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THE CALKIN REPRESENTATION FOR A CERTAIN CLASS OF ALGEBRAS OF UNBOUNDED OPERATORS

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1. Introduction

In his classical paper /2/ Calkin constructed explicitly a class of faithful representations of the quotient C^*-algebra $\mathfrak{B}(\mathcal{H})/\mathfrak{J}_\infty(\mathcal{H})$ (Calkin algebra) of the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators on a separable Hilbert space $\mathcal{H}$ by the closed two-sided $*$-ideal $\mathfrak{J}_\infty(\mathcal{H})$ of all compact operators. The Calkin algebra plays an important role in operator theory and also in the theory of C^*- and W^*-algebras.

In the last 15 years it became clearer and clearer that for applications in quantum field theory and quantum statistics topological algebras and their realizations as algebras of unbounded operators arise in a natural way (cf./7/,/9/,/17/ and the references there). So there has been extensive studies of the structure of such algebras and related objects.

The aim of this paper is to generalize the situation described above to the class of algebras of unbounded operators $\mathcal{L}^+(\mathfrak{D})$ with $\mathfrak{D} = \mathfrak{D}^\infty(T)$. Section 2 contains the necessary definitions and notations (cf.also /7/). In section 3 there are given some of the possible generalizations of $\mathfrak{J}_\infty(\mathcal{H})$ in the unbounded case /16/. It is proved that the closure of the set of finite dimensional operators on $\mathfrak{D}$ with respect to the uniform topology $\tau_{\mathfrak{D}}$ is the only proper two-sided closed $*$-ideal $\mathfrak{C}$ in $\mathcal{L}^+(\mathfrak{D})$, except $\mathfrak{D}$ is of type (I) (see section 2). There are given several equivalent characterizations of $\mathfrak{C}$ which are useful for applications. The main result is contained in section 4. There it is shown that the quotient algebra $\mathcal{L}^+(\mathfrak{D})/\mathfrak{C} = \mathfrak{R}$ equipped with the factor topology τ arising from $\tau_{\mathfrak{D}}$ is an $A\hat{O}^*$-algebra, i.e. $\mathfrak{R}[\tau]$ is algebraically and topologically isomorphic to an operator $*$-algebra $\mathcal{A}(\hat{\mathfrak{D}})[\tau_{\hat{\mathfrak{D}}}]$. A class of such isomorphisms is explicitly given by adapting the classical Calkin construction to the unbounded case. Section 5 indicates some directions of further investigations which are now possible.

2. Preliminaries

For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathcal{H}$ let $\mathcal{L}^+(\mathfrak{D})$ denote the $*$-algebra of all operators A with $A\mathfrak{D}\subset\mathfrak{D}, A^*\mathfrak{D}\subset\mathfrak{D}$. The involution is given by $A \longrightarrow A^+ = A^* \upharpoonright \mathfrak{D}$. An Op^*-algebra $\mathcal{A}(\mathfrak{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathfrak{D})$ with unit I (identity operator).
The topology t on $\mathfrak{D}$ given by the seminorms $\varphi \longrightarrow \|A\varphi\|$ for all

$A \in \mathcal{L}^+(\mathcal{D})$ is called the graph topology. This gives rise to a canonical rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}[t']$ and a canonical dual pair $(\mathcal{D}, \mathcal{D}')$, where t' denotes the strong topology in $\mathcal{D}[t]'$. By $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\bar{A}$ we denote the range, the kernel, the closure of A resp. For $A \in \mathcal{L}^+(\mathcal{D})$, A bounded we often denote $\bar{A} \in \mathcal{B}(\mathcal{H})$ also by A. Among the many possible topologies in $\mathcal{L}^+(\mathcal{D})$ we need the uniform topology $\tau_{\mathcal{D}}$ given by the seminorms

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi \in \mathcal{M}} |\langle \varphi, A\psi \rangle|$$

and the quasi-uniform topology $\tau^{\mathcal{D}}$ given by the seminorms

$$A \longrightarrow \|A\|_B^{\mathcal{M}} = \sup_{\varphi \in \mathcal{M}} \|BA\varphi\| \quad \text{for all } B \in \mathcal{L}^+(\mathcal{D}).$$

In both cases $\mathcal{M}$ runs over the family of t-bounded subsets of $\mathcal{D}$. $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ is a topological *-algebra, $\mathcal{L}^+(\mathcal{D})[\tau^{\mathcal{D}}]$ is a topological algebra.

The most important domains $\mathcal{D}$ are those where t is a Fréchet-topology. Then t can be given by a set of seminorms $\|A_n \cdot\|$, where $A_n = A_n^+$ and $I \le A_1 \le A_2 \le \dots$. A special but very useful type of such (F)-spaces is of the form

$$\mathcal{D} = \mathcal{D}^\infty(T) = \bigcap_{n \ge 0} \mathcal{D}(T^n), \quad T = T^* \ge I \text{ - an unbounded selfadjoint operator.}$$

Without loss of generality one can suppose that $T\varphi_n = t_n \varphi_n$, (φ_n) an appropriate orthonormal basis in $\mathcal{H}$, $t_n \in \mathbb{N}$, i.e., $\sigma(T) = \{t_n\}$. Then the essential spectrum $\sigma_{ess}(T)$ consists of the set of eigenvalues with infinite multiplicity. They are denoted by t'_n and arranged in increasing order, $t'_1 \le t'_2 \le \dots$. Let $\mathcal{H}_n$ be the corresponding eigenspace, P_n the ortho-projection onto $\mathcal{H}_n$. In /10/ there was given a complete classification of such domains. We need here the following rough description of the three basic types (I),(II) and (III):

$\mathcal{D} \in$ (I) if and only if $\lim t_n = \infty$, i.e., $\sigma_{ess}(T) = \emptyset$

$\mathcal{D} \in$ (II) if and only if $\sigma_{ess}(T)$ is finite, i.e., $\mathcal{D}$ has a representation $\mathcal{D} = \mathcal{H}_0 \oplus \mathcal{D}_0$, $\mathcal{H}_0$ an infinite dimensional Hilbert subspace, $\mathcal{D}_0 \in$ (I).

$\mathcal{D} \in$ (III) if and only if $\lim t'_n = \infty$, i.e., $\mathcal{D}$ has a representation $\mathcal{D} = \sum' \oplus \mathcal{H}_n \oplus \mathcal{D}_0$, where $\mathcal{D}_0 = (0)$ (the so-called type (III_A)) or $\mathcal{D}_0 \in$ (I) and

$$\sum{}' \oplus \mathcal{H}_n = \{\psi = \sum \psi_n : \psi_n \in \mathcal{H}_n, \ \sum (t'_n)^{2k} \|\psi_n\|^2 < \infty$$

for all $k \in \mathbb{N}\}$.

In what follows we consider only the case $\mathcal{D} = \mathcal{D}^\infty(T)$ unless otherwise explicitly mentioned. This case is very important for applications for many reasons. So one has for example almost explicit descriptions of the bounded sets and the topologies $\tau_{\mathcal{D}}$ and $\tau^{\mathcal{D}}$ /8,9/. We collect here some of these results in an appropriate form.

2.1. Lemma

i) Let $\mathcal{D} \in (III_A)$, then a total system of bounded sets in $\mathcal{D}[t]$ is given by

$$\mathcal{M}_{(a_n)} = \{\psi = \sum a_n \psi_n, \ \psi_n \in \mathcal{H}_n, \ \|\psi_n\| \le 1\},$$

where (a_n) runs over all sequences of positive numbers with $\sum a_n^2 (t'_n)^{2k} < \infty$ for all $k \in \mathbb{N}$.

ii) Let F be the set of all positive, bounded and continuous functions $f(x)$ on $\mathbb{R}_+$ such that $\sup_{x \ge 0} x^k f(x) < \infty$ for all $k \in \mathbb{N}$ and $f(T)\mathcal{D}$ is $\| \ \|$-dense in $\mathcal{D}$. Then $\{\mathcal{M}_f = f(T)\mathcal{K}, \ f \in F\}$, $\mathcal{K}$ -unit ball in $\mathcal{H}$ is a total system of bounded sets in $\mathcal{D}[t]$.

iii) The topologies $\tau_{\mathcal{D}}$ and $\tau^{\mathcal{D}}$ are given by the following seminorms:

$\tau_{\mathcal{D}}$: $\quad A \longrightarrow \|A\|_f = \|f(T)Af(T)\|$ for all $f \in F$

$\tau^{\mathcal{D}}$: $\quad A \longrightarrow \|A\|^{f,k} = \|T^k A f(T)\|$ for all $f \in F$ and $k \in \mathbb{N}$.

For a similar description in the case of a general (F)-space $\mathcal{D}[t]$ the following *-ideal in $\mathcal{L}^+(\mathcal{D})$ is useful /15/:

$$\mathcal{B}(\mathcal{D}) = \{A \in \mathcal{L}^+(\mathcal{D}) : BAC \text{ bounded for all } B, C \in \mathcal{L}^+(\mathcal{D})\} =$$
$$= \{A \in \mathcal{L}^+(\mathcal{D}) : A\mathcal{H} \subset \mathcal{D}, \ A^*\mathcal{H} \subset \mathcal{D}\} = (\text{for } \mathcal{D} = \mathcal{D}^\infty(T)) =$$
$$= \{A \in \mathcal{L}^+(\mathcal{D}) : T^n A, \ T^n A^* \text{ bounded for all } n \in \mathbb{N}\}.$$

Then in /4/ and in a more general context in /5/ there was proved that $\{A\mathcal{K}, \ A \in \mathcal{B}(\mathcal{D})\}$ forms a total system of bounded sets in $\mathcal{D}[t]$. Remark that $f(T) \in \mathcal{B}(\mathcal{D})$ for $f \in F$. We summarize some properties of $\mathcal{B}(\mathcal{D})$ which we need in the following.

2.2 Lemma

i) If $B = U|B|$ is the polar decomposition of $B \in \mathcal{B}(\mathcal{D})$, then $|B| \in \mathcal{B}(\mathcal{D})$.

ii) If $\{E(\lambda)\}$ denotes the spectral family of $|B|$ and $J = (a,b]$, $0 < a < b$, then $E(J)\mathcal{H} \subset \mathcal{D}$, $UE(J)\mathcal{H} \subset \mathcal{D}$, $U^* E(J)\mathcal{H} \subset \mathcal{D}$.

Proof: i) follows from /15/. ii) is also simple, for example $E(J)\mathcal{H} \subset \mathcal{D}$ is a consequence of the estimation

$$\|T^n E(J)\varphi\| = \|(T^n|B|)(|B|^{-1}E(J))\varphi\| \le C_n \|\varphi\| \quad \text{for all } n \in \mathbb{N}.$$

Here $|B|^{-1}$ denotes the bounded inverse of $|B|$ on $E(J)\mathcal{H}$.

The next Lemma is folklore.

2.3. Lemma

$\omega \in \mathfrak{D}[t]'$ if and only if $\omega(\varphi) = \langle \chi, A\varphi \rangle$ for some $\chi \in \mathcal{H}$, $A \in \mathcal{L}^+(\mathfrak{D})$.

In sections 3 and 4 we need some simole facts which we also formulate as Lemmata. For completeness the short proofs are included. Let w denote the weak topology in $\mathcal{H}$, $\sigma = \sigma(\mathfrak{D}, \mathfrak{D}')$ is the weak topology in $\mathfrak{D}$ with respect to the dual pair $(\mathfrak{D}, \mathfrak{D}')$. The corresponding convergences will be denoted by $\xrightarrow{w}$, $\xrightarrow{\sigma}$ resp.

2.4. Lemma

A sequence $(\psi_n) \subset \mathfrak{D}$ is σ-convergent to zero if and only if (ψ_n) is t-bounded and $\langle \psi, \psi_n \rangle \longrightarrow 0$ for all $\psi \in \mathfrak{D}$ (hence for all $\psi \in \mathcal{H}$).

Proof: Let $\psi_n \xrightarrow{\sigma} 0$, i.e. $\omega(\psi_n) \longrightarrow 0$ for all $\omega \in \mathfrak{D}[t]'$, and by Lemma 2.3 $\langle \chi, A\psi_n \rangle \longrightarrow 0$ for all $A \in \mathcal{L}^+(\mathfrak{D})$, $\chi \in \mathcal{H}$. That means $(A\psi_n)$ is w-convergent in $\mathcal{H}$, hence $\|\ \|$-bounded. Thus (ψ_n) is t-bounded. On the other hand, if (ψ_n) is t-bounded and $\langle \psi, \psi_n \rangle \longrightarrow 0$ for all $\psi \in \mathfrak{D}$, then for arbitrary $A \in \mathcal{L}^+(\mathfrak{D})$ and $\psi = A^+\varrho$, $\varrho \in \mathfrak{D}$ one gets $\langle \varrho, A\psi_n \rangle \longrightarrow 0$. Since $(A\psi_n)$ is $\|\ \|$-bounded, $\langle \chi, A\psi_n \rangle \longrightarrow 0$ for all $\chi \in \mathcal{H}$, i.e., $\psi_n \xrightarrow{\sigma} 0$. q.e.d.

2.5. Lemma

A sequence $(\psi_n) \subset \mathfrak{D}$ is t-convergent to zero if and only if (ψ_n) is t-bounded and $\|\psi_n\| \longrightarrow 0$.

Proof: One direction is trivial, the other follows from the estimation:

$\|A\psi_n\|^2 = |\langle A^+A\psi_n, \psi_n \rangle| \leq \|A^+A\psi_n\| \cdot \|\psi_n\| \leq C(A)\|\psi_n\|$ with

$C(A) = \sup_n \|A^+A\psi_n\| < \infty$. q.e.d.

2.6. Remarks

i) From the proofs it is obvious that Lemmata 2.3 - 2.5. are valid for arbitrary $\mathfrak{D}$.

ii) Lemma 2.5. is false if one considers an operator algebra $\mathfrak{A}$ which is not a $*$-algebra and t replaced by the graph topology induced by $\mathfrak{A}$. This can be seen from the following example. Let (φ_n) be an orthonormal basis in $\mathcal{H}$, $T\varphi_n = n\varphi_n$ for $n = 2k+1$, $T\varphi_n = \varphi_n$ for $n = 2k$ for all k, $\mathfrak{D} = \mathfrak{D}^\infty(T)$. Denote by $\mathfrak{A}(\mathfrak{D})$ the algebra generated by I and A with $A\varphi_{2k} = (2k+1)\varphi_{2k+1}$, $A\varphi_{2k+1} = 0$. $\mathfrak{A}(\mathfrak{D})$ is not a $*$-algebra, and the sequence $\psi_k = \varphi_{2k}/(2k+1)$ is $t_{\mathfrak{A}}$-bounded, $\|\psi_k\| \longrightarrow 0$ but of course $\|A\psi_k\|$ does not go to zero.

3. Ideals of compact operators

It is a well-known fact that some of the equivalent characterizations of compact (completely continuous) operators in Hilbert space do not coincide in more general locally convex spaces. This led to the definition of different subsets of $\mathcal{L}^+(\mathfrak{D})$ in /16/ all of them being candidates for compact or completely continuous operators in $\mathcal{L}^+(\mathfrak{D})$. We repeat the definitions and some of the needed properties for arbitrary $\mathfrak{D}$.

$\mathcal{F}(\mathfrak{D}) = \{ F \in \mathcal{L}^+(\mathfrak{D}) : \dim \mathcal{R}(F) < \infty \}$

$\mathrm{Com}(t,t) = \{ A \in \mathcal{L}^+(\mathfrak{D}) : \exists$ t-neighbourhood U such that AU is relatively t-compact$\}$

$\mathrm{Com}(t,\|\ \|) = \{ A \in \mathcal{L}^+(\mathfrak{D}) : \exists$ t-neighbourhood U such that AU is relatively $\|\ \|$-compact$\}$

$\mathrm{Vol}(t,t) = \{ A \in \mathcal{L}^+(\mathfrak{D}) : A\mathfrak{M}$ is relatively t-compact for all t-bounded $\mathfrak{M} \subset \mathfrak{D}[t] \}$

$\mathrm{Vol}(t,\|\ \|) = \{ A \in \mathcal{L}^+(\mathfrak{D}) : A\mathfrak{M}$ is relatively $\|\ \|$-compact for all t-bounded $\mathfrak{M} \subset \mathfrak{D}[t] \}$.

(Vol is taken from the German word "vollstetig" = completely continuous).

3.1. Lemma /16/

i) $\mathcal{F}(\mathfrak{D}) \subset \mathrm{Com}(t,t) \subset \mathrm{Com}(t,\|\ \|) \subset \mathrm{Vol}(t,\|\ \|)$
$\mathrm{Com}(t,t) \subset \mathrm{Vol}(t,t) \subset \mathrm{Vol}(t,\|\ \|)$.

ii) all sets are algebras.

iii) $\mathrm{Com}(t,t)$ and $\mathrm{Vol}(t,t)$ are two-sided ideals, $\mathrm{Com}(t,\|\ \|)$ and $\mathrm{Vol}(t,\|\ \|)$ are right ideals, $\mathcal{F}(\mathfrak{D})$ is a two-sided $*$-ideal.

iv) $\mathrm{Com}(t,t) \subset \mathcal{S}_\infty(\mathfrak{D}) = \{ A \in \mathcal{L}^+(\mathfrak{D}) : BAC \in \mathcal{S}_\infty(\mathcal{H})$ for all $B, C \in \mathcal{L}^+(\mathfrak{D}) \}$
If t is metrizable, equality holds.

v) If $\mathfrak{D} = \mathfrak{D}^\infty(T)$, then $\mathrm{Vol}(t,t)$ and $\mathrm{Vol}(t,\|\ \|)$ are $\tau_{\mathfrak{D}}$-closed.

3.2 Remarks

i) Using the fact that $I \in \mathrm{Com}(t,t)$ implies that the underlying space is finite dimensional, it is easy to see that $\mathrm{Com}(t,t)$ is different from $\mathrm{Vol}(t,t)$ and $\mathrm{Com}(t,\|\ \|)$.

ii) Because $\mathfrak{D}^\infty(T) \in (I)$ is equivalent to $T^{-1} \in \mathcal{S}_\infty(\mathcal{H})$, one sees that in this case $\mathcal{L}^+(\mathfrak{D}) = \mathrm{Com}(t,\|\ \|) = \mathrm{Vol}(t,t) = \mathrm{Vol}(t,\|\ \|) = \overline{\mathcal{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}}$ ($\overline{}^{\tau_{\mathfrak{D}}}$ means the $\tau_{\mathfrak{D}}$-closure in $\mathcal{L}^+(\mathfrak{D})$).
Moreover $\mathfrak{B}(\mathfrak{D}) = \mathcal{S}_\infty(\mathfrak{D})$.

The next Lemma gives an equivalent characterization of $\mathrm{Vol}(t,t)$ and $\mathrm{Vol}(t,\|\ \|)$ which is standard in Hilbert space.

3.3. Lemma

i) $A \in Vol(t,t)$ if and only if $(A\psi_n)$ is t-convergent to zero for any sequence (ψ_n) which is σ-convergent to zero.
ii) $A \in Vol(t,\|\ \|)$ if and only if $(A\psi_n)$ is $\|\ \|$-convergent to zero for any sequence (ψ_n) which is σ-convergent to zero.
Proof: i) Let $A \in Vol(t,t)$ and $\psi_n \xrightarrow{\sigma} 0$. Then $A\psi_n \xrightarrow{\sigma} 0$ because A is σ-σ-continuous. (ψ_n) is t-bounded and $(A\psi_n)$ is relatively t-compact. If $A\psi_n \not\xrightarrow{t} 0$ then there is a subsequence (ψ_{n_k}) such that $A\psi_{n_k} \xrightarrow{t} \psi \neq 0$ but then $A\psi_{n_k} \xrightarrow{\sigma} \psi$ which is a contradiction.
On the other hand, let A map σ-zero-sequences onto t-zero-sequences. Suppose $A \notin Vol(t,t)$. Then there is a t-bounded set $\mathcal{M}$ such that $A\mathcal{M}$ is not relatively t-compact. Hence there is a sequence $(\psi_n) \subset \mathcal{M}$ such that $(A\psi_n) = (\chi_n)$ does not contain a t-convergent subsequence. Since (ψ_n) is $\|\ \|$-bounded it is relatively w-compact. Consequently there is a subsequence (ψ_{n_k}) with $\langle \psi, \psi_{n_k} \rangle \longrightarrow \langle \psi, \chi \rangle$ for all $\psi \in \mathcal{H}$ and some $\chi \in \mathcal{H}$. This means exactly that (ψ_{n_k}) is a σ-Cauchy sequence, and the σ-sequentially completeness of $\mathcal{D}$ implies $\chi \in \mathcal{D}$, i.e., $(\psi_{n_k} - \chi) \xrightarrow{\sigma} 0$. Hence $A\psi_{n_k} \xrightarrow{t} A\chi$, that is, $(A\psi_{n_k})$ is a t-convergent subsequence of $(A\psi_n)$ which is a contradiction.
ii) The proof is similarly to i).

q.e.d.

3.4. Corollary

$Vol(t,\|\ \|)$ is a two-sided $*$-ideal.

Proof: $A \in Vol(t,\|\ \|)$ implies $AA^+ \in Vol(t,\|\ \|)$. Let $\psi_n \xrightarrow{\sigma} 0$, then the estimation $\|A^+\psi_n\|^2 = \langle AA^+\psi_n, \psi_n \rangle \leq \|AA^+\psi_n\| \|\psi_n\|$ leads to $(A^+\psi_n)$ converges to zero with respect to $\|\ \|$, hence $A^+ \in Vol(t,\|\ \|)$. That $Vol(t,\|\ \|)$ is a left ideal follows from $BA = (A^+B^+)^+$.

q.e.d.

3.5. Proposition

$Vol(t,\|\ \|) = Vol(t,t)$.
Proof: Let $A \in Vol(t,\|\ \|)$, $\psi_n \xrightarrow{\sigma} 0$, then by Lemma 3.3 $A\psi_n \xrightarrow{\|\ \|} 0$ and $A\psi_n$ is t-bounded (Lemma 2.4). The same Lemma 2.4 implies now $A\psi_n \xrightarrow{t} 0$, i.e., $A \in Vol(t,t)$.

q.e.d.

The next two Lemmata give equivalent characterizations of $Com(t,\|\ \|)$ and $Vol(t,\|\ \|)$ which imply that $Com(t,\|\ \|) = Vol(t,t) = Vol(t,\|\ \|)$.

3.6. Lemma

$A \in Com(t,\|\ \|)$ if and only if there is a $k \in \mathbb{N}$ such that $AT^{-k} \in \mathcal{J}_\infty(\mathcal{H})$.
Proof: Let $A \in Com(t,\|\ \|)$. Without loss of generality we may assume that $\overline{AU}$ is $\|\ \|$-compact with $U = \{\varphi \in \mathcal{D} : \|T^k\varphi\| \leq 1\}$ for some $k \in \mathbb{N}$.

That means AT^{-k} maps the unit ball of $\mathcal{H}$ on a relatively $\|\ \|$-compact set. On the other hand, $AT^{-k} \in \mathcal{J}_\infty(\mathcal{H})$ implies that A U is relatively $\|\ \|$-compact, i.e., $A \in Com(t,\|\ \|)$.

q.e.d.

Now let $E(\lambda)$ be the spectral family to $T = T^* \geq I$ and denote by $\mathcal{H}_\mu$ the space $\mathcal{H}_\mu = E((1,\mu])\mathcal{H}$. Clearly, $\mathcal{H}_\mu \subset \mathcal{D}^\infty(T)$ for all $\mu < \infty$.

3.7. Lemma

$A \in Vol(t,\|\ \|)$ if and only if $A: \mathcal{H}_\mu \longrightarrow \mathcal{H}$ is compact for all $\mu < \infty$.
Proof: Suppose $A \in Vol(t,\|\ \|)$. $\mathcal{H}_\mu \subset \mathcal{D}^\infty(T)$ implies that the t-bounded and $\|\ \|$-bounded sets in $\mathcal{H}_\mu$ coincide. Hence A is compact from $\mathcal{H}_\mu$ to $\mathcal{H}$. To see the other direction consider the decompositions $\mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{H}_\mu^\perp$, $\mathcal{D} = \mathcal{H}_\mu \oplus \mathcal{D}_\mu$, $\overline{\mathcal{D}}_\mu = \mathcal{H}_\mu^\perp$. The sets of the form $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{M}_1 \subset \mathcal{H}_\mu$ $\|\ \|$-bounded, $\mathcal{M}_2 \subset \mathcal{D}_\mu$, t-bounded form a fundamental system of t-bounded sets in $\mathcal{D}$. Let $\mathcal{M}$ be such a set with $\|\varphi\| \leq C$ for all $\varphi \in \mathcal{M}$ and suppose $A: \mathcal{H}_\mu \longrightarrow \mathcal{H}$ is compact for all finite μ. Since $T^{-r}\mathcal{H}_\mu = \mathcal{H}_\mu$ for all r, the operator AT^{-r} is compact from $\mathcal{H}_\mu$ to $\mathcal{H}$. Furthermore we may suppose that r is chosen in such a way that AT^{-r} is bounded. Moreover, for any $\delta > 0$ there is a $\mu > 1$ with $\|AT^{-r-1}\varphi\| \leq \delta\|\varphi\|$ for all $\varphi \in \mathcal{D}_\mu$. We show that $AT^{-r-1}\mathcal{M}$ is relatively $\|\ \|$-compact. This proves the lemma because $T^{-r-1}\mathcal{M}$ runs over a fundamental system of t-bounded sets if $\mathcal{M}$ does. We use the following criterion for compactness of $\mathcal{N} \subset \mathcal{H}$ /3/: Let (φ_n) be an arbitrary orthonormal basis in $\mathcal{H}$. $\mathcal{N} \subset \mathcal{H}$ is relatively compact if and only if for all $\varepsilon > 0$ there is a finite set $n_1,\dots,n_k$ of indices such that $\sum_{n \neq n_i} |\langle \varphi, \varphi_n \rangle|^2 < \varepsilon$ for all $\varphi \in \mathcal{N}$.
Thus, let $(\varphi_n) \subset \mathcal{D}$, $\varepsilon > 0$ be given. Then there are $\varphi_{n_1},\dots,\varphi_{n_k}$ with

$$\sum_{n \neq n_i} |\langle \varphi_n, AT^{-r-1}\psi_1 \rangle|^2 < \varepsilon/4 \quad \text{for all } \psi_1 \in \mathcal{M}_1 .$$

For $\psi = \psi_1 + \psi_2$, $\psi_i \in \mathcal{M}_i$ one gets the estimation:

$$\sum_{n \neq n_i} |\langle \varphi_n, AT^{-r-1}\psi \rangle|^2 = \sum |\langle \varphi_n, AT^{-r-1}\psi_1 + AT^{-r-1}\psi_2 \rangle|^2 \leq$$

$$\leq 2 \sum \{ |\langle \varphi_n, AT^{-r-1}\psi_1 \rangle|^2 + |\langle \varphi_n, AT^{-r-1}\psi_2 \rangle|^2 \} \leq \varepsilon/2 + 2\|AT^{-r-1}\psi_2\|^2 \leq$$

$= \varepsilon/2 + 2\delta^2 C^2 \leq \varepsilon$ for suitable δ. This proves that $AT^{-r-1}\mathcal{M}$ is relatively $\|\ \|$-compact.

q.e.d.

3.8. Proposition

Let $\mathcal{D} = \mathcal{D}^\infty(T)$, then $\overline{\mathcal{F}(\mathcal{D})}^{\tau_\mathcal{D}} = Com(t,\|\ \|) = Vol(t,\|\ \|) = Vol(t,t)$. This two-sided $*$-ideal will be denoted by $\mathcal{C}$.

Proof: We show $Vol(t,\| \|) \subset Com(t,\| \|)$, then the two identities on the right-hand side follow and the rest is a consequence of /16/, esp. Propositions 5 and 7. Thus suppose $A \in Vol(t,\| \|)$. We prove the existence of an $s > 0$ with AT^{-s} compact, i.e., $A \in Com(t,\| \|)$ by Lemma 3.6. First there is an $r > 0$ with AT^{-r} is bounded. For all $\varepsilon > 0$ there is a $\mu > 1$ so that $\|T^{-1}\varphi\|^2 \leq (\varepsilon/4C^2)\|\varphi\|^2$, $C = \|AT^{-r}\|$ for all $\varphi \in \mathfrak{D}_\mu$ (see proof of Lemma 3.7). A and AT^{-r-1} are compact from $\mathcal{H}_\mu$ to $\mathcal{H}$ by the foregoing Lemma. Let $\mathcal{K}$ be the unit ball in $\mathcal{H}$, then as in Lemma 3.7. one proves that $\overline{AT^{-s}\mathcal{K}}$ is compact for $s = r+1$.

q.e.d.

3.9. Corollary

$A \in \mathcal{C}$ if and only if $\mathcal{R}(A)$ does not contain a $\| \|$-closed infinite dimensional subspace (i.e., $\mathcal{R}(A)$ is of type I /10/).

Proof: Suppose $\mathcal{H}_0'$ is such a subspace and $\mathcal{H}_0' \subset A\mathfrak{D} = AT^{-r}\mathfrak{D}$. This means $AT^{-r}\mathcal{H}$ also contains $\mathcal{H}_0'$. For r so that AT^{-r} is compact this is a contradiction to the well-known fact about compact operators in Hilbert space. On the other hand, if $A \notin \mathcal{C}$ then there is $\mu > 1$ so that $A: \mathcal{H}_\mu \longrightarrow \mathcal{H}$ is not $\| \|$-compact. Therefore $A\mathcal{H}_\mu$ contains such a subspace $\mathcal{H}_0'$. $\mathcal{H}_\mu \subset \mathfrak{D}$ implies $\mathcal{H}_0' \subset A\mathcal{H}_\mu \subset A\mathfrak{D} = \mathcal{R}(A)$.

q.e.d.

3.10. Remark

$A \in \mathcal{C}$ if and only if $AB \in \mathfrak{S}_\infty(\mathfrak{D})$ for all $B \in \mathfrak{B}(\mathfrak{D})$.

Proof: Let $A \in \mathcal{C}$, then T^mABT^n is compact. Indeed, if $\mathcal{K}$ is the unit ball in $\mathcal{H}$, then $BT^n\mathcal{K}$ is t-bounded, T^mA is in $\mathcal{C}$, hence $T^mABT^n\mathcal{K}$ is relatively $\| \|$-compact. Let $AB \in \mathfrak{S}_\infty(\mathfrak{D})$ for all $B \in \mathfrak{B}(\mathfrak{D})$ and $\mathcal{M} = B\mathcal{K}$ be a t-bounded set (cf. Lemma 2.1) then $A\mathcal{M} = AB\mathcal{K}$ is relatively $\| \|$-compact, i.e., $A \in \mathcal{C}$.

q.e.d.

The next result is the analogous one to the Hilbert space case. In a more general context but by other methods it was independently obtained by Kürsten /5/.

3.11. Theorem

Let $\mathfrak{D} = \mathfrak{D}^\infty(T)$, $\mathfrak{D} \notin (I)$. Then $\mathcal{C}$ is the only proper $\tau_{\mathfrak{D}}$-closed two-sided $*$-ideal in $\mathcal{L}^+(\mathfrak{D})$.

Proof: Let $\mathfrak{J}$ be an arbitrary $\tau_{\mathfrak{D}}$-closed two-sided $*$-ideal in $\mathcal{L}^+(\mathfrak{D})$. $\mathcal{F}(\mathfrak{D}) \subset \mathfrak{J}$ implies $\mathcal{C} \subset \mathfrak{J}$. Suppose $A \in \mathfrak{J}$ but $A \notin \mathcal{C}$. Then there is a $\| \|$-closed, infinite dimensional subspace $\mathcal{H}_0' \subset \mathcal{R}(A)$, $\mathcal{H}_0' \subset \mathfrak{D}^\infty(T)$. Without loss of generality we may suppose that A is bounded (if not, choose AT^{-k} for suitable k). Denote the restriction of A to $\mathcal{H} \ominus \mathcal{N}(A)$

by A_1. Then A_1^{-1} exists and is bounded on $\mathcal{H}_0'$. $\mathcal{H}_0' \subset \mathcal{R}(A)$ implies $\mathcal{H}_1' = A_1^{-1}\mathcal{H}_0' \subset \mathfrak{D}^\infty(T)$ and $B = A_1^{-1}|\mathcal{H}_0' \oplus 0 \in \mathcal{L}^+(\mathfrak{D})$ as well as $P_1 = P_{\mathcal{H}_1'}$ is in $\mathcal{L}^+(\mathfrak{D})$. Consequently $AP_1 \in \mathfrak{J}$ which gives $BAP_1 = P_1 \in \mathfrak{J}$.
In the Hilbert space case one has the fact that $P \in \mathfrak{J}$ and dim P = dim Q, Q an orthoprojection, implies $Q \in \mathfrak{J}$. It is easy to see that now we have a slightly modified result:

(+) $P \in \mathfrak{J}$, dim P = dim Q and $\overline{\mathcal{R}(Q)} \subset \mathfrak{D}^\infty(T)$ imply $Q \in \mathfrak{J}$.

Now we distinguish the cases $\mathfrak{D}^\infty(T) \in (II)$ and $\mathfrak{D}^\infty(T) \in (III)$ to derive $\mathfrak{J} = \mathcal{L}^+(\mathfrak{D})$.

1. $\mathfrak{D}^\infty(T) \in (II)$: This means $\mathfrak{D}^\infty(T) = \mathcal{H}_1 \oplus \mathfrak{D}_1 = \mathcal{H}_1 \oplus \mathfrak{D}^\infty(T_1)$ with $\mathfrak{D}^\infty(T_1) \in (I)$. Then $0 \oplus \mathcal{L}^+(\mathfrak{D}_1) \in \mathcal{C}$ and especially $0 \oplus I_1 \in \mathcal{C}$ where $I_1 = I|\mathfrak{D}_1$.
Moreover by (+) applied to P_1 and $P_{\mathcal{H}_1}$ we have $P_{\mathcal{H}_1} \in \mathfrak{J}$, Therefore $I = P_{\mathcal{H}_1} \oplus I_1 \in \mathfrak{J}$, i.e., $\mathfrak{J} = \mathcal{L}^+(\mathfrak{D})$.

2. $\mathfrak{D}^\infty(T) \in (III)$: This means $\mathfrak{D}^\infty(T) = \Sigma' \oplus \mathcal{H}_n \oplus \mathfrak{D}^\infty(T_1)$ for some $\mathfrak{D}^\infty(T_1) \in (I)$. Because I restricted to $\mathfrak{D}^\infty(T_1)$ is in $\mathfrak{J}$ it is enough to show $I' = I$ restricted to $\Sigma' \oplus \mathcal{H}_n$ is in $\mathfrak{J}$ since this gives again $I \in \mathfrak{J}$. It is $I' = \Sigma \oplus P_n$ where the sum is understood in the strong operator topology. Applying again (+) we see that $P_n \in \mathfrak{J}$.
The proof will be completed if we prove $I' = \tau_{\mathfrak{D}}\text{-}\lim_{N\to\infty} \sum_{n=1}^{N} \oplus P_n$.
Let $\mathcal{M}$ be a t-bounded set, without restriction of generality let $\mathcal{M} \subset \Sigma' \oplus \mathcal{H}_n$, then

$$\|(I' - \sum_{n=1}^{N} \oplus P_n)\|_{\mathcal{M}} = \sup_{\varphi,\psi\in\mathcal{M}} |\langle T\cdot T^{-1}(I' - \sum_{n=1}^{N} \oplus P_n)\varphi, \psi\rangle| \leq$$
$$\leq \sup_{\varphi\in\mathcal{M}} \|T^{-1}(I' - \sum_{n=1}^{N} \oplus P_n)\varphi\| \cdot \sup_{\psi\in\mathcal{M}} \|T\psi\| \leq (\sup_{\varphi\in\mathcal{M}}\|\varphi\| \cdot \sup_{\psi\in\mathcal{M}}\|T\psi\|)(t_{N+1}')^{-1}$$

which goes to zero for N to infinity.

q.e.d.

We conclude this section with some remarks concerning the validity of the results for more general domains.

3.12. Remarks

i) (3.3) - (3.5) are valid for arbitrary (F)-domains.
ii) Proposition 3.8. is in general false for arbitrary (F)-domains, i.e., in general $Com(t,\| \|) \neq Vol(t,\| \|)$ as it can be seen by examples of Montel spaces which are not Schwartz spaces (cf. /13/).

4. The Calkin construction

In this section the main result of the paper is proved, namely that for $\mathfrak{D} = \mathfrak{D}^\infty(T)$ the quotient algebra $\mathcal{L}^+(\mathfrak{D})/\mathcal{C} = \mathfrak{R}$ equipped with the factor topology τ arising from the uniform topology $\tau_{\mathfrak{D}}$ is an $A\hat{O}^*$-algebra, i.e., $\mathfrak{R}$ is algebraically and topologically isomorphic to an appropriate Op^*-algebra $\mathcal{A}(\hat{\mathfrak{D}})[\tau_{\hat{\mathfrak{D}}}]$. This is the analogous result to the classical one obtained by Calkin /2/. We also construct the isomorphism explicitly because the only abstract inner characterization of an $A\hat{O}^*$ - algebra given by Schmüdgen /14/ is not applicable to $\mathfrak{R}[\tau]$. Let us repeat briefly Calkin's construction.

Remember that a Banach limit, LIM, is a singular state on l^∞ (the bounded sequences), i.e., a positive (continuous) normed linear funtional on l^∞ vanishing on the space c_0 of zero-sequences. Consider the linear space $\tilde{\tilde{\mathfrak{H}}}$ of all sequences (ψ_n), $\psi_n \in \mathfrak{H}$, $\psi_n \xrightarrow{w} 0$. An inner product on $\tilde{\tilde{\mathfrak{H}}}$ is given by $((\psi_n),(\chi_n)) = \mathrm{LIM}\langle\psi_n,\chi_n\rangle$. Let $\tilde{\mathfrak{H}}$ be the quotient space of $\tilde{\tilde{\mathfrak{H}}}$ by the subspace of all sequences (ψ_n) with $\|(\psi_n)\|^2 = \mathrm{LIM}\|\psi_n\|^2 = 0$. So $\tilde{\mathfrak{H}}$ is a pre-Hilbert space with scalar product $(\ , \ )$.

It seems that there is some confusion in the literature concerning the completeness of $\tilde{\mathfrak{H}}$. Let us therefore first clarify this point. Reid /11/ has shown that in the case of a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, i.e., $\mathrm{LIM} = \lim_{\mathfrak{U}}$, $\tilde{\mathfrak{H}}$ is already complete. He concludes that Calkin's theorem 4.1 (" $\tilde{\mathfrak{H}}$ is incomplete") is wrong. But the matter is somewhat more delicate. The proof of Calkin's theorem 4.1 is clearly false, but $\tilde{\mathfrak{H}}$ is not for all Banach limits complete. The following interesting result of Kürsten /6/ gives a complete answer:

> $\tilde{\mathfrak{H}}$ is complete if and only if LIM is a finite convex linear combination of pure singular states, i.e.,
>
> $\mathrm{LIM} = \sum_{k=1}^{n} a_k \lim_{\mathfrak{U}_k}$, $a_k \geq 0$, $\sum a_k = 1$, $\mathfrak{U}_k$ - free ultrafilter.

For the translation invariant Banach limits $\tilde{\mathfrak{H}}$ is not complete. The existence of such limits is an exercise in /12/.

Now we describe our representation of $\mathcal{C}$ and start with the construction of $\hat{\mathfrak{D}}$ assuming that LIM is so as in the result above, that is, $\tilde{\mathfrak{H}}$ is complete.

On the linear space

$$\hat{\hat{\mathfrak{D}}} = \{ (\psi_n) = \hat{f} : \psi_n \in \mathfrak{D}, \ \psi_n \xrightarrow{\sigma} 0 \}$$

we consider the sesquilinear forms

$$(\hat{f},\hat{g})_k = \mathrm{LIM}\langle T^k\psi_n, T^k\chi_n\rangle, \quad \hat{f} = (\psi_n), \ \hat{g} = (\chi_n), \ k = 0,1,2,\dots$$

The proof of Lemma 2.5 implies that the following two equivalence

relations coincide, i.e., lead to the same quotient space:

1. $\hat{f} \sim \hat{g}$ if and only if $(\hat{f}-\hat{g}) \sim 0$ if and only if $\mathrm{LIM}\|\psi_n - \chi_n\|^2 = 0$
2. $f \approx g$ if and only if $(\hat{f}-\hat{g}) \approx 0$ if and only if $\mathrm{LIM}\|T^k(\psi_n - \chi_n)\|^2 = 0$ for all $k = 0,1,2,\dots$

Thus $\hat{\mathfrak{D}} = \hat{\hat{\mathfrak{D}}}/\sim$ $(\equiv \hat{\hat{\mathfrak{D}}}/\approx)$ is in a natural way equipped with scalar products induced from $(\ , \ )_k$ and denoted in the same way. Let $\hat{\mathfrak{H}}$ be the completion of $\hat{\mathfrak{D}}$ with respect to the norm $| \ |$ derived from the scalar product $(\ , \ )_0 \equiv (\ , \ )$. For the elements of $\hat{\mathfrak{D}}$ we use the symbols f, g, ... with $f = (\widehat{\psi_n})$ or $(\psi_n) \in f$.

4.1. Remarks

i) If $\mathfrak{D}^\infty(T) \in (I)$ then $\hat{\mathfrak{D}} = (0)$. This follows from the fact that in this case $\psi_n \xrightarrow{\sigma} 0$ implies $\|\psi_n\| \longrightarrow 0$, i.e. $(\psi_n) \sim 0$.

ii) If $\mathfrak{D}^\infty(T) \notin (I)$ then similar considerations as in /2/ give that $\hat{\mathfrak{H}}$ has Hilbert space dimension c.

iii) If $\mathfrak{D} = \mathfrak{H}$ then $\hat{\mathfrak{H}} = \tilde{\mathfrak{H}}$.

Next we define a $*$-representation π of $\mathcal{L}^+(\mathfrak{D})$ on $\hat{\mathfrak{D}}$, i.e., a $*$-homomorphism from $\mathcal{L}^+(\mathfrak{D})$ into $\mathcal{L}^+(\hat{\mathfrak{D}})$.

For $B \in \mathcal{L}^+(\mathfrak{D})$, $f = (\widehat{\psi_n}) \in \hat{\mathfrak{D}}$ put $\pi(B)f = (\widehat{B\psi_n})$. The justification of such a definition and some properties of π are summarized in the following proposition.

4.2. Proposition

i) $\pi(B)$ is correctly defined, i.e., if $(\psi_n) \in \hat{\hat{\mathfrak{D}}}$ then $(B\psi_n) \in \hat{\hat{\mathfrak{D}}}$ and if $(\psi_n) \sim 0$ then $(B\psi_n) \sim 0$.

ii) π is a $*$-representation.

iii) The kernel of π is $\mathcal{C}$.

iv) If $B \in \mathcal{L}^+(\mathfrak{D})$ is bounded then $\pi(B)$ is bounded and $\|\pi(B)\| \leq \|B\|$.

Proof: i) The first part follows from the fact that the t-continuity of B implies the σ-continuity. For the second part suppose $(\psi_n) \sim 0$, i.e., $\psi_n \xrightarrow{\sigma} 0$ and $\mathrm{LIM}\|\psi_n\|^2 = 0$. From this $\mathrm{LIM}\|\psi_n\| = 0$ follows and the same estimation as in the proof of Lemma 2.5 gives $\mathrm{LIM}\|B\psi_n\|^2 \leq C(B)\,\mathrm{LIM}\|\psi_n\|$.

ii) We only show the $*$-property. For $f = (\widehat{\psi_n})$, $g = (\widehat{\chi_n}) \in \hat{\mathfrak{D}}$ it is $(\pi(A^+)f,g) = \mathrm{LIM}\langle A^+\psi_n,\chi_n\rangle = \mathrm{LIM}\langle\psi_n, A\chi_n\rangle = (f, \pi(A)g)$. This means $f \in \mathfrak{D}(\pi(A)^*)$ and $\pi(A^+)f = \pi(A)^*f$ for all $f \in \hat{\mathfrak{D}}$.

Remark that this also implies the closability of $\pi(A)$ for all $A \in \mathcal{L}^+(\mathfrak{D})$.

iii) Let $A \in \mathfrak{C}$, then A maps σ-zero-sequences onto t-zero-sequences, i.e., $(A\psi_n) \sim 0$ for all $(\psi_n) \in \hat{\mathfrak{D}}$, hence $\pi(A) = 0$ on $\hat{\mathfrak{D}}$, so $\mathfrak{C} \subset \mathrm{Ker}\,\pi$. On the other hand, if $A \notin \mathfrak{C}$, then there exists a sequence (ψ_n) with $\psi_n \xrightarrow{\sigma} 0$ but $\|A\psi_n\| \not\rightarrow 0$. Consequently one can find a subsequence (ψ_{n_k}) such that $\|\psi_{n_k}\| \geq c > 0$ and $\|A\psi_{n_k}\| \geq c > 0$, $\psi_{n_k} \xrightarrow{\sigma} 0$. Then $f = (\widehat{\psi_{n_k}}) \neq 0$ and also $\pi(A)f \neq 0$ because $|\pi(A)f| \geq c$, hence $A \notin \mathrm{Ker}\,\pi$.
iv) follows as in the classical case.

q.e.d.

So π induces an isomorphism between $\mathfrak{R}$ and $\pi(\mathcal{L}^+(\mathfrak{D}))$ also denoted by π. For the element of $\mathfrak{R}$ corresponding to $A \in \mathcal{L}^+(\mathfrak{D})$ we use the symbol $[A]$.

To investigate continuity properties of π it is useful to describe the structure of $\hat{\mathfrak{D}}$ in more detail.

Let $\mathfrak{D} \in$ (II), i.e., $\mathfrak{D} = \mathcal{H}_0 \oplus \mathfrak{D}_0$ (see section 2). Then Remark 4.1 immediately gives $\hat{\mathfrak{D}} = \{(\hat{\psi}_n) : (\psi_n) \subset \mathcal{H}_0, \psi_n \xrightarrow{w} 0\}$, that is, $\hat{\mathfrak{D}}$ is in essential identical with $\hat{\mathcal{H}}_0 = \tilde{\mathcal{H}}_0$.

Moreover, let $A = (A_{ij})$, $i,j = 1,2$ with $A_{11} = P_0AP_0$, $A_{22} = (1-P_0)A \cdot (1-P_0)$, etc., where P_0 is the orthoprojection from $\mathcal{H}$ onto $\mathcal{H}_0$.
Then it is easy to see that (the natural extensions of) A_{12}, A_{21} and A_{22} belong to $\mathfrak{C}$, $\pi(\mathcal{L}^+(\mathfrak{D})) = \pi(\mathfrak{B}(\mathcal{H}_0))$ by $\pi(A) = \pi(A_{11})$.
Therefore $\pi(A)$ is bounded for all $A \in \mathcal{L}^+(\mathfrak{D})$. The fact that π is an algebraical and topological isomorphism between $\mathfrak{R}[\tau]$ and $\pi(\mathcal{L}^+(\mathfrak{D}))[\tau_{\hat{\mathfrak{D}}}]$ can be derived as in the classical case.

Let $\mathfrak{D} \in$ (III): In the same way as for type (II) it is enough to consider the case where $\mathfrak{D} = \sum' \oplus \mathcal{H}_n$, i.e., $\mathfrak{D} \in (\mathrm{III}_A)$.

4.3. Lemma

$$\hat{\mathfrak{D}} = \sum{}' \oplus \hat{\mathcal{H}}_n = \mathfrak{D}^\infty(\pi(T)) \;;\; \hat{\mathcal{H}} = \sum \oplus \hat{\mathcal{H}}_n .$$

Proof: To prove the first equality we show

i) $\hat{\mathfrak{D}} \subset \sum' \oplus \hat{\mathcal{H}}_n$; ii) $\sum' \oplus \hat{\mathcal{H}}_n \subset \hat{\mathfrak{D}}$.

ad i) Let $f = (\hat{\psi}_n) \in \hat{\mathfrak{D}}$, $\psi_n \in \mathfrak{D}$. Then (ψ_n) is a t-bounded set and so it can be represented (cf. section 2) by

$$\psi_n = \sum_i a_i \psi_n^i \quad \text{with } \|\psi_n^i\| \leq 1 \text{ and } \sum_i a_i^2 (t_i^\cdot)^{2r} < \infty \quad \text{for all } r.$$

The estimation $\lim_{N\to\infty} |(\hat{\psi}_n) - \sum_{i=1}^{N} (\widehat{a_i\psi_n^i})|^2 = \lim_{N\to\infty} (\mathrm{LIM}\|\sum_{i>N} a_i\psi_n^i\|^2) \leq$

$\leq \lim_{N\to\infty} (\sup_n \|\sum_{i>N} a_i \psi_n^i\|^2) \leq \lim_{N\to\infty} \sum_{i>N} a_i^2 = 0$ gives that

$f = \sum_i f_i$ with $f_i = (\widehat{a_i\psi_n^i}) \in \hat{\mathcal{H}}_i$ and $\sum_i (t_i^\cdot)^{2r} |f_i|^2 \leq$

$\leq \sum_i a_i^2 (t_i^\cdot)^{2r} < \infty$ for all r. This proves i). Moreover, the inclusion $\hat{\mathcal{H}}_n \subset \hat{\mathfrak{D}}$ for all n and i) immediately give $\hat{\mathcal{H}} = \sum \oplus \hat{\mathcal{H}}_n$.

ad ii) We prove the inclusion for the case $\mathrm{LIM} = \lim_{\mathcal{U}}$, $\mathcal{U}$ a free ultrafilter. The general case of a finite convex linear combination can be treated be standard considerations.

Hence, let $\mathcal{U}$ be a free ultrafilter, $f = \sum \oplus f_n \in \sum' \oplus \hat{\mathcal{H}}_n$, $f_n = (\hat{\psi}_n^l)$
$\sum |f_n|^2 (t_n^\cdot)^{2r} < \infty$ for all r, $(\psi_n^l) \subset \mathcal{H}_n$, $\psi_n^l \xrightarrow{w} 0$ for $l \to \infty$.

It is well known /11/ that for all n (ψ_n^l) can be chosen in such a way that $\|\psi_n^l\|^2 = \mathrm{LIM}_l \|\psi_n^l\|^2 = |f_n|^2$ for all l. Put $\psi_l = \sum_n \oplus \psi_n^l$.

The estimation

$$\|T^r\psi_l\|^2 = \sum_n (t_n^\cdot)^{2r} \|\psi_n^l\|^2 = \sum_n (t_n^\cdot)^{2r} |f_n|^2 < \infty \quad \text{for all } r \in \mathbb{N}$$

gives that (ψ_l) is contained in $\mathfrak{D}$ and t-bounded. Moreover, (ψ_l) converges weakly to zero. Indeed, let $\chi \in \mathfrak{D}$, $\chi = \sum \oplus \chi_n$.

Choose N so that $\sum_{n>N} \|\psi_n^l\|^2 < \varepsilon^2/\|\chi\|^2$ for all l which is possible due to the special choice of (ψ_n^l) mentioned above. Consequently

$$|\langle\psi_l, \chi\rangle| \leq \sum_{n=1}^{N} |\langle\psi_n^l, \chi_n\rangle| + \sum_{n>N} \|\chi_n\| \cdot \|\psi_n^l\| \leq \varepsilon + \|\chi\| (\sum_{n>N} \|\psi_n^l\|^2)^{1/2}$$

$\leq 2\varepsilon$ for sufficiently large l. Thus ii) is proved.

$\hat{\mathfrak{D}} = \mathfrak{D}^\infty(\pi(T))$ follows from the observation that $\pi(T)|\hat{\mathcal{H}}_n = t_n^\cdot \cdot I_n$, I_n the identity operator on $\hat{\mathcal{H}}_n$.

q.e.d.

4.4. Remarks

i) Since Lemma 2.1 does not depend on the separability of the underlying space these results are also valid for $\mathcal{L}^+(\hat{\mathfrak{D}})$ and $\hat{t}$.
Moreover the simple form of T and $\pi(T)$ implies that $f(\pi(T)) = \pi(f(T))$ for the functions $f \in F$.

ii) It seems worthwhile to compare Calkin's and our construction.
In our approach $\mathfrak{D} = \sum' \oplus \mathcal{H}_n$, $\mathcal{H} = \sum \oplus \mathcal{H}_n$ implies $\hat{\mathfrak{D}} = \sum' \oplus \hat{\mathcal{H}}_n$ and $\hat{\mathcal{H}} = \sum \oplus \hat{\mathcal{H}}_n$ ($\hat{\mathcal{H}}$ the completion of $\hat{\mathfrak{D}}$!). While in Calkin's construction one has $\tilde{\mathcal{H}} \supsetneq \sum \oplus \tilde{\mathcal{H}}_n = \hat{\mathcal{H}}$, $\tilde{\mathfrak{D}} \supsetneq \sum' \oplus \tilde{\mathcal{H}}_n = \sum' \oplus \hat{\mathcal{H}}_n = \hat{\mathfrak{D}}$. For example, take $\psi_n \in \mathcal{H}_n$, $\|\psi_n\| = 1$. Then $(\tilde{\psi}_n) \in \tilde{\mathcal{H}}$ but $(\tilde{\psi}_n) \notin \hat{\mathfrak{D}}$ and what is more $(\tilde{\psi}_n) \perp \hat{\mathfrak{D}}$ as it can be seen from the fact that for all $(\hat{\chi}_n) \in \hat{\mathfrak{D}}$: $|\langle\psi_n, \chi_n\rangle| = |\langle\psi_n, P_n\chi_n\rangle| \leq \|\psi_n\| \cdot \|P_n\chi_n\| \leq \sup_k \|P_n\chi_k\|$ which goes to zero as $n \to \infty$. Hence, the orthogonality follows.

Now we prove that the algebraic isomorphism π between $\mathcal{R}[\tau]$ and $\pi(\mathcal{L}^+(\mathcal{D}))[\tau_{\hat{\mathcal{D}}}]$ is a topological one.

4.5. Lemma

π is a continuous map from $\mathcal{R}[\tau]$ to $\pi(\mathcal{L}^+(\mathcal{D}))[\tau_{\hat{\mathcal{D}}}]$.

Proof: Let $\hat{\mathcal{M}} \subset \hat{\mathcal{D}}[\hat{t}]$ be bounded. So we may suppose that $\hat{\mathcal{M}} = f(\pi(T))\hat{\mathcal{K}}$ where $\hat{\mathcal{K}}$ is the unit ball in $\hat{\mathcal{H}}$ and f an appropriate function. Thus,

$\|\pi([A])\|_{\hat{\mathcal{M}}} = \|f(\pi(T))\pi(A)f(\pi(T))\| = \|\pi(f(T)Af(T))\| \leq \|f(T)Af(T)\| = \|A\|_{\mathcal{M}}$ by Proposition 4.2.iv) and Remark 4.4.i), $\mathcal{M} = f(T)\mathcal{K}$.

q.e.d.

4.6. Lemma

π is a continuous map from $\pi(\mathcal{L}^+(\mathcal{D}))[\tau_{\hat{\mathcal{D}}}]$ onto $\mathcal{R}[\tau]$.

Proof: The proof is an adaption of Calkin's Theorem 5.5. to the unbounded case. It is to show that for each bounded $\mathcal{M} \subset \mathcal{D}[t]$ there is a bounded $\hat{\mathcal{M}} \subset \hat{\mathcal{D}}[\hat{t}]$ so that

(1) $\|[A]\|_{\mathcal{M}} \leq \|\pi([A])\|_{\hat{\mathcal{M}}}$.

Take $\mathcal{M} = f(T)\mathcal{K}$ and remark that

(2) $\|[A]\|_{\mathcal{M}} = \inf_{K\in\mathcal{C}} \|A + K\|_{\mathcal{M}} = \inf_{K\in\mathcal{C}} \|f(T)Af(T) + f(T)Kf(T)\| =$

$= \inf_{G\in\mathcal{J}_\infty(\mathcal{H})} \|f(T)Af(T) + G\|$.

The last equality follows from the fact that for fixed f the set $\{f(T)Kf(T) : K\in\mathcal{C}\}$ is $\|\ \|$-dense in $\mathcal{J}_\infty(\mathcal{H})$. For the simple proof one uses that $f(T)\mathcal{D}$ is dense in $\mathcal{D}$. Now put $B = f(T)Af(T)$ and let $B = U|B|$ be the polar decomposition. Then B and $|B| \in \mathcal{B}(\mathcal{D})$ by Lemma 2.2. Let b be the supremum of the essential spectrum of $|B|$.
If $b = 0$, then $|B|$ and B are compact, hence $B\in\mathcal{C}$ and (1) is trivially fulfilled. Thus suppose $b > 0$. Let $\{b_n\}$ be the set of points of $\sigma(|B|)$ with $b < b_n$. Then the b_n are eigenvalues of finite multiplicity with the only possible limit point b. Let Q_n denote the orthoprojection on the eigenspace corresponding to b_n. Then the operator $K = \sum (b_n - b) Q_n$ is compact and

(3) $b = \||B| - K\| = \|U(|B| - K)\| = \|B - UK\| \geq \inf_{G\in\mathcal{J}_\infty(\mathcal{H})} \|B + G\|$.

On the other hand, $\|\pi(B)\| \geq b$ as can be seen as follows. Choose $J = (a, c]$, $0 < a$, $b \in J$, so that $E(J)$ is infinite dimensional, where $E(.)$ is the spectral family of $|B|$. From Lemma 2.2 there follows the existence of a t-bounded orthonormal system $(\varphi_n) \subset E(J)\mathcal{H} \subset \mathcal{D}$ and a monotone sequence $0 < a_n \leq a_{n+1} \leq \ldots, a_n \longrightarrow b$ so that $a_n \leq \||B|\varphi_n\| \leq a_{n+1}$. For the element $f = (\hat{\varphi}_n) \in \hat{\mathcal{D}}$ it is

(4) $|\pi(B)f| = \mathrm{LIM}\|B\varphi_n\| = \mathrm{LIM}\,\||B|\varphi_n\| = b$.

Here the definition of U in the polar decomposition was used. Thus

(5) $b \leq \|\pi(B)\| = \|\pi(f(T)Af(T))\| = \|\pi([A])\|_{\hat{\mathcal{M}}}$

with $\hat{\mathcal{M}} = \pi(f(T))\hat{\mathcal{K}}$, $\hat{\mathcal{K}}$ the unit ball in $\hat{\mathcal{H}}$. From (2) - (5) one gets the desired estimation (1).

q.e.d.

Now Lemmata 4.5 and 4.6 give the main result:

4.7. Theorem

Let $\mathcal{D} = \mathcal{D}^\infty(T)$. Then $\mathcal{R}[\tau] = \mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]/\mathcal{C}$ is algebraically and topologically isomorphic to $\pi(\mathcal{L}^+(\mathcal{D}))[\tau_{\hat{\mathcal{D}}}]$, i.e., $\mathcal{R}[\tau]$ is an $A\hat{O}^*$-algebra.

4.8. Remark

If one looks at the proofs in this section one sees that in many places $\tau_{\mathcal{D}}$ can be replaced by $\tau^{\mathcal{D}}$. But such results would be not so nice because $\mathcal{L}^+(\mathcal{D})[\tau^{\mathcal{D}}]$ is only a topological algebra but not a topological *-algebra.

5. Concluding remarks

The results in this paper suggest a lot of problems for further investigations. We mention some of them. Results in these directions will be published in forthcoming papers.

i) Irreducibility of the representations constructed will be considered in dependence of the limit used in forming $\hat{\mathcal{D}}$. It appears that the approach of /11/ is a good guide.

ii) In a natural way one can define Fredholm operators in terms of $\mathcal{L}^+(\mathcal{D})$ and the unbounded Calkin algebra $\mathcal{L}^+(\mathcal{D})/\mathcal{C}$.

iii) Very promising is the study of singular states on $\mathcal{L}^+(\mathcal{D})$, i.e., $\tau_{\mathcal{D}}$-continuous states which are zero on $\mathcal{C}$. There can be proved results concerning extensions from the maximal abelian subalgebra of diagonal operators to $\mathcal{L}^+(\mathcal{D})$ along the line of /1/. Here one has to take those diagonal operators which commute with T.

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Представление Калкина для некоторого класса алгебр неограниченных операторов

Предполагается, что $\mathfrak{D} = \mathfrak{D}^{\infty}(T)$, $T = T^{*} \geq I$ – самосопряженный оператор. Доказано, что замыкание конечномерных операторов на $\mathfrak{D}$ относительно равномерной топологии $\tau_{\mathfrak{D}}$ является единственным двусторонним замкнутым *-идеалом в $\mathfrak{K}$ максимальной операторной *-алгебре $\mathfrak{L}^{+}(\mathfrak{D})$. Более того, фактор-алгебра $\mathfrak{L}^{+}(\mathfrak{D})/\mathcal{C}$, снабженная фактор-топологией, индуцированной от $\tau_{\mathfrak{D}}$, алгебраически и топологически изоморфна некоторой Op*-алгебре $A(\mathfrak{D})[\tau_{\hat{\mathfrak{D}}}]$. Изоморфизм строится явным образом. Это обобщает классический результат Калкина на неограниченный случай.

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The Calkin Representation for a Certain Class of Algebras of Unbounded Operators

Let $\mathfrak{D} = \mathfrak{D}^{\infty}(T)$, $T = T^{*} \geq I$ selfadjoint. It is proved that the closure of the finite dimensional operators on $\mathfrak{D}$ with respect to the uniform topology $\tau_{\mathfrak{D}}$ is the only twosided $\tau_{\mathfrak{D}}$-closed *-ideal $\mathcal{C}$ in the maximal operator *-algebra $\mathfrak{L}^{+}(\mathfrak{D})$ on $\mathfrak{D}$. Moreover the quotient algebra $\mathfrak{L}^{+}(\mathfrak{D})/\mathcal{C}$ equipped with the factor topology induced by $\tau_{\mathfrak{D}}$ is algebraically and topologically isomorphic to an appropriate Op*-algebra $A(\mathfrak{D})[\tau_{\hat{\mathfrak{D}}}]$. The isomorphism is constructed explicitly. This generalizes the classical Calkin result to the unbounded case.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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