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ON THE TIME INTERVAL BETWEEN THE MOMENTS OF PARTICLE REGISTRATIONS BY A MODIFIED COUNTER WITH PROLONGING DEAD TIME

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1. INTRODUCTION

Suppose that a counter before registration process is idle and particles arrive at the counter at instants $0 = \tau_1 < \tau_2 < \ldots$, where the interarrival times, $T_n = \tau_{n+1} - \tau_n$, $n \geq 1$, are independent positive random variables with the distribution functions

$$F_n(t) = P(T_n < t), \quad n \geq 1. \tag{1.1}$$

Any arriving particle generates an impulse, χ_n, of a random length (may be constant, too). Due to inertia of the counting device, it is possible that all particles will not be counted. The time during that the device is unable to record is called the dead time. A counter with prolonging dead time is one in which the dead time is produced after the registration of all impulses of emitted particles. An example of that counter is the electron multiplier.

For a modified counter with prolonging dead time (a modified counter for short), we suppose that the first dead time is produced by interarrival times, $\{T_n\}_{n=1}^{\infty}$, and by impulse lengths, $\{\chi_n\}_{n=1}^{\infty}$, which are assumed to be independent positive random variables with the distribution functions

$$H_n(t) = P(\chi_n < t), \quad n \geq 1, \tag{1.2}$$

independent of $\{T_n\}_{n=1}^{\infty}$. Any successive dead time is resumed with initial conditions, independently of the previous dead times. This counter will be denoted by $\mathcal{C} = (F_1, F_2, \ldots; H_1, H_2, \ldots)$. Its basic properties were established in ref.[1].

Our main aim is to determine some asymptotic properties of the cycle, C, that is, the dead time period between the beginnings of two successive dead times. We recall that, as has been noticed by several authors[2-4], the determination of the distribution function of the cycle, G, or its Laplace transform γ, respectively, is an extremely difficult problem even in the case when $F_1 = F_2 = \ldots$ and $H_1 = H_2 = \ldots$ (that is a non-modified counter). However, there are some solutions to that problem in the form of the complicated contour integrals[5] and some integral equations[6,3] which formally, but not always in practice, determine G or γ. For the counter with the recurrent input process and i.i.d. impulse lengths, the Laplace transform has been derived in paper[7], and, for a modified counter

$(F, F, \dots; H_1, H_2, \dots)$ the joint Laplace transform of the cycle and the generating function of the number of particles arriving at the counter during the dead time, ν, may be found in /1/.

We note that similar problems arise in the theory of queues with infinitely many servers /8,9/, film or filmless measurements of track ionization in high energy physics /10/, random walks, etc.

2. CYCLE

For the given counter $\mathcal{C} = (F_1, F_2, \dots; H_1, H_2, \dots)$ we define a sequence of counters, $\{\mathcal{C}_k\}_{k=1}^{\infty}$, where $\mathcal{C}_k = (F_k, F_{k+1}, \dots; H_k, H_{k+1}, \dots)$, $k \geq 1$. For any k, let ν_k, be the number of particle arriving at the counter $\mathcal{C}_k$ during the dead time. It is clear that $\nu = \nu_1$.

Suppose that

$$A_n^k = \{\chi_k < T_k + \dots + T_{k+n-1}, \chi_{k+1} < T_{k+1} + \dots + T_{k+n-1}, \dots, \chi_{k+n-1} < T_{k+n-1}\}, \quad n \geq 1, \quad k \geq 1. \tag{2.1}$$

Then $\{A_n^k : n \geq 1, k \geq 1\}$ are semirecurrent events /11/ that is, for any k and integers i_j with

$$1 \leq i_0 < i_1 < \dots < i_n, \quad n \geq 1, \tag{2.2}$$

we have

$$P(A_{i_1}^k \dots A_{i_n}^k \mid A_{i_0}^k) = P(A_{i_1 - i_0}^{k+i_0} \dots A_{i_n - i_0}^{k+i_0}). \tag{2.3}$$

As has been proved in /1/, for $P_n^k = P(\nu_k = n)$ we have

$$P_n^k = P(\bar{A}_1^k \dots \bar{A}_{n-1}^k A_n^k), \quad n \geq 1, \tag{2.4}$$

where $\bar{A}$ denotes the complement of A. Using (2.3) we may obtain

$$\left.\begin{aligned} P_1^k &= P(A_1^k), \\ P_n^k &= P(A_n^k) - \sum_{j=1}^{n-1} P(A_j^k) P_{n-j}^{k+j}, \quad n \geq 2. \end{aligned}\right\} \tag{2.5}$$

For every ν_k we denote by $f_k(z)$ its generating function, that is, $f_k(z) = \sum_{n=1}^{\infty} P_n^k z^n$, $|z| < 1$. For $k = 1$ we put $f(z) = f_1(z)$. It is clear that

$$f_k(z) = U_k(z) - 1 - \sum_{n=1}^{\infty} P(A_n^k) z^n, \quad P(A_0^k) \equiv 1, \quad |z| < 1. \tag{2.6}$$

Suppose that for the counter $\mathcal{C} = (F_1, F_2, \dots; H_1, H_2, \dots)$ we have $F = F_1 = F_2 = \dots$ $(F(0) < 1, F(\infty) = 1)$.

Then we may determine the joint Laplace transform, $\Phi(s,z)$, of the cycle C and the generating function of ν as follows.

The associated distribution function, F_s, to the distribution function F is defined via the density

$$dF_s(t) = e^{-st}\, dF(t)/a(s), \quad t \geq 0, \quad s \geq 0, \tag{2.7}$$

where $a(s) = \int_0^{\infty} e^{-st} dF(t)$.

For a given counter $\mathcal{C} = (F, F, \dots; H_1, H_2, \dots)$ we define an associated counter, $\mathcal{C}_s$, such that $\mathcal{C}_s = (F_s, F_s, \dots; H_1, H_2, \dots)$, $s \geq 0$. Let $f^s(z)$ be the generating function of the number of particles, ν_s, in the counter $\mathcal{C}_s$. Then, according to /1/, we have, for $\Phi(s,z) = M(e^{-sC} z^{\nu})$, $s \geq 0$, $|z| < 1$,

$$\Phi(s,z) = f^s(a(s)z). \tag{2.8}$$

If

$$\mu = \int_0^{\infty} t\, dF(t) < \infty, \tag{2.9}$$

then

$$M(C) = \mu M(\nu). \tag{2.10}$$

To illuminate the relationship between the probabilities related to random variables ν_s and ν, respectively, we prove the next lemma.

<u>Lemma 2.1</u>. Let $\mathcal{C} = (F, F, \dots; H_1, H_2, \dots)$ be a modified counter. For any $s \geq 0$ we define the probability measure, P_s, on the Borel σ-algebra $\mathcal{B}(R^{\infty})$ via $P_s = \prod_{n=1}^{\infty} \mu_{F_s} \times \prod_{n=1}^{\infty} \mu_{H_n}$, where μ_H is the probability measure on $\mathcal{B}(R^1)$ corresponding to the distribution function H. Then P_{s_0} is equivalent to P_0, for some $s_0 > 0$, iff there is a constant $b > 0$ such that $F(t) = 0$, if $t \leq b$, and $F(t) = 1$, if $t > b$. In this case $P_s = P_0$ for any $s > 0$. In the other case P_s and P_0 are singular measures, for any $s > 0$.

<u>Proof</u>. In order to prove the lemma we examine the Kakutani theorem on equivalence of the infinite direct product of probability measures /12/. From (2.7) we have that, for any $s_0 > 0$, $\mu_{F_{s_0}}$ and μ_F are equivalent, and the Radon-Nikodym derivative of $\mu_{F_{s_0}}$ with respect to μ_F is $d\mu_{F_{s_0}} / d\mu_F(t) = e^{-s_0 t} / a(s_0)$.

Hence $\rho(\mu_{F_{s_0}}, \mu_F) := \int_0^{\infty} (e^{-s_0 t} / a(s_0))^{1/2}\, dF(t) = a(s_0/2)/a(s_0)^{1/2}$.

Therefore $\rho(\mu_{F_{s_0}}, \mu_F) = 1$ iff $a^2(s_0/2) = a(s_0)$ iff $F_{s_0} = F$. The Laplace transform of F_{s_0}, $a^{s_0}(u)$ has the form $a^{s_0}(u) = a(u)a(s_0)$, for any $u \geq 0$. Therefore $a(ns_0) = a^n(s_0)$ for any integer $n \geq 1$. Putting $b = -\ln a(s_0)/s_0$ we have that $a(ns_0) = e^{-ns_0}$. It is known that a completely monotone function $a(s)$ is uniquely determined by its value for $\{s_n\}_{n=1}^{\infty}$, when the series $\sum_{n=1}^{\infty} 1/s_n = \infty$ [13], so that $a(s) = e^{-sb}$ for any s.

The same fact may be proved using the property that, for any $s_1 \geq 0$, $s_2 \geq 0$, $\mu_{F_{s_1}}$ and $\mu_{F_{s_2}}$ are equivalent measured on $B(R^1)$. Let now $s_1 \geq s_2 \geq 0$. Then, for any t

$$F_{s_1}(t) = \int_0^t \frac{dF_{s_1}}{dF_{s_2}}(x)\, dF_{s_2}(x) = a(s_2)\, a(s_1)^{-1} \int_0^{\infty} e^{-(s_1 - s_2)x} \cdot I_{(0,t)}(x)\, dF_{s_2}(x),$$

where I_S is the indicator function of the set S. The Chebyshev inequality entails

$$F_{s_1}(t) \geq a(s_2)a(s_1)^{-1} \int_0^{\infty} e^{-(s_1 - s_2)x}\, dF_{s_2}(x) \int_0^{\infty} I_{(0,t)}(x)\, dF_{s_2}(x) = F_{s_2}(t).$$

Therefore, if $\rho(\mu_{F_{s_0}}, \mu_F) = 1$ then $F_{s_0}(t) = F(t)$ for any $t \geq 0$, and, consequently, $F_v(t) = F(t)$ for any $v \in [0, s_0)$ and $t \geq 0$, that is, $a(v+s) = a(v)a(s)$ for any $v \in [0, s_0)$. Let us put $a(s_0) = e^{-bs_0}$, then, for any n, $a(s_0) = a(ns_0/n) = a^n(s_0/n)$, so that $s(s_0/n) = e^{-bs_0/n}$. If $m \leq n$, then $a(ms_0/n) = e^{-bs_0 m/n}$. From the continuity of $a(s)$ we see that $a(v) = e^{-bv}$ for any $v \in [0, s_0)$. Since $a(s)$ is an analytic function in $\operatorname{Re} s > 0$, $a(s)$ is uniquely determined in $[0, \infty)$ by the values it assumes in $[0, s_0)$ (see ref. [14], Sect. 2.6.1).

Appealing to the Kakutani theorem the proof of theorem is finished. Q.E.D.

3. COUNTER OF ORDER m

We obtain a very important class of modified counters if there is an integer m such that $F_m = F_{m+1} = \ldots$, $H_m = H_{m+1} = \ldots$. In this case the counter $\mathcal{C} = (F_1, F_2, \ldots; H_1, H_2, \ldots)$ is said to be the counter of order m and we write simply $\mathcal{C} = (F_1, \ldots, F_m; H_1, \ldots, H_m)$. If, particularly, $F = F_1 = \ldots = F_m$, then we write $\mathcal{C} = (F; H_1, \ldots, H_m)$.

From (2.1) we conclude that for any counter $\mathcal{C} = (F_1, F_2, \ldots; H_1, H_2, \ldots)$ we have

$$P(A_n^k) = \int_0^{\infty} \cdots \int_0^{\infty} H_k(t_1 + \ldots + t_n)\, H_{k+1}(t_2 + \ldots + t_n) \ldots H_{k+n-1}(t_n) \tag{3.1}$$

$dF_k(t_1) \ldots dF_{k+n-1}(t_n)$, $k, n \geq 1$.

Hence there follows, for the counter of order m that

$$P(A_n^k) = P(A_n^m), \quad n \geq 1, \tag{3.2}$$

for any $k \geq m$.

This case corresponds to m-semirecurrent events studied in [11].

It is evident that the counter $(F_1, F_2, \ldots; H_1, H_2, \ldots)$ is of order m iff $(F_m, F_{m+1}, \ldots; H_m, H_{m+1} \ldots)$ is of order 1 (that is, a non-modified counter). Similarly if $(F_1, F_2, \ldots; H_1, H_2, \ldots)$ is the counter of order m, $m \geq 2$, then $(F_2, F_3, \ldots; H_2, H_3, \ldots)$ is the counter of order $m-1$.

For the modified counter of order m, $(F_1, \ldots F_m; H_1, \ldots, H_m)$, we have from (2.6)

$$U_k(z) = \Big(1 + \sum_{j=0}^{m-k-1} P(A_j^k)\, z^j \big(f_{k+1}(z) - f_m(z)\big)\Big)/\big(1 - f_m(z)\big), \tag{3.3}$$
$$|z| < 1, \quad k = 1, \ldots, m.$$

Here the sum over the empty set is defined as 0.

The equations (3.1) and (3.2) entail that $P(A_1^m) \geq P(A_2^m) \geq \ldots$ and that $\{P(A_n^m)\}_{n=1}^{\infty}$ is a sequence of non-periodic recurrent events (for the definition of recurrent events see, for example, [15]). Therefore there exists the limit

$$p = \lim_n P(A_n^m). \tag{3.4}$$

Due to Dvurečenskij [11] we have that if $p > 0$, then, for any $k = 1, \ldots m$, $p = \lim_n P(A_n^k)$ and $\sum_{n=1}^{\infty} P_n^k = 1$. In paper [8] there are the conditions which entail $p > 0$. For example, if $\int_0^{\infty} t\, dF_m(t) < \infty$, then

$$p > 0 \quad \text{iff} \quad \int_0^{\infty} t\, dH_m(t) < \infty \quad \text{and} \quad P(A_1^m) > 0. \tag{3.5}$$

4. LIMIT PROPERTIES

In this section we derive the asymptotic exponential law of the cycle of a modified counter of order m when $p \to 0$. Similar problems were solved in [16], in the queueing theory language for the case $G/D/\infty$, that is, for a particular modified counter of order 1, in our presentation.

In the following we will suppose that

$$H_1(t) \geq H_2(t) \geq \ldots \geq H_m(t), \quad t \geq 0. \tag{4.1}$$

Then, for $P(A_n^k)$, $k,n \geq 1$, we have

$$\text{(i)}\quad P(A_n^1) \geq P(A_n^2) \geq \dots > P(A_n^m), \quad n \geq 1,$$

$$\text{(ii)}\quad P(A_1^k) \geq P(A_2^k) \geq \dots, \quad k = 1, \dots, m, \tag{4.2}$$

$$\text{(iii)}\quad P(A_{n+1}^k) \leq P(A_n^{k+1}), \quad n \geq 1, \; k = 1, \dots, m-1, \quad \text{if} \quad m \geq 2.$$

For any $k = 1, \dots, m$, we define a cycle, C_k, as the cycle of the modified counter $(F; H_k, \dots, H_m)$.

Introduce, for any $k = 1, \dots, m$, a function

$$\phi_k(z) = 1 - \sum_{n=0}^{\infty} a_n^k z^n, \quad |z| < 1, \tag{4.3}$$

where $a_n^k = P(A_n^k) - P(A_{n+1}^k)$, $n > 0$. It is clear that $\phi_k(1) = p$. Using the property $U_k(z) - 1 = z\phi_k(z)/(1-z)$, $|z| < 1$, we conclude from (3.2) that

$$f_k(z) = z(\phi_k(z) + \sum_{j=1}^{m-k-1} p_j^k z^j (\phi_m(z) - \phi_{k+j}(z)))/(1 - z + z\phi_m(z)),$$

$$|z| < 1, \quad k = 1, \dots, m. \tag{4.4}$$

From [11] there follows that if $p > 0$, then

$$-\phi_k'(1) = \sum_{n=1}^{\infty} (P(A_n^k) - p), \quad k = 1, \dots, m. \tag{4.5}$$

Hence using (4.2) we obtain that

$$|\phi_1'(1)| < \infty \;\text{ iff }\; |\phi_2'(1)| < \infty \;\text{ iff, etc., iff }\; |\phi_m'(1)| < \infty \;\text{ iff }\; M(\nu_m^2) < \infty.$$

In this case, for the mean value of ν_k, we have

$$M(\nu_k) = (\phi_k'(1) - \phi_m'(1) + 1 - \sum_{j=1}^{m-k-1} P_j^k (\phi_m'(1) - \phi_{k+j}'(1)))/p. \tag{4.6}$$

Let us, similarly as in Section 2, study the counter of order m, $\mathcal{C}_s = (F_s; H_1, \dots, H_m)$, where F_s is the distribution function associated to F, defined by (2.7). Let $P(A_n^k(s))$, p_s, $P_n^k(s)$ and $a_n^k(s)$ be the quantities defined for the counter $\mathcal{C}_s$, $s \geq 0$. Then using the form of (3.1) and the Chebyshev inequality we may obtain, for any $s_1 \geq s_2 \geq 0$ and $k = 1, \dots, m$,

$$P(A_n^k(s_1)) \leq P(A_n^k(s_2)) \leq P(A_n^k), \quad n \geq 1, \quad p_{s_1} \leq p_{s_2} \leq p. \tag{4.7}$$

Using (3.4) we have that if (2.9) holds, then

$$p > 0 \quad \text{iff} \quad p_s > 0, \quad s > 0. \tag{4.8}$$

Due to $(F_{s_1})_{s_2}(t) = F_{s_1+s_2}(t)$, $s_1, s_2 \geq 0$, $(p_{s_1})_{s_2} = p_{s_1+s_2}$. If

$$D = \int_0^{\infty} t\, dH_m(t) < \infty, \tag{4.9}$$

and $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$, for some $\lambda > 0$, then, due to Takács [17]

$$p = e^{-\lambda D}. \tag{4.10}$$

Hence $p_s = e^{-(\lambda+s)D} < p$, and p_s is strictly decreasing function of s.

Varying the parameter $p \in (0,1]$ we may obtain, in general, different functions $\phi_k(z)$, $k = 1, \dots, m$. Taking into account this dependence on p we shall write $\phi_k(z) = \phi_{pk}(z)$. Similarly we write $P(A_n^k) = P(A_n^k)(p)$.

<u>Theorem 4.1</u>. Let $(F; H_1, \dots, H_m)$ be a modified counter of order m with (4.1) and (i) $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$; (ii) $\int^{\infty} t^2\, dH_m(t) < \infty$; (iii) $H_m(0+) = 0$. Then $\lim_{\lambda \to \infty} P(\lambda a_k(p) C_k > t) = e^{-t}$, $t \geq 0$, $k = 1, \dots, m$, where $a_k = a_k(p) = 1/M(\nu_k)$.

<u>Proof</u>. Since $\mu = 1/\lambda$, due to (2.8)-(2.10) and (4.4), we may write, independently of the form of F

$$M(e^{-sa_k C_k/\mu}) = f_k^{\bar{s}}(a(\bar{s})),$$

where $\bar{s} = s a_k(p)/\mu$.

Therefore

$$M(e^{-sa_k C_k/\mu}) = [a(\bar{s})(\phi_k^{\bar{s}}(a(\bar{s})) + \sum_{j=1}^{m-k-1} P_j^k(\bar{s})\, a(\bar{s})^j$$

$$(\phi_m^{\bar{s}}(a(\bar{s})) - \phi_{k+j}^{\bar{s}}(a(\bar{s})))] / [1 - a(\bar{s}) + a(s)\, \phi_m^{\bar{s}}(a(\bar{s}))] =$$

$$= a(\bar{s})[1 + \sum_{j=1}^{m-k-1} P_j^k(\bar{s})\, a(\bar{s})^j (\phi_m^{\bar{s}}(a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s})) - \tag{4.11}$$

$$- \phi_{k+j}^{\bar{s}}(a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s}))] / [(1 - a(\bar{s}))/\phi_k^{\bar{s}}(a(\bar{s})) + a(\bar{s}) \quad \phi_m^{\bar{s}}(a(\bar{s})/\phi_k^{\bar{s}}(a(\bar{s}))],$$

where ϕ_k^s is the function defined by (4.3) for the associated counter $\mathcal{C}_s$.

In the following the proof is divided into four steps.

I. First of all we prove that under our conditions, for any $\lambda > 0$, $|\phi_m^{\bar{s}\prime}(1)| < \infty$ and $\lim_{\lambda \to \infty} \phi_m^{\bar{s}\prime}(1) = 0$. In order to prove this, it is sufficient to show the same for $\phi_m'(1)$. Indeed, $\phi_m^{\bar s}$ corresponds to ϕ_m when $F = 1 - e^{-\Lambda t}, t \geq 0$, where

$$\Lambda = \lambda(1 + se^{-\lambda D}), \tag{4.12}$$

and if $\lambda \to \infty$, then $\Lambda \to \infty$.

It is clear that

$$P(A_1^1) = \lambda \int_0^\infty H_1(t) e^{-\lambda t}\, dt, \quad P(A_n^m) = \lambda^{n+1}/n! \int_0^\infty \Big(\int_0^t H_m(x)\,dx\Big)^n e^{-\lambda t}\, dt, \; n \geq 1, \tag{4.13}$$

$$\phi_m(z) = 1 - \lambda \int_0^\infty \exp\Big(-\lambda \int_0^y (1 - zH_m(u))\,du\Big)(1 - H_m(y))\,dy. \tag{4.14}$$

Since $D_2 = \int_0^\infty y(1-H_m(y))\,dy < \infty$, the integral

$$\int_0^\infty (1 - H_m(y)) \exp\Big(-\lambda \int_0^y (1 - zH_m(u))\,du\Big) \int_0^y H_m(u)\,du\,dy$$

converges uniformly in $z \in [0,1]$. Therefore we may take the derivative of (4.14) with respect to $z = 1$, and obtain

$$\phi_m'(z) = -\lambda^2 \int_0^\infty (1-H_m(y)) \exp\Big(-\lambda \int_0^\infty (1 - zH_m(u))\,du \int_0^y H_m(u)\,du\,dy.$$

Let $A > 0$ be arbitrary. Denote

$$I_1(\lambda, A) = \lambda^2 \int_0^A (1-H_m(y)) \exp\Big(-\lambda \int_0^\infty (1-H_m(u))\,du\Big) \int_0^y H_m(u)\,du\,dy,$$

$$I_2(\lambda, A) = \phi_m'(1) - I_1(\lambda, A).$$

It is obvious that $I_2(\lambda, A) \leq \lambda^2 \exp\Big(-\lambda \int_0^A (1 - H_m(u))\,du\, D_2$.

Because of $D < \infty$, for an arbitrary $\epsilon > 0$, there is $\Lambda_2(\epsilon, A) > 0$ so that $I_2(\lambda, A) < \epsilon/4$ whenever $\lambda > \Lambda_2(\epsilon, A)$.

Using the per-partes integration method we conclude

$$I_1(\lambda, A) = I_3(\lambda, A) + I_4(\lambda, A) + I_5(\lambda, A),$$

where

$$I_3(\lambda, A) = -\lambda \exp\Big(-\lambda \int_0^A (1 - H_m(u))\,du\Big) \int_0^A H_m(u)\,du,$$

$$I_4(\lambda, A) = \exp\Big(-\lambda \int_0^A (1 - H_m(u))\,du\Big),$$

$$I_5(\lambda, A) = \lambda \int_0^A \exp\Big(-\lambda \int_0^y (1 - H_m(u))\,du\Big)\,dy - 1.$$

It is clear that $|I_3(\lambda, A)| \leq \lambda A \exp\Big(-\lambda \int_0^A (1 - H_m(u))\,du\Big)$.

Hence there is $\Lambda_3(\epsilon, A) > 0$ so that $|I_3(\lambda, A)| < \epsilon/4$ whenever $\lambda > \Lambda_3(\epsilon, A)$. Similarly there is $\Lambda_4(\epsilon, A) > 0$ so that $|I_4(\lambda, A)| < \epsilon/4$ when $\lambda > \Lambda_4(\epsilon, A)$.

The condition $H_m(0+) = 0$ entails that, for any $\epsilon_1 > 0$, there is $A(\epsilon_1)$ with $1 - \epsilon_1 \leq 1 - H_m(u) \leq 1$ whenever $0 < u < A(\epsilon_1)$. Therefore

$$\lambda \int_0^{A(\epsilon_1)} e^{-\lambda y}\,dy \leq I_5(\lambda, A) \leq \lambda \int_0^{A(\epsilon_1)} e^{-\lambda(1-\epsilon_1)y}\,dy - 1$$

and

$$-e^{-\lambda A(\epsilon_1)} \leq I_5(\lambda, A) \leq (1 - e^{-\lambda A(\epsilon_1)})/(1 - \epsilon_1) - 1 < \epsilon_1/(1 - \epsilon_1).$$

Using the inequality $\epsilon_1/(1-\epsilon_1) < 2\epsilon_1$ which holds for $0 < \epsilon_1 < 1/2$ we get $|I_5(\lambda, A)| \leq \max(e^{-A(\epsilon_1)}, 2\epsilon)$.

Now, for a given $\epsilon_1 > 0$, we may choose $\Lambda_5(\epsilon_1) > 0$ so that $e^{-\lambda A(\epsilon_1)} < 2\epsilon_1$ whenever $\lambda > \Lambda_5(\epsilon_1)$. From this restriction we may find ϵ_1 and A so that $\epsilon_1 = \epsilon/8$ and $A = A(\epsilon/8)$. Hence if $\lambda > \max_{2 \leq i \leq 5}(\Lambda_i(\epsilon, A(\epsilon/8)))$, then $|\phi_m'(1)| < \epsilon$.

II. From (4.13) we have that $\lim_{\lambda\to\infty} P(A_1^1) = 0$ and (iii) of (4.2) entails $P(A_n^1) \leq P(A_{n-m+1}^m), n \geq m$. Therefore, this, (4.5) and $\lim_{\lambda\to\infty} \phi'(1) = 0$ yield $\lim_{\lambda\to\infty} \phi_k'(1) = 0, k = 1, \ldots,$, and, consequently, $\lim_{\lambda\to\infty} \phi_k^{\bar s\,\prime}(1) = 0, \; k = 1, \ldots, m$.

Using (4.6) we may check that $\lim_{\lambda\to\infty} a_k(p) = 0$, $\lim_{\lambda\to\infty} a_k(p)/p = 1$.

Therefore $\lim_{\lambda\to\infty} (1 - a(\bar s))/p = s$.

III. Here we show that

$$\lim_{\lambda\to\infty} \phi_i^{\bar s}(a(\bar s))/p_{\bar s} = 1, \quad i = 1, \ldots, m. \tag{4.15}$$

We note that, for any fixed $p \in (0,1]$, $\phi_i^{\bar s}(z)$ is non-decreasing function for each $0 \leq z \leq 1$. Using that and the elementary inequality $e^{-x} \geq 1 - x, x \geq 0$, we obtain $p_{\bar s} = \phi_i^{\bar s}(1) \leq \phi_i^{\bar s}(a(s)) \leq \phi_i^{\bar s}(1 - sa_k(p))$. Due to the inequality $(1-x)^n \geq 1 - nx, |x| \leq 1$, we have for sufficiently small $p > 0$

$$\phi_i^{\bar s}(1 - sa_k(p)) = 1 - \sum_{n=0}^\infty (1 - sa_k(p))^n a_n^i(\bar s) = p_{\bar s} - sa_k(p)\,\phi_i^{\bar s\,\prime}(1).$$

Hence

$$1 \leq \phi_i^{\bar s}(a(\bar s))/p_{\bar s} \leq 1 - sa_k(p)p_{\bar s}^{-1}\,\phi_i^{\bar s\,\prime}(1) \leq 1 - sa_k(p)p^{-1}\,\phi_i^{\bar s\,\prime}(1),$$

where we use (4.7), so that (4.14) holds.

IV. Using this fact we have

$$\lim_{\lambda\to\infty} \phi_i^{\bar s}(a(\bar s))/\phi_j^{\bar s}(a(\bar s)) = 1, \quad i,j = 1,\dots,m.$$

Finally, it is easy to show that in our case

$$\lim_{\lambda\to\infty} p/p_{\bar s} = 1, \qquad (4.16)$$

which completely proces the theorem. Q.E.D.

Theorem 4.2. Let $(F; H_1,\dots,H_m)$ be a modified counter of order m for which the conditions of Theorem 4.1 are fulfilled. Then

$$\lim_{\lambda\to\infty} M(e^{-sa_kC_k/\mu}\, e^{-qa_k\nu_k}) = 1/(1+s+q), \quad s,q\geq 0, \quad k=1,\dots,m. \qquad (4.17)$$

Proof. The proof is similar to that of Theorem 4.1 and it may be obtained from (4.11) replacing $M(e^{-sa_kC_k/\mu})$ by the left-hand side of (4.17) and $a(\bar s)$ by $a(\bar s)e^{-qa_k}$. Here we notice only the following.

It is clear that $\lim_{\lambda\to\infty} (1-a(s)e^{-qa_k})/p = s+q$. In order to prove

$$\lim_{\lambda\to\infty} \phi_i^{\bar s}(a(\bar s)e^{-qa_k(p)})/p_{\bar s} = 1 \qquad (4.18)$$

we proceed by analogous manner as in the proof of (4.15). We notice that $a(\bar s)\geq 1 - sa_k$, $e^{-qa_k}\geq 1-qa_k$.

Therefore if $a(\bar s) = \min(a(\bar s), e^{-qa_k})$, then

$$p_{\bar s} \leq \phi_i^{\bar s}(a(\bar s)e^{-qa_k}) \leq \phi_i^{\bar s}(a(\bar s)^2) \leq \phi_i^{\bar s}((1-sa_k(p))^2) \leq$$

$$\leq 1 - \sum_{n=0}^{\infty}(1-2sa_k(p)n)a_n^i(\bar s) = p_{\bar s} - 2sa_k(p)\,\phi_i^{\bar s\,\prime}(1).$$

If $e^{-qa_k} = \min(a(\bar s), e^{-qa_k})$, then similarly we have

$$p_{\bar s} \leq \phi_i^{\bar s}(a(s)e^{-qa_k}) \leq \phi_i^{\bar s}(e^{-2qa_k}) \leq p_{\bar s} - 2qa_k(p)\,\phi_i^{\bar s\,\prime}(1).$$

Hence $1 \leq \phi_i^{\bar s}(a(\bar s)e^{-qa_k})/p_{\bar s} \leq 1 - 2a_k(p)p^{-1}\,\phi_i^{\bar s\,\prime}(1)\max(s,q)$.

Taking into account that (4.18) yields

$$\lim_{\lambda\to\infty} \phi_i^{\bar s}(a(\bar s)e^{-qa_k})/\phi_j^{\bar s}(a(\bar s)e^{-qa_k}) = 1, \quad i,j=1,\dots,m,$$

we have proved (4.17). Q.E.D.

Corollary 4.2.1. Under the conditions of Theorem 4.2

$$\lim_{\lambda\to\infty} P(a_k\nu_k > t) = e^{-t}, \quad t\geq 0, \quad k=1,\dots,m.$$

Proof. It is evident. Q.E.D.

Theorem 4.3. Let $(F; H_1,\dots,H_m)$ be a counter of order m such that F is the distribution function of the constant equal to 1. Let $H_1,\dots,H_m$ vary so that (i) there is an integer n_0 such that, for any H_m,

$$0 < H_m(1) \leq H_m(2) \leq \dots \leq H_m(n_0) < 1, \quad H_m(n_0+1) = 1;$$

(ii) $H_1(n)\geq H_2(n)\geq\dots\geq H_m(n)$, $n\geq 1$.

If $H_1(1)\to 0$ whenever $H_m(1)\to 0$, then

$$\lim_{p\to 0+} M(e^{-sa_kC_k}\, e^{-qa_k\nu_k}) = 1/(1+s+q), \quad s,q>0, \quad k=1,\dots,m.$$

Proof. In this case $p = \prod_{i=1}^{n_0} H_m(i)$. Due to Lemma 2.1, $p_s = p$ for any $s\geq 0$. It is evident that $p\to 0$ iff $H_m(1)\to 0$. Therefore $\phi_k^{\bar s} = \phi_k$ and $\phi_k^{\bar s\,\prime}(1)\to 0$ when $p\to 0+$. Hence, using the analogous method as in the proof of Theorem 4.1, namely the part where we prove (4.15), we may obtain

$$\lim_{p\to 0+} \phi_k(a(\bar s)e^{-qa_k})/p = 1.$$

Appealing to (4.11) we prove the theorem. Q.E.D.

Theorem 4.4. Let $(F; H_1,\dots,H_m)$ be a counter of order m, where $F(t)<1$, for any $t\geq 0$, and (2.9) holds. Let H_m be the distribution function of the constant $D>0$. If $H_1,\dots,H_m$ vary so that (4.1) holds and $\lim_{D\to\infty} P(A_1^1)=0$, then

$$\lim_{D\to\infty} M(e^{-sa_kC_k/\mu}\, e^{-qa_k\nu_k}) = 1/(1+s+q), \quad s,q\geq 0, \quad k=1,\dots,m.$$

Proof. It is evident that, for any $s>0$, $\phi_m^s(z) = 1 - F_s(D) > 0$ and $\phi_m^{s\,\prime}(z) = 0$. Therefore $\phi_m^{\bar s}(a(\bar s)e^{-qa_k})/p_{\bar s} = 1$.

Using the per-partes integration method we easily check

$$p_{\bar s}/p = (a(sp/\mu) - \int_0^D e^{-s(1-F(D))t/\mu}\, dF(t)/(a(sp/\mu)(1-F(D))) =$$

$$= (a(sp/\mu)-1)/(a(sp/\mu)p) + e^{-s(1-F(D))D\mu}/a(sp/\mu) +$$

$$+ s\int_0^D e^{-s(1-F(D))t/\mu}(1-F(t))\,dt/(a(sp/\mu)\mu).$$

If $D\to\infty$, then the first term converges to $-s$, the second one to 1 and the third to s. Hence $\lim_{D\to\infty} p_{-s}/p=1$. Using the similar arguments as in the proof of Theorem 4.2 we finish the proof of the theorem. Q.E.D.

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О временном интервале между моментами регистрации частиц модифицированным счетчиком с мертвым временем продлевающегося типа

Изучается модифицированный счетчик с мертвым временем продлевающегося типа, когда первые m-частицы, приходящие на счетчик за период мертвого времени, имеют, вообще говоря, различные функции распределений. Выведены функция Лапласа для временного периода между началами двух соседних моментов регистрации, а также асимптотические показательные свойства в основном для случая пуассоновского процесса приходящих частиц.

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On the Time Interval between the Moments of Particle Registrations by a Modified Counter with Prolonging Dead Time

The modified counter with prolonging dead time, when the first m particles arriving at the counter during any dead time have, in general, different distribution functions of their impulse lengths, is studied. The Laplace transform of the time period between the beginnings of two successive registration moments is derived, and some asymptotic exponential properties are established mainly for the case of the Poisson recurrent input process of arriving particles.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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