \\ \cdots & n_{sk}-1 & \cdots \\ n_{r+1,r} & \cdots & n_{n+1,r} \end{vmatrix} \otimes v_i \end{aligned} \tag{14a}$$

(notice the normalization convention we use). Moreover, we define the operators $\tilde{E}_{ik}$, $i,k=1,2,\ldots,r$ and $\tilde{E}_{st}$, $s,t=r+1,r+2,\ldots,n+1$, by

$$\begin{aligned} \tilde{E}_{ik} &= 1 \otimes \sigma_r(E_{ik}) \\ \tilde{E}_{st} &= 1 \otimes \sigma_r(E_{st}). \end{aligned} \tag{14b}$$

3.4 Theorem 3.6 of Ref. 1 states that the induced representation $\varrho_r \equiv \mathrm{ind}(g, \sigma_n)$ can be rewritten using the above defined operators (14a-b). By a simple calculation which employs the commutation relations (6), we get the formulae

$$\varrho_r(E_{ik}) = \sum_{t=r+1}^{n+1} -\bar{a}_{tk}\, a_{ti} + \tilde{E}_{ik}, \qquad i,k=1,2,\ldots,r,$$

$$\varrho_r(E_{st}) = \sum_{i=1}^{r} \bar{a}_{si}\, a_{ti} + \tilde{E}_{st}, \qquad s,t=r+1,r+2,\ldots,n+1,$$

$$\varrho_r(E_{r+1,r}) = \bar{a}_{r+1,r} ,$$

$$\varrho_r(E_{r,r+1}) = \sum_{j=1}^{r} \varrho_r(E_{rj})\, a_{r+1,j} - \sum_{t=r+1}^{n+1} a_{tr}\, \tilde{E}_{t,r+1} . \qquad (15)$$

3.5 Now the sought realizations are obtained easily by replacing the operators in the above expressions by suitable algebraic objects. The mapping

$$\varphi(p_{ti}) = a_{ti}$$
$$\varphi(q_{ti}) = \bar{a}_{ti}$$
$$\varphi(E_{ik}) = \tilde{E}_{ik}$$
$$\varphi(E_{st}) = \tilde{E}_{st} ,$$

where $i,k=1,2,\dots,r$, $t=r+1,r+2,\dots,n+1$ extends naturally to a faithful representation $W_{2N_r} \otimes U(\tilde{g}_o^{b_r})$ on W. Thus there exists the inverse φ^{-1}, and we can define the mappings $\tau_r : g \to W_{2N_r} \otimes U(\tilde{g}_o^{b_r})$ by

$$\tau_r = \varphi^{-1} \circ \varrho_r .$$

According to Proposition 3.8 of Ref. 1, every such τ_r is a realization of $gl(n+1,R)$. We obtain

$$\tau_r(E_{ik}) = \sum_{t=r+1}^{n+1} -q_{tk}\, p_{ti} + E_{ik} , \qquad i,k=1,2,\dots,r ,$$

$$\tau_r(E_{st}) = \sum_{i=1}^{r} q_{si}\, p_{ti} + E_{st} , \qquad s,t=r+1,r+2,\dots,n+1 ,$$

$$\tau_r(E_{r+1,r}) = q_{r+1,r} , \qquad (16)$$

$$\tau_r(E_{r,r+1}) = \sum_{j=1}^{r} \tau_r(E_{rj}) p_{r+1,j} - \sum_{t=r+1}^{n+1} p_{tr}\, E_{t,r+1} .$$

3.6 The realizations (16) are not skew-Hermitean with respect to the conjugation (3a-c), but skew-Hermitean realizations τ_r' can be obtained using the method described in Ref. 1, sec. 3.9; They are given by the following formulae:

$$\tau_r'(E_{ik}) = \tau_r(E_{ik}) - \frac{n+1-r}{2}\, \delta_{ik}$$

$$\tau_r'(E_{st}) = \tau_r(E_{st}) + \frac{r}{2}\, \delta_{st}$$

$$\tau_r'(E_{r+1,r}) = \tau_r(E_{r+1,r}) \qquad (17)$$

$$\tau_r'(E_{r,r+1}) = \sum_{j=1}^{r} \left(\tau_r'(E_{rj}) - \frac{1}{2} \right) p_{r+1,j} - \sum_{t=r+1}^{n+1} p_{tr}\, E_{t,r+1} .$$

3.7 For any $r=1,2,\dots,n$, the elements b_r have the same meaning as the element b from sec. 4 of Ref. 1. Therefore we can apply theorem 4.3 of Ref. 1 to the realizations (17) and (18) obtaining in this way

Lemma: τ_r and τ_r' are Schur-realizations of $gl(n+1,R)$ in the $W_{2N_r} \otimes U(gl(r,R) \oplus gl(n+1-r,R))$ for any $r=1,2,\dots,n$.

4. Discussion

4.1 In the last section we are going to compare our results to the realizations of $gl(n+1,R)$ given in Ref. 4. They are expressed by the formulae

$$\tilde{\tau}(E_{ij}) = q_i p_j + E_{ij} + \tfrac{1}{2}\delta_{ij}\, 1$$

$$\tilde{\tau}(E_{n+1,n}) = -p_n$$

$$\tilde{\tau}(E_{n,n+1}) = q_n \left(\sum_{j=1}^{n} q_j p_j + \frac{n+1}{2} - E_{n+1,n+1} \right) + \sum_{j=1}^{n} q_j\, E_{nj} \qquad (18)$$

$$\tilde{\tau}(E_{n+1,n+1}) = - \sum_{i=1}^{n} q_i p_i - \frac{n}{2} + E_{n+1,n+1} .$$

If we put $r=n$ in the formulae (17), we get

$$\tau_n'(E_{ij}) = -q_{n+1,j}\, p_{n+1,i} - \tfrac{1}{2}\delta_{ij}\, 1 + E_{ij} , \qquad i,j=1,2,\dots,n$$

$$\tau_n'(E_{n+1,n}) = q_{n+1,n} ,$$

(19)

$$\tau_n'(E_{n,n+1}) = \Big(-\sum_{j=1}^{n} q_{n+1,j}\,p_{n+1,j} - \frac{n+1}{2} - E_{n+1,n+1}\Big)p_{n+1,n} + {} + \sum_{j=1}^{n} p_{n+1j}\,E_{nj}\,,$$

$$\tau_n'(E_{n+1,n+1}) = \sum_{i=1}^{n} q_{n+1,i}\,p_{n+1,i} + \frac{n}{2} + E_{n+1,n+1}\,.$$

Comparing the formulae (18) and (19), we see that the realizations $\tilde{\tau}$ and τ_n' are related(cf. the definition 2.5) and that the corresponding automorphism θ is generated by

$$\theta(q_{n+1,i}) = -p_i$$

$$\theta(p_{n+1,i}) = q_i \qquad (20)$$

$$\theta(X) = X \quad \text{for } X \in g_0^{b_n}.$$

4.2 The formulae (18) give, in particular, pure boson realizations with $\frac{1}{2}$ d(2n+1-d) canonical pairs for d=1,2,...,n; for a detailed information see Ref. /4/. An analogons result obviously holds to the other realizations given by (17) , i.e., with r=1,...,n. In this way, we can obtain purely boson realizations with a number of pairs nonequal to $\frac{1}{2}$ d(2n+1-d), for any d=1,2,...,n.

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Бурдик Ч. E5-84-813
Новый класс реализаций алгебр Ли gl(n+1,R)

Метод конструкции антиэрмитовых реализаций, обоснованный в работе /1/, применяется к построению бозонных реализаций для алгебры Ли gl(n+1,R). Полученные реализации выражаются рекуррентными формулами при помощи r(n+1-r) бозонных пар и генераторов подалгебры gl(r,R)⊕gl(n+1-r,R), где r = 1,2,...,n. Они антиэрмитовы и шуровские.

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A New Class of Realizations of the Lie Algebra gl(n+1,R)

The method of Ref. /1/ is applied to the construction of boson realizations for Lie algebras gl(n+1,R). These realizations are expressed by means of certain recurrent formulae in terms of r(n+1-r) canonical pairs and generators of the subalgebra gl(r,R)⊕gl(n+1-r,R), where r = 1,2,...,n. They are skew-Hermitean and Schurean.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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