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ON ENTROPY FOR A CLASS OF CONTINUOUS PARTICLE SYSTEMS

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1. PRELIMINARIES AND THE THEOREM

Let R^d denote the d-dimensional Euclidean space with the metric $|x-y|$; $x,y \in R^d$. We let $\mathcal{B}^d$ and $\mathcal{B}^d_0$ denote the Borel σ-field on R^d and the class of all bounded Borel sets in R^d. The symbol $\Lambda \uparrow R^d$ will mean that Λ are bounded convex sets in $\mathcal{B}^d$ such that $d(\Lambda) \to \infty$, where $d(\Lambda)$ is the supremum of radii of spheres contained in Λ. Let X denote the set of all points $t \in R^d$ having integral coordinates (with respect to some, not necessarily orthogonal, base in R^d).

Our basic measurable space $(\Omega, \mathcal{F})$ will consist of all Radon (=locally finite) point measures in R^d, with $\mathcal{F}$ being the natural Kolmogorov σ-field. We consider $\omega \in \Omega$ either as a measure or as a subset of R^d. Thus, $\omega \in \Omega$ iff $\omega \cap \Lambda$ is finite for each $\Lambda \in \mathcal{B}^d_0$. Let $G = \{T_u ; u \in X\}$ denote the action of X on Ω:

$$(T_u \omega)(A) = \omega(A+u); \quad u \in X, \quad A \in \mathcal{B}^d. \tag{1}$$

If $\Lambda \subset R^d$, we let $\Lambda^c = R^d \Lambda$, and put

$$\pi_\Lambda(\omega) = \omega \cap \Lambda; \quad \omega \in \Omega. \tag{2}$$

If $\Lambda \in \mathcal{B}^d$, then $\mathcal{F}_\Lambda \subset \mathcal{F}$, where $\mathcal{F}_\Lambda = \pi_\Lambda^{-1} \mathcal{F}$. Further, let $\Omega_\Lambda = \{\omega \in \Omega : \omega \subset \Lambda\}$. If $\Lambda \in \mathcal{B}^d_0$, then $\omega(\Lambda) < \infty$. Hence, we may identify Ω_Λ with the union $\Lambda_{symm} \cup \Lambda^2_{symm} \cup \ldots$ (here, symmetrization of Λ^n means we take instead of Λ^n the set of equivalence classes modulo the permutations of elements in the supports of measures $\omega \in \Lambda^n$).

A natural reference measure should correspond to the "free" field. Let $z>0$ denote the specific density of particles, and let $\rho = z.\lambda$, where λ is the d-dimensional Lebesgue measure. Put

$$\nu_\Lambda = \exp(-\rho(\Lambda))\tilde{\nu}_\Lambda = \exp(-z.|\Lambda|)\tilde{\nu}_\Lambda, \tag{3}$$

where $\tilde{\nu}_\Lambda$ is the Liouville measure in Ω_Λ:

$$\tilde{\nu}_\Lambda = \bigotimes_{n=0}^{\infty} z^n \lambda^n / n! \tag{4}$$

and $|\Lambda|$ = volume of Λ. That is, ν describes the Poisson point process in R^d with intensity measure ρ /3/.

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Let Φ be a bounded, translationally invariant, and regular potential, i.e., (Σ^* means summation over finite sets only):

$$||\Phi|| = \sup_{0\in\omega\in\Omega} \sum_{0\in A\subset\omega}^* |\Phi(A)| < \infty ; \tag{5}$$

$$\Phi = \Phi \circ T_u \quad \text{for each } u \in X \quad (\text{cf. } (1)) ; \tag{6}$$

and for each $\epsilon > 0$ there exists an $R > 0$ such that

$$\Sigma^*\{|\Phi(A)| : 0 \in A \subset \omega, \ \operatorname{diam}(A) > R\} < \epsilon . \tag{7}$$

Let $\Omega_e = \{\omega \in \Omega : \omega$ is finite$\}$ and, for a fixed $a > 0$, let

$$\Omega_a = \{\omega \in \Omega : x, y \in \omega, \ x \neq y \Rightarrow |x - y| \geq a\} . \tag{8}$$

Then put

$$U(\omega) = (+\infty) 1_{\Omega\setminus\Omega_a}(\omega) + \Phi(\omega), \quad \omega \in \Omega_e , \tag{9}$$

where 1_E stands for the indicator function of the set E. It is easy to check that the set $g^{inv}(U)$ of all G-invariant limit Gibbs distributions for the potential U is non-empty [2]. Recall the relevant notions from Ref.2 (see also [4]). Let $\phi \neq \xi \in \Omega_\Lambda$, $\phi \in \Omega_{\Lambda^c}$, and $\omega = \xi + \phi$. The energy of ξ given the boundary condition ϕ is given by

$$E_{\Lambda,\phi}(\xi) = (+\infty) 1_{\Omega\setminus\Omega_a}(\omega) + [\sum_{\substack{A\subset\omega \\ A\cap\xi\neq\phi}}^* \Phi(A)] 1_{\Omega_a}(\omega) . \tag{10}$$

The conditional Gibbs distribution is then given by

$$\mu_{\Lambda,\phi}(A) = \int_A \frac{\exp\{-E_{\Lambda,\phi}\circ\pi_\Lambda\}}{\int \exp\{-E_{\Lambda,\phi}\circ\pi_\Lambda\} d\nu_\Lambda} d\nu ; \quad A \in \mathfrak{F}_\Lambda . \tag{11}$$

By definition, $P \in g^{inv}(U)$ if $P = P \circ T_u (u \in X)$ and

$$\operatorname{Prob}_P[\pi_\Lambda(\omega) \in A \,|\, \mathfrak{F}_{\Lambda^c}] = \mu_{\Lambda,\pi_{\Lambda^c}(\omega)}(A) \ P-\text{a.e.} \tag{12}$$

Let us formally introduce the entropy of a G-invariant probability measure P on $(\Omega, \mathfrak{F})$. Put

$$F_n = \{(x^1, \dots, x^d) \in R^d : -(2n+1)/2 \leq x^i < (2n+1)/2\} .$$

If $\Lambda \in \mathfrak{B}_0^d$ we define

$$S_\nu(P|\Lambda) = \int -\log(dP_\Lambda / d\nu_\Lambda) dP , \tag{13}$$

where $P_\Lambda = P|\Omega_\Lambda$. Then

$$S(P) = \inf_{n\geq 0} |F_n|^{-1} S_\nu(P|F_n) . \tag{14}$$

<u>Theorem.</u> Let $P \in g^{inv}(U)$. For any $0 < \epsilon < 1$ put

$$D_\Lambda(\epsilon; P, \nu) = \inf\{\nu_\Lambda(E) : E \in \mathfrak{F}_\Lambda, \ P_\Lambda(E) > 1 - \epsilon\} , \tag{15}$$

where $\Lambda \in \mathfrak{B}_0^d$. The limit

$$D(\epsilon; P, \nu) = \lim_{\Lambda\uparrow R^d} |\Lambda|^{-1} \log D_\Lambda(\epsilon; P, \nu) \tag{16}$$

exists except an at most countable set of values ϵ, $0 < \epsilon < 1$. If P is indecomposable (i.e., G-ergodic [2]), then $D(\epsilon; P, \nu)$ exists for any $\epsilon \in (0,1)$, does not depend on ϵ, and

$$D(\epsilon; P, \nu) = S(P) . \tag{17}$$

Observe that if $P = \nu$, the free field, then for each $\Lambda \in \mathfrak{B}_0^d$, $D_\Lambda(\epsilon; P, \nu) = 1 - \epsilon$ so that $S(P) = 0$. Since we shall see that $S(P) \leq 0$ (this follows readily also from (17) and (16), for $\nu_\Lambda(E) \leq 1$ for each Λ and each E), the free system has the maximum entropy. In fact, if $\Phi \equiv 0$, then $g^{inv}(\Phi) = \{\nu\}$.

2. THE PROOF

Since the technical details of the proof follow the proof for discrete-time stationary processes [1], we restrict ourselves to merely sketching the main ideas and pointing out the differences. A general framework for McMillan's theorem is described as follows [5,6,4]:

(A1) $(\Omega, \mathfrak{F}, \mu)$ is an abstract probability space admitting for an Abelian group $G = \{T_u ; u \in X\}$ of automorphisms.

(A2) To any $B \in \mathfrak{B}^d$ there is assigned a σ-field $\mathfrak{F}_B \subset \mathfrak{F}$, $\mathfrak{F}_B = \sigma(\cup \mathfrak{F}_{B_n})$ if $B = \cup B_n$; $\mathfrak{F}_{B_1\cup B_2} = \mathfrak{F}_{B_1} \times \mathfrak{F}_{B_2}$ if $B_1 \cap B_2 = \phi$.

(A3) There is a G-invariant probability measure ν on $(\Omega, \mathfrak{F})$ such that $\nu_{\Lambda_1\cup\Lambda_2} = \nu_{\Lambda_1} \times \nu_{\Lambda_2}$ if $\Lambda_1, \Lambda_2 \in \mathfrak{B}_0^d$ and $\Lambda_1 \cap \Lambda_2 = \phi$.

(A4) $T_u \mathfrak{F}_B = \mathfrak{F}_{B+u}$ for $B \in \mathfrak{B}^d$ and $u \in X$.

(A5) $S(\mu) > -\infty$ (cf. (14) for $P = \mu$).

Under (A1) through (A5) the McMillan theorem is true; that is, there exist a G-invariant, μ-integrable function s on such that in $L^1(\mu)$ we have

$$\lim_{\Lambda\uparrow R^d} |\Lambda|^{-1} \log(d\mu_\Lambda / d\nu_\Lambda) = s ; \tag{18}$$

$$S(\mu) = -\int s(\omega)\mu(d\omega). \qquad (19)$$

The reader may check easily that (A1)-(A4) are satisfied in our situation. In order to prove (A5) we shall use (19) for $\mu = P \in g^{inv}(U)$ together with the following fact:

Lemma 1. Let $P \in g^{inv}(U)$. Then the function s from (18) (for $\mu = P$) is essentially bounded with respect to P.

Proof. It follows from (10), (11), and (12) that any $P \in g^{inv}(U)$ is concentrated on Ω_a (cf. (8)). Let $\rho_a(\Lambda) = \sup\{\omega(\Lambda): \omega \in \Omega_a\}$. Then it is straightforward to check directly that

$$|E_{\Lambda\phi} \circ \pi_\Lambda(\omega)| \le \rho_a(\Lambda)||\Phi|| < \infty ; \qquad (20)$$

independently of ϕ and ω. Define the specific energy of $\omega \in \Omega$ by

$$e(\omega) = \lim_{\Lambda \uparrow R^d} |\Lambda|^{-1} E_{\Lambda\phi}(\omega \cap \Lambda) \qquad (21)$$

(the limit exists and does not depend on the actual choice of ϕ's [4]). Thus, $e(\cdot)$ is essentially bounded with respect to any $P \in g^{inv}(U)$. As in Ref.4 we may find a finite constant $p = p(U)$ (the "pressure") such that the following microscopic variational principle is valid: $s(\omega) + e(\omega) = -p$ P-a.e. But then s must be essentially bounded, too. This proves Lemma 1 and, at the same time, (A5) and thereby (18).

Knowing McMillan's theorem (i.e., (18) and (19)) one can proceed exactly as in Ref.1, and prove the following assertions:

Lemma 2. Let $c \in R^1$ and $P\{\omega \in \Omega: s(\omega) \le c\} = 1$. Then

$$\lim_{\Lambda \uparrow R^d} \sup [-|\Lambda|^{-1} \log D_\Lambda(\epsilon; P, \nu)] \le c, \; 0 < \epsilon < 1.$$

Lemma 3. Let $c \in R^1$ and $P\{\omega \in \Omega: s(\omega) \ge c\} = 1$. Then

$$\lim_{\Lambda \uparrow R^d} \inf [-|\Lambda|^{-1} \log D_\Lambda(\epsilon; P, \nu)] \ge c, \quad 0 < \epsilon < 1.$$

The first assertion in our theorem follows easily from the two lemmas and the following fact (not needed within the frame of discrete-time processes). Let $E = [s \le c]$ or $E = [s \ge c]$, $c \in R^1$, and let $P(\cdot|E)$ denote the usual conditional probability when $P(E) > 0$.

Lemma 4. The probability measure $P_1(\cdot) = P(\cdot|E)$ is G-invariant, s in (18) is P_1- essentially bounded, and $S(P_1) > -\infty$.

Proof. The first two assertions are trivial. The latter one can be proved as follows. By definition, $P_1 \ll P$ so that for any $\Lambda \in \mathfrak{B}_0^d$, $P_{1\Lambda} \ll P_\Lambda \ll \nu_\Lambda$ (note that $P_\Lambda \ll \nu_\Lambda$ is a consequence of (A5) for P). Hence $dP_{1\Lambda}/d\nu_\Lambda = (dP_{1\Lambda}/dP_\Lambda)\cdot(dP_\Lambda/d\nu_\Lambda)$. An easy calculation gives $-\int \log(dP_1/dP)dP = \log P(E)$ so that $S(P_1) > -\infty$ as follows from (13).

If P is G-ergodic, then we may take $c = -S(P)$ in Lemmas 2 and 3 above showing the second assertion of the theorem. As in Ref.1 we may obtain also the detailed information concerning the values $\epsilon \in (0,1)$ for which the limit (16) does not exist as well as the formula

$$D(P,\nu) = \lim_{\epsilon \to 0+} D(\epsilon; P, \nu) = \operatorname*{ess\,inf}_{\omega \in \Omega[P]} s(\omega) \qquad (22)$$

valid for non-ergodic P.

As pointed out on p.47 [2], it is possible to show that $g^{inv}(U) \neq \phi$ also under weaker assumptions concerning the character of U at small a. This leads to an interesting problem, namely, to find conditions under which Lemma 1 is true for any $P \in g^{inv}(U)$, i.e., to find when the specific energy is essentially bounded relative to any translationally invariant limit Gibbs distribution.

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Шуян Ш. E5-84-623
Об энтропии для одного класса непрерывных систем частиц

В настоящей заметке обобщаются результаты /полученные автором ранее/ о скорости дивергенции для пар стационарных процессов на случай непрерывных систем частиц в d -мерном евклидовом пространстве. Получено новое определение энтропии для трансляционно-инвариантных предельных распределений Гиббса, которые соответствуют инвариантным и регулярным потенциалам с твердой сердцевиной.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Šujan Š. E5-84-623
On Entropy for a Class of Continuous Particle Systems

In this note previous author's results on the divergence rate of pairs of stationary processes are extended to the case of continuous particle systems in the d-dimensional Euclidean space. A new definition of entropy is obtained for translationally invariant limit Gibbs distributions which correspond to invariant and regular potentials of hard-core type.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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