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ON SEMIRECURRENT EVENTS

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1. INTRODUCTION

We suppose that during the k th experiment, $k = 1,2,\ldots$, the condition A^k either may be fulfilled or not. The fulfilment of A^k at the n th trial, $n = 1,2,\ldots$, we denote by A^k_n and its nonfulfilment by $\bar{A}^k_n$. The events $\{A^k_n;\ n \geq 1,\ k \geq 1\}$ are said to be semirecurrent if for any integers i_j with

$$1 \leq i_0 < i_1 < \ldots < i_n,\ n \geq 1, \tag{1.0}$$

we have

$$P(A^k_{i_1} \ldots A^k_{i_n} \mid A^k_{i_o}) = P(A^{k+i_o}_{i_1-i_o} \ldots A^{k+i_o}_{i_n-i_o}). \tag{1.1}$$

An equivalent condition to (1.1) is

$$P(A^k_{i_o} \ldots A^k_{i_n}) = P(A^k_{i_o})\, P(A^{k+i_o}_{i_1-i_o}) \ldots P(A^{k+i_{n-1}}_{i_n-i_{n-1}}) \tag{1.2}$$

whenever i_j fulfil (1.0).

Denote by ν_k, $k \geq 1$, an integer-valued random variable saying that the condition A^k is fulfilled for the first time in the kth experiment and put $P^k_n = P(\nu_k = n) = P(\bar{A}^k_1 \ldots \bar{A}^k_{n-1} A^k_n)$. Using (1.1) we may prove that

$$\begin{aligned} P^k_1 &= P(A^k_1), \\ P^k_n &= P(A^k_n) - \sum_{j=1}^{n-1} P(A^k_j)\, P^{k+j}_{n-j},\quad n \geq 2. \end{aligned} \tag{1.3}$$

Let us define for $k > 1$ and $|z| < 1$ $U_k(z) = \sum_{n=0}^{\infty} P(A^k_n) z^n$, where $P(A^k_0) \equiv 1$, $P_k(z) = \sum_{n=1}^{\infty} P^k_n z^n$. Due to (1.3) we have

$$U_k(z) = 1 + \sum_{n=0}^{\infty} P(A^k_n)\, P_{k+n}(z)\, z^n. \tag{1.4}$$

An interesting case is obtained when there is an integer m so that $P_m(z) = P_{m+1}(z) = \ldots$, $|z| < 1$. Then the semirecurrent events are said to be m-semirecurrent. It is clear that if $\{A^k_n : n \geq 1,\ k \geq 1\}$ are m-semirecurrent events, $m \geq 2$, then $\{B^k_n \equiv A^{k+1}_n : n \geq 1,\ k \geq 1\}$ are $(m-1)$-semirecurrent.

If $m = 1$, then (1.1) and (1.3) do not depend on the superscripts, and they are the recurrent events (for the defini-

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tion of the recurrent events see, e.g., [6,7]) iff $m = 1$. In this case (1.3) reduces into the familiar formula for the recurrent events [6,7]

$$P_1 = P(A_1),$$
$$P_n = P(A_n) - \sum_{j=1}^{n-1} P(A_j)P_{n-j}, \quad n \geq 2, \tag{1.5}$$

where $P_n = P_n^1 = P_n^2 = \ldots$, $P(A_n) = P(A_n^1) = P(A_n^2) = \ldots$ for each n. It is evident that semirecurrent events $\{A_n^k: n \geq 1, k \geq 1\}$ are m-semirecurrent iff $\{B_n^k: n \geq 1, k \geq 1\}$, where $B_n^k = A_n^{k+m-1}$, are recurrent events.

If $m = 2$, then (1.1) has the following form: for any $k = 1,2$ and integers i_j with (1.0)

$$P(A^k_{i_1} \ldots A^k_{i_n} | A^k_{i_o}) = P(A^2_{i_1-i_o} \ldots A^2_{i_n-i_o}). \tag{1.6}$$

Therefore for (1.3) we conclude that

$$P_1^k = P(A_1^k),$$
$$P_n^k = P(A_n^k) - \sum_{j=1}^{n-1} P(A_j^k) P^2_{n-j}, \quad n \geq 2, \; k = 1,2. \tag{1.7}$$

This case of semirecurrent events is also known in literature as recurrent events with delay [7,13].

Example 1.1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with $P(X_n < 0) > 0$ and $P(X_n > 0) > 0$, $n \geq 1$. Put $S_n^k = X_k + \ldots + X_{k+n-1}$, $n \geq 1$, $k \geq 1$. Then $\{A_n^k: n \geq 1, k \geq 1\}$ are semirecurrent events, where $A_n^k = \{S_n^k = 0\}$. If, particularly, $P(X_n = -1) = p_n > 0$, $P(X_n = -1) = 1 - p_n = q_n > 0$, then

$$P(A^k_{2n-1}) = 0,$$

$$P(A^k_{2n}) = \sum_{\substack{j_1 + \ldots + j_{2n} = n \\ j_i \in \{0,1\}}} \prod_{i=k}^{k+2n-1} p_i(j_i), \quad n \geq 1, \; k \geq 1,$$

where $p_i(1) = p_i$, $p_i(0) = q_i$.

In the particular we deal with a generalization of random walk when the particle moves from any stage with not necessarily equal probabilities, and returning to the initial stage, the movement of the particle resumes with initial conditions.

Example 1.2. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of positive independent random variables, independent of a sequence of positive independent random variables $\{\chi_n\}_{n=1}^{\infty}$. Then $\{A_n^k: n \geq 1, k \geq 1\}$, where

$$A_n^k = \{\chi_k < T_k + \ldots + T_{k+n-1}, \; \chi_{k+1} < T_{k+1} + \ldots + T_{k+n-1}, \ldots, \; \chi_{k+n-1} < T_{k+n-1}\}, \tag{1.8}$$

is a sequence of semirecurrent events.

It plays an important role in the mathematical theory of particle counters, especially, for a modified counter with prolonging dead time [3,2,4,14]. Here T_n denotes an interrival time between the arrivals of particles at the counter, where it is assumed

$$F_n(t) = P(T_n < t), \quad n \geq 1, \tag{1.9}$$

and χ_n denotes an impulse length generated by the nth particle (may be non-proper, too) with the distribution function

$$H_n(t) = P(\chi_n < t), \quad n \geq 1. \tag{1.10}$$

Analogous questions may arise in the modified queueing systems with infinitely many servers [1]. The random variable ν_1 may be interpreted as the number of particles arriving at the counter during the dead time [3], respectively, the number of customers served during the busy period [1].

2. SEMERECURRENT SEQUENCES

A sequence of non-negative numbers, $\{u_n^k: n \geq 0, k \geq 1\}$, is said to be semirecurrent if, for any $k \geq 1$,

$$u_0^k = 1, \tag{2.0}$$

and if there is a sequence of non-negative numbers $\{f_n^k\}_{n=1}^{\infty}$ with

$$\sum_{n=1}^{\infty} f_n^k \leq 1, \tag{2.1}$$

such that

$$u_n^k = f_n^k + \sum_{j=1}^{n-1} u_j^k f_{n-j}^{k+j}, \quad n \geq 1. \tag{2.2}$$

Theorem 2.1. Let the sequences of non-negative numbers $\{f_n^k\}_{n=1}^{\infty}$, $k \geq 1$, with (2.1) be given. Define recurrently a sequence $\{v_n^k: n \geq 0, k \geq 1\}$ via $v_0^k = 1$, $v_n^k = f_n^k + \sum_{j=1}^{n-1} f_j^k v_{n-j}^{k+j}$, $n \geq 1, k \geq 1$. Then $u_n^k = v_n^k$, $k, n \geq 1$, that is

$$u_n^k = f_n^k + \sum_{j=1}^{n-1} f_j^k u_{n-j}^{k+j}, \quad n, k \geq 1. \tag{2.3}$$

Proof. It sufficies to take into account the formulae (2.2), (2.3) and use the method of mathematical induction. Q.E.D.

Denote by $\mathcal{S}$ the set of all semirecurrent sequences. If $\{A_n^k: n \ge 1, k \ge 1\}$ are semirecurrent events, then $\{u_n^k: n \ge 0, k \ge 1\}$, where

$$u_n^k = P(A_n^k), \quad n \ge 0, \ k \ge 1 \tag{2.4}$$

is, due to (1.7), the semirecurrent sequence. Indeed, it is sufficient to put

$$f_n^k = P_n^k, \quad k,n \ge 1. \tag{2.5}$$

If we denote by $\mathcal{E}$ the set of all semirecurrent sequences for which there are semirecurrent events defined on an appropriate probability space with (2.4) and (2.5), then $\mathcal{E} \subset \mathcal{S}$. Therefore in the sequel without ambiguity we shall write U_k and P_k, $k \ge 1$, for the corresponding generating functions for semi-recurrent events as well as for semirecurrent sequences.

We say that a semirecurrent sequence, $\{u_n^k: n \ge 0, k \ge 1\}$, is m-semirecurrent if there an integer $m \ge 1$ so that $P_m(z) = P_{m+1}(z) = \dots$, $|z| < 1$, or, equivalently, $U_m(z) = U_{m+1}(z) = \dots$, $|z| < 1$, and denote by $\mathcal{S}_m(\mathcal{E}_m)$ the set of all m-semirecurrent sequences (the same for which there are m-semirecurrent events with (2.4)). It is clear that $\mathcal{E}_m \subset \mathcal{S}_m \subset \mathcal{S}_{m+1}$, $\mathcal{E}_m \subset \mathcal{E}_{m+1}$.

If $\{u_n^k: n \ge 0, k \ge 1\}$ is an m-semireccurent sequence, then without ambiguity we shall write for it $\{u_n^k: n \ge 0, k = 1, \dots, m\}$. Analogically we shall write $\{A_n^k: n \ge 1, k = 1, \dots, m\}$ for m-semirecurrent events $\{A_n^k: n \ge 1, k \ge 1\}$.

It is known[7] that $\mathcal{E}_1 = \mathcal{S}_1$, $\mathcal{E}_2 = \mathcal{S}_2$. More precisely, $\{u_n^1\}_{n=0}^{\infty}$ is a recurrent sequence (that is, $m=1$) iff there are independent integer-valued random variables $Y_1, Y_2, \dots$ (may be non-proper, too) with a common probability function $\{f_n^1\}_{n=1}^{\infty}$ such that

$$u_n^1 = P\Big(\bigcup_{i=1}^{n} \{Y_1 + \dots + Y_i = n\}\Big), \quad n \ge 1; \tag{2.6}$$

iff there is a sequence of recurrent events $\{A_n\}_{n=1}^{\infty}$ such that $u_n^1 = P(A_n^1)$, $n \ge 1$; iff there is a Markov chain $\{X_n\}_{n=0}^{\infty}$ on a countable state space such that $u_n^1 = P(X_n = i | X_0 = i)$, $n \ge 1$, for some state i (this is a result of K.L.Chung recorded by Feller[7]).

Analogically, $f_n^2 = f_n^3 = \dots$, $n \ge 1$, iff there is a sequence of independent integer-valued random variables $\{Y_n\}_{n=1}^{\infty}$ such that $Y_2, Y_3, \dots$ have the probability function $\{f_n^2\}_{n=1}^{\infty}$, and Y_1 has the probability function $\{f_n^1\}_{n=1}^{\infty}$ so that (2.5) is fulfilled and $u_n^2 = P(\bigcup_{i=2}^{n+1} \{Y_2 + \dots + Y_i = n\})$; iff there are semirecurrent events with delay, $\{A_n^1, A_n^2\}_{n=1}^{\infty}$, so that $u_n^k = P(A_n^k)$, $n \ge 1$, $k = 1,2$.

The problem of the existence of a Markov chain $\{X_n\}_{n=0}^{\infty}$ on a countable state space having at least two states, for a given

recurrent sequence with delay, such that $u_n^1 = P(X_n = i | X_0 = j)$, $u_n^2 = P(X_n = i | X_0 = i)$ for all $n \ge 1$ and some pair $i \ne j$ of states is studied by Kingman[12]. He notes that he does not know an effective characterization of this problem.

We remark that for m-semirecurrent sequences with $m \ge 3$ the characterizations analogous to those in the above do not hold. Indeed, let $m = 3$ and $f_1^1 = f_2^1 = 1/2$, $f_1^2 = 1/3$, $f_2^2 = 2/3$, $f_1^3 = 1/4$, $f_2^3 = 3/4$. Then $u_3^1 = 1/2 < 13/24 = P(\bigcup_{i=1}^{3} \{Y_1 + \dots + Y_i = 3\})$.

Therefore it would be interesting to give the similar description of m-semirecurrent sequences for $m \ge 3$ via m-semirecurrent events as they are known in the case when $m = 1,2$.

The sequence $\{u_n^k: n \ge 0, k \ge 1\}$ is far from being an arbitrary sequence of numbers between 0 and 1. Its behaviour is restricted by inequalities which are consequences of (2.2) and (2.3), or simple probabilistic argument for semirecurrent events based on (1.2). For example, if m and n are integers, then for any $k \ge 1$

$$u_m^k u_n^{k+m} \le u_{m+n}^k \le 1 - u_m^k + u_m^k u_n^{k+m}. \tag{2.7}$$

These are the simplest of a whole class of inequalities, which are completely described in Theorem 2.4. We note that (2.7) was known only for recurrent sequences[10].

First of all we prove the following two lemmas.

<u>Lemma 2.2.</u> Let $\{A_n^k: n \ge 1, k \ge 1\}$ be semirecurrent events. If $1 \le i_1 < i_2 < \dots < i_n$, $n \ge 2$, then for any $k \ge 1$, $s \le n-1$, and $j_i \in \{0,1\}$

$$\begin{aligned} &P({}^{j_1}A_{i_1}^k \dots {}^{j_{s-1}}A_{i_{s-1}}^k \, A_{i_s}^k \, {}^{j_{s+1}}A_{i_{s+1}}^k \dots {}^{j_n}A_{i_n}^k) = \\ &= P({}^{j_1}A_{i_1}^k \dots {}^{j_{s-1}}A_{i_{s-1}}^k \, A_{i_s}^k)\, P({}^{j_{s+1}}A_{i_{s+1}-i_s}^{k+i_s} \dots {}^{j_n}A_{i_n-i_s}^{k+i_s}), \end{aligned} \tag{2.8}$$

where ${}^0A = \bar{A}$ and ${}^1A = A$.

Proof. It is evident. Q.E.D.

<u>Lemma 2.3.</u> For all semirecurrent sequences $\{u_n^k: n \ge 0, k \ge 1\}$ satisfying $u_1^k \ge 0$, $k \ge 1$, there hold (i)

(i) $u_n^k > 0$, $k,n \ge 1$;

(ii) $U_j(1) = \infty$ implies $U_{j-1}(1) = \infty$.

Proof. (i) Let there be two integers k_0 and n such that $u_n^{k_0} = 0$. Denote $n_0 = \min\{n: u_n^{k_0} = 0\}$. From our assumption we have

$n_0 > 1$. (2.7) implies that $0 = u^{k_0}_{n_0} \geq u_1^{n_0} u_1^{n_0+1} \ldots u_1^{k+n_0-1} > 0$ which contradicts with the assumption.

(ii) (2.7) yields $U_k(z) \geq 1 + u_1^k z U_{k+1}(z)$, $0 < z < 1$, $k \geq 1$. Using this inequality we have proved the second part of the proposition.

Theorem 2.4. Let $\{u_n^k : n \geq 1,\ k \geq 1\}$ be non-negative numbers. For any $k \geq 1$ and integers i_j with

$$1 \leq i_1 < \ldots < i_m \tag{2.9}$$

write

$$\phi_k(i_1,\ldots,i_m) = 1 - \sum_{1 \leq j_1 \leq m} u^k_{i_{j_1}} + \sum_{1 \leq j_1 \leq j_2 \leq m} u^k_{i_{j_1}} u^{k+i_{j_1}}_{i_{j_2}-i_{j_1}} - \ldots +$$

$$+ (-1)^m \sum_{1 \leq j_1 < \ldots < j_m \leq m} u^k_{i_{j_1}} u^{k+j_1}_{i_{j_2}-i_{j_1}} \ldots u^{k+i_{j_{m-1}}}_{i_{j_m}-i_{j_{m-1}}}.$$

Then there are semirecurrent events $\{A_n^k : n \geq 1,\ k \geq 1\}$ with $u_n^k = P(A_n^k)$ for each k,n if and only if

$$0 \leq \phi_k(i_1,\ldots,i_{m-1}) \leq \phi_k(i_1,\ldots,i_m) \tag{2.10}$$

for any $m \geq 1$, $i_1,\ldots,i_m$ satisfying (2.9) and each $k \geq 1$.

Proof. The necessity of (2.10) we obtain from the observation that if i_j satisfy (2.9), then $\phi_k(i_1,\ldots,i_m) = P(\bar{A}^k_{i_1} \ldots \bar{A}^k_{i_m})$.

Conversely, suppose that (2.10) hold. For any $k \geq 1$ we have to construct a probability space $(\Omega_k, \mathcal{A}_k, P_k)$ and a sequence $\{\bar{A}_n^k\}_{n=1}^{\infty} \subset \mathcal{A}_k$ such that $u_n^k = P_k(\bar{A}_n^k)$, $n \geq 1$. To do this we must verify the Kolmogorov consistence conditions[15]. Due to Lemma 2.2 and (2.8), we see that, for any m-tuple of events, we need not verify all 2^m inequalities, but only two which are guaranteed by (2.10).

Hence if we construct the direct probability space $(\Omega, \mathcal{A}, P) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, P_k)$, then $A_n^k = \pi_k^{-1}(\bar{A}_n^k)$, where $\pi_k : \Omega \to \Omega_k$ is the kth projection function, are semirecurrent events in question.

We note that the author does not know whether the semirecurrent sequence $\{u_n^k : n \geq 0,\ k \geq 1\}$ satisfy the condition (2.10), or, equivalently, $\mathcal{E} = \mathcal{S}$.

3. THE KALUZA SEQUENCES

In 1928 Kaluza[8] noticed that if the formal power series $U(z) = \sum_{n=0}^{\infty} u_n z^n$ with real coefficients satisfies the condition

$$u_0 = 1,\quad u_n \geq 0,\quad u_n^2 \leq u_{n-1} u_{n+1},\ n \geq 1, \tag{3.1}$$

then the formal power series defined via $P(z) = \sum_{n=1}^{\infty} f_n z^n = 1 - 1/U(z)$ has also non-negative coefficients.

The sequences with (3.1), as it was shown much later on, play an important role in the theory of infinitely divisible sequences. We recall[9] that a recurrent sequence $\{u_n\}_{n=0}^{\infty}$ is said to be infinitely divisible if $\{u_n^t\}_{n=0}^{\infty}$ is a recurrent sequence for each real non-negative t. It is known[11, p. 439] that if $\{u_n\}_{n=0}^{\infty}$ is an infinitely divisible recurrent sequence, then there is an integer h such that $u_n > 0$ iff n is a multiple of h, and $\{u_{nh}\}_{n=0}^{\infty}$ is the sequence satisfying (3.1).

In the sequel we shall study the sequences for which a condition similar to (3.1) holds.

Firstly we note that the following proposition is true.

Proposition 3.1. (i) If $\{u_n^k : n \geq 0,\ k \geq 1\} \in \mathcal{E}$, then $\{u_{nh}^{kh} : n \geq 0,\ k \geq 1\} \in \mathcal{S}$ for any integer h.

(ii) If $\{u_n^k : n \geq 0,\ k \geq 1\}$ and $\{v_n^k : n \geq 0,\ k \geq 1\}$, then $\{u_n^k v_n^k : n \geq 0,\ k \geq 1\}$.

(iii) If $\{u_n^k(j) : u \geq 0,\ k \geq 1\} \subset \mathcal{S}(\mathcal{E})$ or any j, and if $u_n^k(j) \to u_n^k$ as $j \to \infty$, for each n and k, then $\{u_n^k : n \geq 0,\ k \geq 1\} \in \mathcal{S}(\mathcal{E})$.

(iv) If $\{u_n^k : n \geq 0,\ k \geq 1\} \in \mathcal{S}$, and there is $u_n = \lim_k u_n^k$, $n \geq 1$, then $\{u_n\}_{n=0}^{\infty} \in \mathcal{S}$.

Proof. Part (i) may be proved using (1.1). For (ii) we note that we may find two sequences of semirecurrent events, $\{A_n^k : n \geq 1,\ k \geq 1\}$ and $\{B_n^k : n \geq 1,\ k \geq 1$ with $u_n^k = P(A_n^k)$, $v_n^k = P(B_n^k)$, $n,k \geq 1$, and assuming the independence of $\{A_n^k\}$ on $\{B_n^k\}$. Hence $\{C_n^k = A_n^k B_n^k : n \geq 1,\ k \geq 1\}$ are semirecurrent events such that (ii) holds.

Using (2.2) and Theorem 2.4 the parts (iii) and (iv) may be easily proved. Q.E.D.

A sequence of real non-negative numbers, $\{u_n^k : n \geq 0,\ k \geq 1\}$, is said to be Kaluza's if

$$u_0^k = 1,\quad u_n^k u_n^{k+1} \leq u_{n+1}^k u_{n-1}^{k+1} \quad \text{for any } k,n \geq 1. \tag{3.2}$$

We remark that if $u_1^k > 0$, $k \leq 1$, then from (3.2) we have $u_n^k > 0$ for every $k,n \geq 1$.

Example 3.2. Let $0 \leq H_1(1) \leq H_2(2) \leq \ldots \leq 1$, $0 \leq H_2(1) \leq H_2(2) \leq \ldots \leq 1$. Put $u_0^k = 1$, $k \geq 1$; $u_1^1 = H_1(1)$, $u_2^1 = H_1(2)H_2(1)$, $u_n^1 = H_1(n)H_2(n-1)/(n-1)$ if $n \geq 3$; $u_1^2 = H_2(1)$, $u_n^2 = H_2(n)/n$ if $n \geq 2$; $u_n^k = 1/(n+1)$ for any $k \geq 3$, $n \geq 1$. Then $\{u_n^k : n \geq 0,\ k \geq 1\}$ is the Kaluza sequence.

Let $\{u_n\}_{n=0}^{\infty}$ be a sequence with (3.1) and let $\{a_k\}_{k=1}^{\infty}$ be a sequence of (i) positive numbers between 0 and 1; (ii) with $1 \geq a_1 \geq a_2 \geq \ldots \geq 0$. Then $\{u_n^k : n \geq 0,\ k \geq 1\}$, where (i) $u_n^k = a_k u_n$; (ii) $u_n^k = (a_k)^n u_n$, is the Kaluza sequence.

The last example is a particular case of the following result saying that any Kaluza's sequence with $u_n^k \leq u_{n-1}^{k+1}$, $n,k \geq 1$, corresponds to some semirecurrent events of the form (1.8) for special modified counter with prolonging dead time.

Theorem 3.3. Let $\{u_n^k: n \geq 0, k \geq 1\}$, for which (3.2) holds, be given. If, moreover,

$$u_n^k \leq u_{n-1}^{k+1}, \quad n \geq 1, \; k \geq 1, \tag{3.3}$$

then $\{u_n^k: n \geq 0, \; k \geq 1\} \in \mathfrak{S}$.

Proof. First of all we conclude from (3.3) that any u_n^k is a real number between 0 and 1. Now we shall construct the sequences of real numbers, $\{p_k(n)\}_{n=1}^{\infty}$, $k \geq 1$, as follows.

For every $k \geq 1$ we put $p_k(1) = u_1^k$, and for each $n \geq 2$ we recurrently define $p_k(n)$ so that $u_n^k = p_k(n)\, u_{n-1}^{k+1}$ holds, where $p_k(n) = u_n^k/u_{n-1}^{k+1}$ if $u_{n-1}^{k+1} > 0$, and $p_k(n) = p_k(n-1)$ otherwise. From our construction of $p_k(n)$, using (3.3) and (3.2), we have that $0 \leq p_k(1) \leq p_k(2) \leq \dots \leq 1$.

Define a sequence of distribution functions, $\{H_k\}_{k=1}^{\infty}$, so that $H_k(t) = 0$, if $t < 1/2$, $H_k(t) = p_k(n)$, if $n - 1/2 \leq t < n + 1/2$, $n \geq 1$. Now we consider a modified counter with prolonging dead time (see Example 1.1) for which the impulse lengths, χ_k, of arriving particles are distributed according to $\{H_k\}_{k=1}^{\infty}$, and interarrival times, T_k, are constants equal to 1. Then for this particular case (1.8) has the form

$$P(A_n^k) = u_n^k = p_k(n)\, p_{k+1}(n-1) \dots p_{k+n-1}(1), \quad n, k \geq 1, \tag{3.4}$$

and this completes the proof of the theorem. Q.E.D.

We write $\mathfrak{S}_0$ for the set of all infinitely divisible semirecurrent sequences $\{u_n^k: n \geq 0, k \geq 1\}$ defined by the requirement $\{(u_n^k)^t: n \geq 0, k \geq 1\}$ is to belong to $\mathfrak{S}$ for each real non-negative t.

Corrollary 3.3.1. If we denote by K the set of all Kaluza's sequences for which (3.3) holds, then $K \subset \mathfrak{S}_0 \subset \mathfrak{S}$.

Proof. If $\{u_n^k: n \geq 0, k \geq 1\} \in K$, then easy checking shows us that $\{(u_n^k)^t: n \geq 0, k \geq 1\}$ fulfils (3.2) and (3.3) for any real $t > 0$. The application of Theorem 3.3 proves the corollary. Q.E.D.

Motived by Kingman[12] and (ii) from Proposition 3.1. we suggest to prove that if $\{u_n^k: n \geq 0, k \geq 1\}$ is a semirecurrent sequence, then so is $\{(u_n^k)^t: n \geq 0, k \geq 1\}$ for any real $t \geq 1$.

We must remark that the solution of this problem is unknown even for recurrent sequences (for details see [12]), and the author of the present note does not know the affirmative answer even for integer-valued t in the case of the semirecurrent sequences.

4. m-SEMIRECURRENT SEQUENCES

In this part we shall study the properties of m-semirecurrent sequences in more detail.

We say that for m-semirecurrent sequence $\{u_n^k: n \geq 0, k = 1, \dots, m\}$ the case of periodicity holds if there is an integer $t > 1$ such that $u_n^m > 0$ if $n = t, 2t, \dots$. In this case $u_n^m = 0$ whenever $n \neq jt$. The greatest integer $t > 1$ with this property is called the period. In the opposite case $\{u_n^m\}_{n=0}^{\infty}$ is said to be non-periodic. The sequence $\{u_n^k\}_{n=0}^{\infty}$, $k = 1, \dots, m$, is called certain or uncertain as $\sum_{n=1}^{\infty} f_n^k = 1$ or $\sum_{n=1}^{\infty} f_n^k < 1$.

It is known[7] that the recurrent sequence $\{u_n^m\}_{n=0}^{\infty}$ is certain iff $\sum_{n=1}^{\infty} u_n^m = \infty$. We note the analogous assertion does not hold, in general, for $\{u_n^k\}_{n=0}^{\infty}$, when $k = 1, \dots, m-1$ $(m \geq 2)$. Indeed, let $m = 2$ and $u_1^1 = 1$, $u_{n+1}^1 = u_n^2 = 0$ for any $n \geq 1$. Then $\sum_{n=1}^{\infty} f_n^1 = 1$ but $\sum_{n=1}^{\infty} u_n^1 = 1$.

However the next result holds:

Theorem 4.1. Let us have for m-semirecurrent sequence $\{u_n^k: n \geq 0, k = 1, \dots, m\}$

$$\begin{aligned} &\text{(i)} \quad u_n^1 \geq u_n^2 \geq \dots \geq u_n^m, \; n \geq 1, \\ &\text{(ii)} \quad u_1^k \geq u_2^k \geq \dots, \quad k = 1, \dots, m, \\ &\text{(iii)} \quad u_{n+1}^k \leq u_n^{k+1}, \quad n \geq 0, \; k = 1, \dots, m. \end{aligned} \tag{4.1}$$

If $\{u_n^m\}_{n=0}^{\infty}$ is the certain sequence, then

$$\lim_{z \to 1^-} U_k(z)/U_{k+1}(z) = 1 \tag{4.2}$$

and $\{u_n^k\}_{n=0}^{\infty}$ is certain, too, $k = 1, \dots, m-1$ $(m \geq 2)$.

Proof. (4.2) follows from the next inequality $U_{k+1}(z) \leq U_k(z) \leq$ $< 1 + u_1^k(U_{k+1}(z) - 1)$, $0 \leq z < 1$, $k = 1, \dots, m$. Since $\{u_n^m\}_{n=0}^{\infty}$ is certain, then $u_1^m > 0$. Hence from (i) of (4.1) we have $u_1^k > 0$, for any k, and Lemma 2.3 entails (4.2). Q.E.D.

We note that m-semirecurrent events with (4.1) appear in the theory of counters[3].

It is easily verifiable that for an arbitrary m-semirecurrent sequence, $\{u_n^k: n \geq 0, k = 1, \dots, m\}$, there holds

$$U_k(z) = \Big(1 + \sum_{j=0}^{m-k-1} u_j^k z^j (P_{k+j}(z) - P_m(z))\Big)/(1 - P_m(z)), \quad |z| < 1, \; k = 1, \dots, m. \tag{4.3}$$

Theorem 4.2. Let $\{u_n^k: n \geq 0,\ k = 1,\dots,m\}$ be an m-semirecurrent sequence with certain and non-periodic $\{u_n^m\}_{n=0}^{\infty}$. Then

$$\lim_n u_n^k = (P_k(1) - \sum_{j=1}^{m-k-1} u_j^k (P_{k+j}(1) - 1))/M(\nu_m), \quad k = 1,\dots,m. \qquad (4.4)$$

(Here, as usually, if $M(\nu_m) = \infty$, then $1/\infty = 0$, that is, if $M(\nu_m) = \infty$, then $u_n^k \to 0$ as $n \to \infty$).

In particular, if every $\{u_n^k\}_{n=0}^{\infty}$, $k = 1,\dots,m$, is certain, then $\lim_n u_n^k = 1/M(\nu_m)$.

Proof. From /10,11/ we have that in our case the limit of u_n^m when $n \to \infty$ exists and is equal to $1/M(\nu_m)$. Using (4.3) and the equality $U_m(z) = 1/(1 - P_m(z))$ we see that $u_n^k = u_n^m b_0 + u_{n-1}^m b_1 + \dots + u_0^m b_n$, where $\{b_n\}_{n=0}^{\infty}$ are the coefficients for the power series

$$P_k(z) + \sum_{j=1}^{m-k-1} u_j^k z^j (P_{k+j}(z) - P_m(z)),$$

and for them we have $\sum_{n=0}^{\infty} |b_n| < \infty$. Let $B = \sum_{n=0}^{\infty} b_n$, $u_\infty = \lim_n u_n^m$ and $B_1 = \max_n\{|b_n|\}$. Then, for any $n > N$,

$$|u_n^k - Bu_\infty| \leq |b_0(u_n^m - u_\infty) + \dots + b_N(u_{n-N}^m - u_\infty)| +$$

$$+ |b_{N+1}(u_{n-N+1}^m - u_\infty) + \dots + b_n(u_0^m - u_\infty)| + u_\infty \sum_{i=n+1}^{\infty} |b_i|.$$

Let $\epsilon > 0$ be given. Then there are sufficiently large integers N and N_1 such that $\sum_{i=n+1}^{\infty} |b_i| < \epsilon/4$, $|u_n^m - u_\infty| < \epsilon/2(N+1)B_1$, whenever $n > N$ and $n > N_1$, respectively. Hence for any $n > \max(2N, N_1)$ $|u_n^k - Bu_\infty| < \epsilon$. It is evident that B is equal to the numerator on the right-hand side of (4.4). Q.E.D.

Now we consider the periodic case of m-semirecurrent sequences, that is, the case when $P_m(z) = \sum_{n=1}^{\infty} f_{nt}^m z^{nt}$, $|z| < 1$, where $t > 1$ is the period. From (4.3) we see that the coefficients u_{nt}^k depend only on the coefficients of the power series in the numerator of (4,3). Hence

$$U_k(z) = U_k^0(z) + zU_k^1(z) + \dots + z^{t-1} U_k^{t-1}(z),$$

$$P_k(z) = P_k^0(z) + z P_k^1(z) + \dots + z^{t-1} P_k^{t-1}(z),$$

where

$$U_k^i(z) = \sum_{n=0}^{\infty} u_{nt+i}^k z^{nt}, \quad P_k^i(z) = \sum_{n=1}^{\infty} f_{nt+i}^k z^{nt},$$

for $i = 0,1,\dots,t-1$. Therefore

$$U_k^0(z) = (1 + \sum_{j=0}^{m-k-1} u_j^k z^j (P_{k+j}^0(z) - P_m^0(z)))/(1 - P_m(z)),$$

$$U_k^i(z) = \sum_{j=0}^{m-k-1} u_j^k z^j P_{k+j}^i(z)/(1 - P_m(z)), \qquad i = 1,\dots,t-1.$$

Using the analogous method as in the proof of the last theorem we may prove:

Theorem 4.3. Let, for m-semirecurrent sequences $\{u_n^k: n \geq 0,\ k = 1,\dots,m\}$, $\{u_n^m\}_{n=0}^{\infty}$ be certain and periodic with a period $t > 1$. Then

$$\lim_n u_{nt}^k = t \sum_{j=0}^{m-k-1} u_j^k (P_{k+j}^0(1) - 1)/M(\nu_m),$$

$$\lim_n u_{nt+i}^k = t \sum_{j=0}^{m-k-1} u_j^k P_{k+j}^i(1)/M(\nu_m), \quad i = 1,\dots,t-1.$$

5. RECURRENT EVENTS WITH DELAY

In this section we derive some information concerning the asymptotic behaviour of $\{u_n^1, u_n^2\}_{n=0}^{\infty}$ in the case of the recurrent events (and, equivalently, sequences) with delay. The similar results for recurrent events only are in /6/.

For convenience we put

$$q_n^k = \sum_{i=n+1}^{\infty} f_i^k, \qquad r_n^k = \sum_{i=n+1}^{\infty} q_i^k,$$

$$Q_k(z) = \sum_{n=0}^{\infty} q_n^k z^n, \quad R_k(z) = \sum_{n=0}^{\infty} r_n^k z^n, \quad |z| < 1,\ k = 1,2.$$

For certain $\{u_n^k\}_{n=0}^{\infty}$, $k = 1,2$, we put $m_k = \sum_{n=1}^{\infty} n f_n^k$, $M_k = \sum_{n=2}^{\infty} n(n-1) f_n^k$. Then

$$1 - P_k(z) = (1 - z) Q_k(z), \quad m_k - Q_k(z) = (1 - z) R_k(z), \qquad (5.1)$$

and hence

$$m_k = P_k'(1) = Q_k(1) = \sum_{n=0}^{\infty} q_n^k, \qquad (5.2)$$

$$M_k = P_k''(1) = 2Q_k'(1) = 2R_k(1) = 2\sum_{n=0}^{\infty} r_n^k . \qquad (5.3)$$

Theorem 4.2 implies that if $\{u_n^1\}_{n=0}^{\infty}$ and $\{u_n^2\}_{n=0}^{\infty}$ are both certain, and $\{u_n^2\}_{n=0}^{\infty}$ is non-periodic, then $\lim_n u_n^1 = u_{\infty} = \lim_n u_n^2$.

Throughout the sequel by $\{u_n^1, u_n^2\}_{n=0}^{\infty}$ we shall understand recurrent sequences with delay.

Theorem 5.1. Let $\{u_n^2\}_{n=0}^{\infty}$ be certain and non-periodic. Then $M_2 < \infty$ iff

$$\sum_{n=1}^{\infty} |u_n^2 - 1/m_2| < \infty . \qquad (5.4)$$

Proof. The necessary condition follows from [6, Th.3]. For the proof of sufficiency we use the equality

$$U_2(z) - 1/m_2(1-z) = R_2(z)/(m_2 Q_2(z)), \ |z| < 1 . \qquad (5.5)$$

Since $\{u_n^2\}_{n=0}^{\infty}$ is certain, then (5.4) implies that $u_{\infty} > 0$. The power series for $R_2(z)$ and $Q_2(z)$ have positive coefficients, and $Q_2(z)$ converges for $z = 1$. Moreover, $Q_2(z) \neq 0$ for any $|z| \leq 1$. From (5.5) we have for $0 \leq z < 1$.

$$0 \leq R_2(z)/m_2 \leq Q_2(z) \sum_{n=0}^{\infty} |u_n^2 - 1/m_2| \leq Q_2(1) \sum_{n=0}^{\infty} |u_n^2 - 1/m_2| < \infty .$$

Hence $\lim_{z \to 1^-} R_2(z) < \infty$, and, consequently, $M_2 < \infty$. Q.E.D.

Theorem 5.2. Let $\{u_n^k\}_{n=0}^{\infty}$, $k = 1,2$, be certain with finite m_1 and m_2, and let $\{u_n^2\}_{n=0}^{\infty}$ be non-periodic. Then

$$\sum_{n=1}^{\infty} |u_n^1 - u_n^2| < \infty , \qquad (5.6)$$

$$\sum_{n=1}^{\infty} (u_n^1 - u_n^2) = 1 - m_1/m_2 . \qquad (5.7)$$

Proof. For the recurrent sequence with delay we have $u_n^1 = f_1^2 u_{n-1}^1 + f_2^2 u_{n-2}^1 + \dots + f_{n-1}^2 u_1^1 + f_n^1 u_0^1$. Due to $q_0^k = 1$, $f_n^k = q_{n-1}^k - q_n^k$, $n \geq 1$, we obtain $q_0^2 u_n^1 = (q_0^2 - q_1^2)u_{n-1}^1 + \dots + (q_{n-2}^2 - q_{n-1}^2) u_1^1 + (q_{n-1}^1 - q_n^1)u_0^1$, and so $q_0^2 u_n^1 + \dots + q_{n-1}^2 u_1^1 + q_n^1 u_0^1 = q_0^2 u_{n-1}^1 + \dots + q_{n-2}^2 u_1^1 + q_{n-1}^1 u_0^1$.

Denote by A_n the left-hand side of this equality. Hence $A_n = A_{n-1} = \dots = A_1 = A_0$. But $A_0 = q_0^1 u_0^1 = 1$, and, consequently, A_n are the coefficients of the power series $Q_1(z) - Q_2(z) + Q_2(z) U_1(z) = 1/(1-z)$. Repeating the same for $\{u_n^2\}_{n=0}^{\infty}$ we have $Q_2(z) U_2(z) = 1/(1-z)$. Hence

$$Q_1(z) - Q_2(z) = Q_2(z)(U_2(z) - U_1(z)) . \qquad (5.8)$$

To prove the proposition we have to show that the power series for $U_2(z) - U_1(z) = (Q_1(z) - Q_2(z))/Q_2(z)$ has absolutely convergent coefficients. Due to our assumption, $Q_1(z) - Q_2(z)$ has absolutely convergent coefficients, too. The same is true for $Q_2^{-1}(z)$, according to the Wiener theorem [5]. Hence (5.6) holds. For (5.7) we may use (5.8) and (5.1). Q.E.D.

Theorem 5.3. Let $\{u_n^k\}_{n=0}^{\infty}$, $k = 1,2$, be both certain, $\{u_n^2\}_{n=0}^{\infty}$ be non-periodic and let $m_1 < \infty$. Then $M_2 < \infty$ iff $\sum_{n=0}^{\infty} |u_n^1 - 1/m_2| < \infty$. In this case $\sum_{n=0}^{\infty} (u_n^2 - 1/m_2) + 1 - m_1/m_2 = \sum_{n=0}^{\infty} (u_n^1 - 1/m_2)$. Particularly,

$$\sum_{n=0}^{\infty} (u_n^1 - 1/m_2) = M_2/2m_2^2 + 1 - m_1/m_2 . \qquad (5.9)$$

Proof. Using (5.5) we may obtain

$$R_2(z)/(m_2 Q_2(z)) + (P_1(z) - P_2(z))/((1-z)Q_2(z)) = U_1(z) - 1/(m_2(1-z)) .$$

If $M_2 < \infty$, then $R_2(z)/m_2 Q_2(z)$ has the absolutely convergent coefficients of its power series. The same is true for the power series for $(P_1(z) - P_2(z))/m_2 Q_2(z)$. Therefore $\sum_{n=0}^{\infty} |u_n^1 - u_n^2| < \infty$, due to Theorem 5.2. Hence $\sum_{n=0}^{\infty} |u_n^1 - 1/m_2| < \infty$ and the Abel theorem entails (5.9); its right-hand side follows from [6, Th.2].

Conversely, suppose $\sum_{n=0}^{\infty} |u_n^1 - 1/m_2| < \infty$. Theoreme $m_2 < \infty$ and Theorem 5.2 guarantees $\sum_{n=1}^{\infty} |u_n^1 - u_n^2| < \infty$. Therefore $\sum_{n=0}^{\infty} |u_n^2 - 1/m_2| < \infty$ and, consequently, $M_2 < \infty$. Q.E.D.

Theorem 5.4. Let $\{u_n^1, u_n^2\}_{n=0}^{\infty}$ be certain, $\{u_n^2\}_{n=0}^{\infty}$ non-periodic and let $m_1 < \infty$, $M_2 < \infty$. Then

$$u_n^1 = 1/m_2 + o(1/n) , \qquad (5.10)$$

$$u_n^2 = 1/m_2 + o(1/n) . \qquad (5.11)$$

Proof. The last formula has been proved by Feller [6, Th.4] (in this case the assumption $m_1 < \infty$ is superfluous, of course). The term $n(u_n^1 - 1/m_2)$ is obviously the coefficients of z^{n-1} on the left-hand side in

$$U_1'(z) - \frac{1}{m_2(1-z)^2} = \frac{1}{1 - P_2(z)} \left[\frac{R_2(z)}{m_2} - \frac{Q_2'(z)}{Q_2(z)} \right] +$$

$$+ [P_1'(z) - \frac{P_2'(z)(1 - P_1(z))}{(1 - P_2(z))}] \} . \qquad (5.12)$$

We have seen that the power series of the term within the first square brackets converges absolutely for $z = 1$ and its value is 0 by (5.3). The coefficients of the factor $(1 - P_2(z))^{-1}$ converge to $1/m_2$, and therefore the coefficients of z^n tend to zero. The analogous result is also true for the coefficients of the term within the second square brackets, consequently, (5.10) is proved. Q.E.D.

Theorem 5.5. Let $\{u_n^1, u_h^2\}_{n=0}^{\infty}$ be certain and let $\{u_n^2\}_{n=0}^{\infty}$ be non-periodic. Then for any $0 \leq z < 1$

(i) $U_1(z) \geq U_2(z)$ iff $Q_1(z) \leq Q_2(z)$ iff $P_1(z) \geq P_2(z)$ iff $m_1 \leq m_2$.

(ii) If $u_n^1 \geq u_n^2$ and $u_{n+1}^1 \leq u_n^2$, $n \geq 1$, then $M_1 \leq M_2$ and $\sum_{n=1}^{\infty} (u_n^1 - u_n^2) \leq 1$.

If $u_1^1 < 1$, then $m_1 < \infty$ iff $m_2 < \infty$, and

$$(1 - u_1^1) m_2 \leq m_1 \leq m_2 . \qquad (5.13)$$

(iii) If $u_n^1 \geq u_n^2$, $u_{n+1}^1 \leq u_n^2$, $n \geq 1$, and $M_2 < \infty$, then

$$\sum_{n=1}^{\infty} n(u_n^1 - u_n^2) < \infty . \qquad (5.14)$$

Proof. Part (i) follows immediately from (5.9), (5.1) and from the properties of the generating functions and their mean values.

Taking the derivative of (5.8) we have

$$Q_1'(z) - Q_2'(z) = Q_2(U_2'(z) - U_1'(z)) + Q_2'(z)(U_2(z) - U_1(z)) . \qquad (5.15)$$

From this equality we have, for any $0 < z < 1$, $Q_1'(z) \leq Q_2'(z)$, which yields $M_1 \leq M_2$.

Similarly we obtain $U_2(z) - U_1'(z) + 1 \geq 0$ and so

$$0 \leq \sum_{n=1}^{\infty} (u_n^1 - u_n^2) \leq 1 .$$

Now we suppose that $m_1 < \infty$. Then

$$\sum_{n=1}^{\infty} (u_n^2 - u_n^1) z^n = \sum_{n=1}^{\infty} (u_n^2 - u_{n+1}^1 z) z^n - u_1^1 z = A(z) - u_1^1 z ,$$

where $A(z)$ denote the first term in the above middle. The formula (5.8) entails $Q_1(z) - Q_2(z) = Q_2(z)A(z) - Q_2(z)u_1^1 z$, $Q_1(z) = Q_2(z)A(z) + (1 - u_1^1 z)Q_2(z)$. Because of $u_1^1 < 1$, the last implies $Q(z) \geq (1 - u_1^1 z)Q_2(z)$ and so, due to (5.2), $m_2 < \infty$ and (5.13) holds.

To prove (iii) it suffices to take into account that (ii) guarantees $M_1 < \infty$. Therefore using (5.14) we can prove (5.14). Q.E.D.

Finally we note, that we hope to study elsewhere a continuous analogue of the semirecurrent events.

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О семирекуррентных событиях

В некоторых проблемах математической теории счетчиков частиц, фильмовых и бесфильмовых измерениях ионизации треков в физике высоких энергий, а также в теории очередей, случайных блужданий появляются классы семирекуррентных и m-семирекуррентных событий, обобщающих рекуррентные события и рекуррентные события с запаздыванием, соответственно. В настоящей работе показаны их основные свойства и соотношения между ними.

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On Semirecurrent Events

In some problems of the mathematical theory of particle counters, film or filmless measurements of track ionization in high energy physics, queueing theory, random walks, etc., the classes of semirecurrent and m-semirecurrent events, which generalize the recurrent events and the recurrent events with delay, appeared. In the present paper we study their basic properties, and some relationships between them are shown.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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