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W.Timmermann

THE CLASSICAL WEYL-VON NEUMANN THEOREM IN ALGEBRAS OF UNBOUNDED OPERATORS

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The classical Weyl-von Neumann theorem states the following:

<u>Theorem /1/:</u> Let H be a selfadjoint operator. Then for any $\varepsilon > 0$ there is an $A = A^* \in \mathcal{S}_2(\mathcal{H})$ (Hilbert-Schmidt operators) such that $\|A\|_2 < \varepsilon$ and $H+A$ has pure point spectrum, i.e., there exist a real sequence (λ_n), an orthonormal basis $(\varphi_n) \subset \mathcal{D}(H)$ with $(H+A)\varphi_n = \lambda_n \varphi_n$.

In this paper we give two variants (Propositions 1 and 3) of this result in algebras of unbounded operators. The proofs are adaptions of the classical one /1/. We give the details of the proof of Prop.1, because we think that the ideas can be used to extend the results further. It is also possible to give a short proof of Propositions 1 and 3 which uses the statement of the classical theorem. We give such a proof for Prop.3, then Prop.1 is obtained as a corollary.
We start with some necessary notions and notations /2/,/4/. Let $\mathcal{D}$ be a dense linear manifold in a separable Hilbert space $\mathcal{H}$. By $\mathcal{L}^+(\mathcal{D})$ we denote the following set of linear operators: $\mathcal{L}^+(\mathcal{D}) = \{A : A\mathcal{D} \subset \mathcal{D} \,;\, A^*\mathcal{D} \subset \mathcal{D}\}$. With the usual operations and the involution $A \longrightarrow A^+ = A^* \restriction \mathcal{D}$ becomes a $*$-algebra. It can be shown that for $1 \leq p \leq \infty$ the sets
$\mathcal{S}_p(\mathcal{D}) = \{A \in \mathcal{L}^+(\mathcal{D}) : \overline{BAC} \in \mathcal{S}_p(\mathcal{H})$ for all $B, C \in \mathcal{L}^+(\mathcal{D})\}$
are two-sided $*$-ideals in $\mathcal{L}^+(\mathcal{D})$. Here $\overline{D}$ denotes the closure of the operator D and $\mathcal{S}_p(\mathcal{H})$ are the usual ideals in $\mathcal{B}(\mathcal{H})$. We need also the two-sided $*$-ideal $\mathcal{B}(\mathcal{D}) = \{A \in \mathcal{L}^+(\mathcal{D}) :$ BAC bounded for all $B, C \in \mathcal{L}^+(\mathcal{D})\}$.
An important class of domains $\mathcal{D}$ is given by $\mathcal{D} = \mathcal{D}^\infty(H) = \bigcap_{n \geq 0} \mathcal{D}(H^n)$, where $H = H^*$ is an (unbounded) selfadjoint operator, without loss of generality $H \geq 1$. In this case $\mathcal{S}_p(\mathcal{D})$ and $\mathcal{B}(\mathcal{D})$ can be characterized as follows /4/: $\mathcal{S}_p(\mathcal{D}) = \{A \in \mathcal{L}^+(\mathcal{D}) : H^n A \,,\, H^n A^+ \in \mathcal{S}_p(\mathcal{H})$ for all $n \in \mathbb{N}\} = \{A \in \mathcal{L}^+(\mathcal{D}) : AH^n \,,\, A^+H^n \in \mathcal{S}_p(\mathcal{H})$ for all $n \in \mathbb{N}\}$. and for $\mathcal{B}(\mathcal{D})$ analogously.
Now we state the result:

<u>Proposition 1:</u> Let $\mathcal{D} = \mathcal{D}^\infty(H)$, $H = H^*$. Then for any $\delta > 0$ there is an $A = A^+ \in \mathcal{S}_2(\mathcal{D})$ such that $\|A\|_2 < \delta$ and $H+A$ has pure point spectrum.

The proof of this proposition is based on an adaption of the corresponding lemma in the classical case /1/.
<u>Lemma 2:</u> Let $\mathcal{D} = \mathcal{D}^\infty(H)$, $H = H^*$. Then for any $\varphi \in \mathcal{D}$, $\varepsilon > 0$, $m \in \mathbb{N}$ there are a finite dimensional projection $P \in \mathcal{L}^+(\mathcal{D})$ and an $Y = Y^+ \in \mathcal{L}^+(\mathcal{D})$ such that
i) $\|(1-P)\varphi\| < \varepsilon$
ii) $\|H^m Y\|_2 < \varepsilon$
iii) $H+Y$ is reduced by $P\mathcal{H}$.

Remark that in what follows we will not indicate the closure $\overline{D}$ for bounded D if there cannot arise confusions. Statement iii) means that $\mathfrak{D}_0 = P\mathcal{H} \cap \mathfrak{D}$ is dense in $P\mathcal{H}$, $(H+Y)\mathfrak{D}_0 \subset \mathfrak{D}_0$ and $\overline{(H+Y)\restriction \mathfrak{D}_0}$ is a selfadjoint operator in $\mathcal{H}_0 = \overline{\mathfrak{D}_0}$, or what is the same, P commutes with H+Y.

<u>Proof of Lemma 2:</u> Let $a > 0$ be so large that

(1) $\quad \| [1 - E(I_a)]\varphi \| < \varepsilon$

where $I_a = (-a,a]$ and $E(.) \in \mathcal{L}^+(\mathfrak{D})$ are the spectral projections corresponding to H. For each $n \in \mathbb{N}$ put $E_k = E((\frac{2k-n-2}{n} a, \frac{2k-n}{n} a])$, $k=1,\ldots,n$. Remark that $E_j E_k = \delta_{jk} E_k$, $E_k \in \mathcal{L}^+(\mathfrak{D})$ and consequently $\mathcal{H}_k = E_k\mathcal{H} \subset \mathfrak{D}$. Put $\varphi_k = E_k\varphi$, $\psi_k = \varphi_k / \|\varphi_k\|$ if $\varphi_k \neq 0$ and $\psi_k = 0$ otherwise. Thus, (ψ_k) is an orthonormal system satisfying

(2) $\quad \sum_{k=1}^{n} \|\varphi_k\| \psi_k = \sum_{k=1}^{n} \varphi_k = E(I_a)\varphi$

and the ψ_k are "almost-eigenvalues" of H in the sense that for

$\lambda_k = \frac{2k-n-1}{n} a$

(3) $\quad \| (H - \lambda_k)\psi_k \| = \| (H - \frac{2k-n-1}{n} a)\psi_k \| \leq a/n.$

Let P denote the orthoprojection on the space spaned by $\psi_1,\ldots,\psi_n$,then

(4) $\quad P \in \mathcal{L}^+(\mathfrak{D})$, $\dim P \leq n$

(5) $\quad (1-P)\psi_k = 0$ implies $\|(1-P)H\psi_k\| = \|(1-P)(H-\lambda_k)\psi_k\| \leq a/n$ and together with (2): $(1-P)E(I_a)\varphi = 0.$

(6) It is $(1-P)H\psi_k \in \mathcal{H}_k$, hence $\langle (1-P)H\psi_j , (1-P)H\psi_k \rangle = 0$ for $j \neq k$.

From (5) the first statement of the Lemma follows: $\| (1-P)\varphi \| =$
$= \|(1-P)(\varphi - E(I_a)\varphi)\| \leq \|(1-P)\| \cdot \|(1-E(I_a))\varphi\| < \varepsilon$.

Put $-Y = (1-P)HP + ((1-P)HP)^+$, then from $H = PHP + (1-P)H(1-P) + (1-P)HP + ((1-P)HP)^+$ one gets $H+Y = PHP + (1-P)H(1-P)$. Clearly, this operator is reduced by $P\mathcal{H}$, i.e., iii) is valid. Next let us estimate $\|H^m Y\|_2$. This is done by estimating $H^m(1-P)HP$ and $H^m PH(1-P)$ separately. Using $H^m(1-P)H\psi_k \in \mathcal{H}_k$ it follows that $\|H^m(1-P)HP\psi\|^2 = \|\sum_k \langle \psi_k, \psi \rangle H^m(1-P)H\psi_k\|^2 =$
$= \sum_k |\langle \psi_k, \psi \rangle|^2 \|H^m(1-P)H\psi_k\|^2 \leq \|\psi\|^2 a^{2m} a^2/n^2$ for all $\psi \in \mathcal{H}$. Here we used (6), $(1-P)H\psi_k \in E(I_a)\mathcal{H}$ and H^m restricted to $E(I_a)\mathcal{H}$ is bounded by a^m. Consequently $\|H^m(1-P)HP\| \leq a^{m+1}/n$ and therefore $\|H^m(1-P)HP\|_2 \leq a^{m+1}/n^{1/2}$ because the dimension of $H^m(1-P)HP$ is smaller than n. To estimate $H^m PH(1-P)$ one considers the adjoint operator, proceeds similarly and obtains the same estimation. Thus, $\|H^m Y\|_2 \leq 2a^{m+1}/n^{1/2}$. Because n can be chosen independent of a, ii) is proved.

q.e.d.

<u>Proof of Proposition 1:</u> Let $(\varphi_n) \subset \mathfrak{D}$ be a countable set, dense in $\mathcal{H}$. Apply Lemma 2 successively:

1. $\varphi = \varphi_1$, $\varepsilon = \delta/2$, $m = 0$. One obtains $P_1, Y_1 \in \mathcal{L}^+(\mathfrak{D})$ with i)-iii) as in Lemma 2.

2. Apply the Lemma to that part of $H+Y_1$ which lies in $(1-P_1)\mathcal{H} \cap \mathfrak{D} = (1-P_1)\mathfrak{D}$ with $\varphi = (1-P_1)\varphi_2$, $\varepsilon = \delta/4$, $m = 1$. This gives P_2, Y_2 which we continue to whole $\mathfrak{D}$ by 0 and denote the resulting operators by the same symbol. Then $H+Y_1+Y_2$ is reduced by $P_1\mathcal{H}$ and $P_2\mathcal{H}$.

Continuing this procedure one gets two sequences of operators in $\mathcal{L}^+(\mathfrak{D})$: $P_1,\ldots,$ and $Y_1, Y_2, \ldots$ with $\|(1-P_1-\ldots-P_k)\varphi_k\| \leq \delta/2^k$ and $\|H^{k-1}Y_k\|_2 \leq \delta/2^k$ for all k. Put $A_n = \sum_{k=1}^{n} Y_i$ and $A = \sum_i Y_i$. To prove that $A = A^+ \in \mathcal{J}_2(\mathfrak{D})$ we show that $H^m A \in \mathcal{J}_2(\mathcal{H})$ for all $m \in \mathbb{N}$. For $r > m$ the estimation

$\|H^m \sum_{k=r}^{r+s} Y_k\|_2 \leq \sum_{k=r}^{r+s} \|H^k Y_k\|_2 \leq \sum_k \delta/2^k$ implies that $(H^m A_n)$ is a Cauchy sequence in $\mathcal{J}_2(\mathcal{H})$. Now it is standard to see that $H^m A_n \xrightarrow{\|\ \|_2} H^m A \in \mathcal{J}_2(\mathcal{H})$. The estimation above also gives $\|A\|_2 < \delta$. It remains to prove that H+A has pure point spectrum. As in the classical case one sees that $\sum P_k = 1$ and $P_k\mathcal{H}$ reduces H+A. Thus the finite dimensionality of each $P_k\mathcal{H}$ and $\sum P_k = 1$ gives the desired result that H+A has a complete system of eigenvectors.

q.e.d.

The following Proposition gives one of the possible generalizations of Proposition 1.

<u>Proposition 3</u>: Let $\mathfrak{D} = \mathfrak{D}^\infty(T)$, $H \in \mathcal{L}^+(\mathfrak{D})$ so that the following conditions are fulfilled

i) H is essentially selfadjoint

ii) The spectral projections E_λ corresponding to $\overline{H}$ are in $\mathcal{L}^+(\mathfrak{D})$.

iii) $E(I)\mathcal{H} \subset \mathfrak{D}$ for any bounded interval $I \subset \mathbb{R}$.

Then for any $\delta > 0$ there is an $A = A^+ \in \mathcal{J}_2(\mathfrak{D})$ with $\|A\|_2 < \delta$ and H+A has pure point spectrum.

<u>Proof:</u> Put $E_n = E((n,n+1])$, $\mathcal{H}_n = E_n\mathcal{H}$, $H_n = H|\mathcal{H}_n$, $\|T^m\|_{\mathcal{H}_n,\mathcal{H}} = a_{nm}$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}$. Since we may suppose $T \geq 1$, it follows that $a_{nm} \leq a_{nm+1} \leq \ldots$ for all m,n. Choose (δ_n) so that $\delta_n > 0$ and $\sum \delta_n < \delta$ $\sum a_{nn}\delta_n < \infty$. On $\mathcal{H}_n$ construct $A_n = A_n^+$ according to the classical Weyl-von Neumann theorem so that $\|A_n\|_2 < \delta_n$ and $H_n + A_n$ has pure point spectrum as an operator on $\mathcal{H}_n$. Put $A = \sum \oplus A_n$. Then $\|T^m A\|_2 \leq \sum_n \|T^m A_n\|_2 \leq$
$\leq \sum a_{nm}\delta_n = \sum_{n=1}^{m} a_{nm}\delta_n + \sum_{n>m} a_{nm}\delta_n \leq \sum_{n=1}^{m} a_{nm}\delta_n + \sum_{n>m} a_{nn}\delta_n < \infty$.

For $m=0$ it follows that $\|A\|_2 < \delta$. Clearly, H+A has pure point spectrum.

q.e.d.

As in the classical case one can get stronger statements about A. This is formulated as a corollary to Propositions 1 and 3.

<u>Corollary 4</u>: Suppose that the assumptions of Propositions 1 or 3 are fulfilled.Then for any $p > 1$ A can be chosen so that $A \in \mathcal{J}_p(\mathfrak{D})$.

Proof: As in the classical case it is possible to choose $A \in \mathcal{J}_q(\mathcal{H})$, $q > 1$ arbitrary. From the polar decomposition $A = |A|V$ it follows that $|A| \in \mathcal{J}_q(\mathcal{H})$. Moreover, considerations similar to those in the proof of Proposition 3 give $\|T^m|A|\| = \|T^m A\| < \infty$, i.e. $|A| \in \mathcal{B}(\mathcal{D})$ and consequently $|A|^\alpha \in \mathcal{B}(\mathcal{D})$ for all $\alpha > 0$ /4/. Now let $p > 1$ be given. Choose $0 < \alpha < 1$ and q so that $p = q/(1-\alpha)$. Construct A so that $A \in \mathcal{J}_q(\mathcal{H})$, then $|A| \in \mathcal{J}_q(\mathcal{H})$ and $|A|^{1-\alpha} \in \mathcal{J}_p(\mathcal{H})$. In the decomposition $T^m A = (T^m |A|^\alpha)(|A|^{1-\alpha} V)$ the first factor is in $\mathcal{B}(\mathcal{H})$, the second in $\mathcal{J}_p(\mathcal{H})$, hence $T^m A \in \mathcal{J}_p(\mathcal{H})$. This means $A \in \mathcal{J}_p(\mathcal{D})$.

q.e.d.

Remark 5 : The condition iii) of Proposition 3 seems to be very strong. It is of course fulfilled if T and H commute strongly. But there is also a large class of operators H non-commuting with T such that iii) holds. Such operators can be constructed as follows. Let $T = \int_1 \mu \, dF_\mu$, $[1,\infty) = \bigcup_n I_n$, I_n bounded and pairwise disjoint intervals, $\mathcal{H}_n = F(I_n)\mathcal{H}$. Then for appropriately chosen operators $H_n = H_n^*$ in $\mathcal{H}_n$ the operator $H = \sum \oplus H_n$ satisfies the assumptions of Proposition 3. That means, H has a kind of block-decomposition with respect to T. It seems that condition iii) cannot be droped completely.

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Классическая теорема Вейля-фон Неймана
в алгебре неограниченных операторов

Дается вариант классической теоремы Вейля-фон Неймана в алгебрах неограниченных операторов с областью определения вида $\mathfrak{D} = \mathfrak{D}^\infty(H)$, $H = H^*$ - неограниченный самосопряженный оператор.

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The Classical Weyl-von Neumann Theorem
in Algebras of Unbounded Operators

There is given a variant of the classical Weyl-von Neumann theorem in algebras of unbounded operator defined on domains $\mathfrak{D} = \mathfrak{D}^\infty(H)$, where $H = H^*$ is an unbounded selfadjoint operator.

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