12/III-84

сообщения
объединенного
института
ядерных
исследований
дубна

E5-83-831

B.Florko, A.Ianovski

ON MAGRI'S THEOREM FOR COMPLETE INTEGRABILITY

1983

INTRODUCTION

Recently there has arisen a considerable interest to some geometrical techniques for investigation of the symmetries and the Hamiltonian structures of the nonlinear evolution equations[3-7]. In a beautiful paper[1] (see also[2]) Magri has made an important suggestion about the geometrical interpretation of the Inverse Scattering Method. But as far as we know the proof of the main result is lacking. The purpose of this paper is to give it.

We must start with a preliminary remark. The proof is made with a full rigour only for the finite-dimensional case, but we give it in a form that allows a possible generalization. An important theorem used in the proof, namely, the Nijenhuis theorem, is based on the Frobenius theorem of integrability. This theorem has an infinite-dimentional generalization[11], and therefore, so has the Nijenhuis theorem. As to the situation described in Magri's theorem, it occurs in the systems solvable through the Inverse Scattering Method (see, for example,[13]), but a careful analysis of the spectral properties should be performed.

Let M be a smooth manifold, let A be a $(1,1)$ tensor field, i.e., A is an operator $A: T(M) \to T(M)$, linear on the fibers of the tangent bundle $T(M)$. A Nijenhuis bracket[8] is a tensor field defined as: $[A, A](X, Y) = A^2[X, Y] + [AX, AY] - A\{[AX, Y] + [X, AY]\}$, X, Y being vector fields on M; and $[X, Y]$, the Lie bracket.

In what follows M will be a symplectic manifold (M, ω), ω being nondegenerate, closed $(d\omega = 0)$ two-form. The operator A and the symplectic structure are coupled if $\omega(X, AY) = \omega(AX, Y)$ for two arbitrary vector fields X and Y. Now we can formulate the Theorem (Magri).

Let (M, ω) be a symplectic manifold, its symplectic structure being coupled with the operator A. Let $[A, A] = 0$ and in addition let A take diagonal form on a set of eigenvectors at every point of M. Let the dimension of the eigenspaces be constant on M. Then:

I) The eigenspaces S_i, corresponding to the eigenvalues λ_i, are orthogonal with respect to the symplectic form ω. The dimension of S_i is even.

II) If X_i are vector fields such that $d\lambda_i = -i_{X_i}\omega = -\omega(X_i, \cdot)$, then $\{d\lambda_i\}_{i \in J}$ is a set of one-forms in involution. If λ_i are nowhere constants, then $d\lambda_i$ are independent, $X_i \in S_i$.

III) 1. If $\dim S_i = 2$, i.e., every eigenvalue is doubly degenerated, and λ_i are nowhere constant, then $\{d\lambda_i\}$ is a complete set of forms in involution. Every X_i is completely integrable Hamiltonian system.

III) 2. Under the conditions of the previous point (III.1)

$$\omega = \sum_{i \in J} \omega_i, \quad \omega_i \equiv \omega|_{S_i}, \quad \omega_i = d\lambda_i \wedge \gamma_i ,$$

γ_i being one-forms; $\wedge$, the exterior product. If Y_i is vector field such that $i_{Y_i}\omega = -\gamma_i$, then S_i is spanned on X_i, Y_i. If in addition X_i and Y_i commute, then the Lie derivative $L_{X_i}A$ vanishes.

IV) Suppose that $\dim S_i = 2$, λ_i are nowhere constant and λ_i has no zeroes. Then the two-forms

$$\omega_A(X, Y) \equiv \omega(AX, Y), \quad \omega_{A^{-1}}(X, Y) \equiv \omega(A^{-1}X, Y)$$

are symplectic forms and the set $\{d\lambda_i\}_{i \in J}$ is still in involution with respect to ω_A and $\omega_{A^{-1}}$.

REMARK

In his original work Magri required one more condition on the tensor field A: if $i_X\omega$ is closed, so must be $i_{AX}\omega$. There is a property of a tensor field with vanishing bracket that, if $i_X\omega$, $i_{AX}\omega$ are closed, so are $i_{A^nX}\omega$, n = 2,3..., see /9/. Although this property is quite important for constructing symmetries from a given one /1,2/, (A being a recursion operator /1,13/) this requirement does not affect the integrability.

PROOF.

The condition $[A,A] = 0$ is equivalent, see /8,9/, to the following two conditions (Nijenhuis theorem):

a) The eigenspaces S_i, and also every sum $\underset{j \in K \subset J}{\oplus} S_j$, are completely integrable in the Frobenius sense, i.e., at every point there exists a submanifold such that S_i coincides with its tangent space. Vector fields that belong to different subspaces S_i commute.

b) $d\lambda_i(X_j) = 0$, i,j, $i \neq j$, $X_j \in S_j$, i.e., λ_i depends only on the coordinates of the integral submanifold.

I) To begin with the proof, let us consider $\omega(X_i, X_j)$, $X_i \in S_i$, $X_j \in S_j$, $i \neq j$, $\omega(AX_i, X_j) = \lambda_i\omega(X_i, X_j) = \omega(X_i, AX_j) = \lambda_j\omega(X_i, X_j)$. As $\lambda_i \neq \lambda_j$, then $\omega(X_i, X_j) = 0$, i.e., S_i and S_j

are orthogonal with respect to ω. Since ω is nondegenerated, it is nondegenerated when restricted to S_i, therefore the dimension of S_i must be even.

II) Suppose $i_{X_i}\omega = -d\lambda_i$. From the point b) it follows that X_i is orthogonal to the subspaces S_j, i.e., $X_i \in S_i$. It is evident now that $d\lambda_i$ are in involution. As $d\lambda_i \neq 0$, then $X_i \neq 0$. X_i belong to different eigenspaces and are linearly independent. Therefore $d\lambda_i$ also are.

III) 1. Now let $\dim S_i = 2$, $\omega = \sum_{i \in J} \omega_i$, $\omega_i = \omega|_{S_i}$. $\{d\lambda_i\}$ is a complete set of one-forms in involution. Moreover, for every field of subspaces $S = S_{i_1} \oplus S_{i_2} \oplus \dots \oplus S_{i_k}$, $i_1, i_2, \dots, i_k \in J$ the two-form $\omega|_S = \omega_{i_1} + \omega_{i_2} + \dots + \omega_{i_k}$ is symplectic on the integral submanifold of the field S, $d\lambda_{i_1}, \dots, d\lambda_{i_k}$ being a complete set in involution. According to the Liouville-Cartan theorem /10/, $\omega|_S$ can be written in the following way: $\omega|_S = d\lambda_{i_1} \wedge \gamma_{i_1} + d\lambda_{i_2} \wedge \gamma_{i_2} + \dots + d\lambda_{i_k} \wedge \gamma_{i_k}$, γ_i are one-forms, $d\gamma_i$ belongs to the ideal generated by $d\lambda_i$ in the algebra of the exterior forms. Every X_i is a completely integrable Hamiltonian system.

III) 2. Suppose that Y_i are vector fields such that $i_{Y_i}\omega = -\gamma_i$. It is evident that $\omega(X_i, Y_i) = 1$, $\omega(X, Y_i) = 0$, $X \in S_j$, $i \neq j$. So $Y_i \in S_i$ and S_i is spanned by X_i, Y_i. Let us calculate $L_{X_i}A$.

$L_{X_i}A(Y) = [X_i, AY] - A[X_i, Y]$. As Y_j, X_j span S_j then it is sufficient to verify that the right-hand side vanishes for X_j, Y_j, $j \in J$. We have $[X_i, Y_j] = 0$ $i, j \in J$ (If γ_i are closed, then $[X_j, Y_j] = 0$ follows from the well-known identity $d(\omega(X_j, Y_j)) = i_{[X_j, Y_j]}\omega$).

$$L_{X_i}A(X_j) = d\lambda_j(X_i)X_j = 0 \quad i \neq j,$$

$$L_{X_i}A(X_i) = d\lambda_i(X_i)X_i = -\omega(X_i, X_i)X_i = 0,$$

$$L_{X_i}A(Y_j) = d\lambda_j(X_i)Y_j = 0.$$

IV) We can now write down the two forms ω_A, $\omega_{A^{-1}}$ in the following way:

$$\omega_A = \sum_{i \in J} \lambda_i \omega_i, \quad \omega_{A^{-1}} = \sum_{i \in J} \lambda_i^{-1} \omega_i .$$

As we have $d\omega_A = d\omega_{A^{-1}} = 0$, then ω_A, $\omega_{A^{-1}}$ are symplectic forms. Finally, the vector field X_i^A corresponding to $d\lambda_i$ with respect to ω_A is $\lambda_i^{-1}X_i$. It follows that $\omega_A(X_i^A, X_j^A) = \lambda_j^{-1}\omega(X_i, X_j) = 0$, i.e., the forms $d\lambda_i$ are still in involution.

We remark that the completeness property of the set $\{d\lambda_i\}_{i \in J}$ takes place even if the dimension of M is infinite. If F is a smooth function on M, dF being in involution with the set $\{d\lambda_i\}_{i \in J}$ then dF is a linear combination of the one-forms $d\lambda_i$. Indeed, let X be a vector field, $i_X \omega = -dF$. As dF is in involution with $d\lambda_i$, $\omega(X, X_i) = 0$, $X = \sum_{j \in J} f_j X_j + g_j Y_j$, then $g_j = 0$,

$X = \sum_{j \in J} f_j X_j$, or equivalently, $dF = \sum_{j \in J} f_j \, d\lambda_j$.

ACKNOWLEDGEMENTS

The authors are grateful to Prof. V.G.Makhankov and V.S.Gergjikov for their interest and constant support.

After the completion of this manuscript the authors came across the paper [14] in which close results are obtained.

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Received by Publishing Department
December 12,1983.

Флорко Б., Яновский А. E5-83-831
К теореме Магри о полной интегрируемости

Представлено доказательство теоремы Магри относительно полной интегрируемости, появившейся в связи с попытками дать геометрическую интерпретацию метода обратной задачи рассеяния. Показано, что даже при выполнении более слабых условий основные выводы теоремы сохраняют свою силу, т.е. что в конечномерном случае собственные значения порождающего оператора есть гамильтонианы вполне интегрируемых систем.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1983

Florko B., Ianovski A. E5-83-831
On Magri's Theorem for Complete Integrability

A rigorous proof of Magri's theorem is given, which may help to understand better the geometrical foundations of the inverse scattering method. It is shown, that even under weaker conditions, the main results of the theorem still hold, i.e., that in the finite-dimensional case the eigenvalues of the generating operator are Hamiltonians of completely integrable systems.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983