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G.A.Emelyanenko, Š.Šujan

A NEW LOOK UPON INFORMATION PROCESSING IN EXPERIMENTAL PHYSICS. GEOMETRY OF RANDOM FACTORS

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1. INTRODUCTION

In a previous paper[II] we studied in detail a geometric representation of Lorentz equation (see (2.1) of Ref.II) in the form of two equations - a differential equation describing the time evolution of the modulus of impulse, and a nonlinear algebraic equation describing a walk on the surface of the unit sphere S_3. In the presence of random factors the geometrical picture becomes much more complicated due to the "memory effects" of these factors. Consequently, in the geometric picture characterizing the direction $\vec{n}(t)$ of the impulse (or velocity) vector a remarkable role is played by information concerning the previous states of the system considered.

In light of this fact it is more appropriate to construct an operator representation of the investigated dynamical system.

2. OPERATORS AND FLOWS FOR LORENTZ EQUATION AND ITS GEOMETRICAL REPRESENTATION

Let the quantities $Q(t)$, $\theta(t)$, and $e(t)$, $t \in [0,T]$ be defined as in (2.18), (2.13), and (2.12) of Ref.II, respectively. The dynamical system describing the particle's motion in the absence of random forces may be described by the following system

$$\text{(a)} \quad \frac{d\vec{p}}{dt} = Q(t)\, p(t)\, \vec{e}(t)\ ;$$

$$\text{(b)} \quad \frac{d\vec{p}}{dt} = \phi(t, X);\ t \in [0, T], \tag{2.1}$$

where (a) is but an equivalent form of the classical Lorentz equation (see [II] Eq. (2.1)). As pointed out in the quoted paper, we may transform (2.1) (under the condition that $p(t) > 0$ for all $t \in [0, T]$) to (2.1b) and

$$\frac{d\vec{n}}{dt} = Q(t)\, \vec{e}(t) + S(t)\, \vec{n}(t)\,. \tag{2.2}$$

Let us fix the same initial conditions $(p, \vec{n})$ for both systems. For (2.1) this should be understood as $\vec{p} = p\vec{n}$. Let $\vec{p}(t)$, $t \in [0, T]$, be a solution to (2.1). By substituting it into (2.1a) we get

$$\frac{dp}{dt}\vec{n}(t) + \frac{d\vec{n}}{dt}p(t) = Q(t)\,p(t)\,\vec{e}(t), \qquad \text{i.e.,} \quad \frac{d\vec{n}}{dt} = Q(t)\,\vec{e}(t) + S(t)\,\vec{n}(t).$$

Conversely, let $(p(t), \vec{n}(t))$, $t \in [0, T]$, be a solution to (2.2). As $\vec{p}(t) = p(t)\,\vec{n}(t)$, we have

$$\frac{d\vec{p}}{dt} = \frac{dp}{dt}\vec{n}(t) + \frac{d\vec{n}}{dt}p(t) = \phi(t, X)\,\vec{n}(t) + [Q(t)\,\vec{e}(t) + S(t)\,\vec{n}(t)]\,p(t) =$$

$$= Q(t)\,p(t)\,\vec{e}(t) + \vec{n}(t)\,[\phi(t, X) + p(t)\,S(t)].$$

But $\phi(t, X) + p(t)\,S(t) = \phi(t, X) - p(t)\frac{1}{p(t)} \cdot \frac{dp}{dt} = 0$

so that $\frac{d\vec{p}}{dt} = Q(t)\,p(t)\,\vec{e}(t)$, and thus any solution to (2.2) is also a solution to (2.1). Let us describe this fact in the language of flows. Let

$$T_t^{(1)}(p, \vec{n}) = (p(t), \vec{n}(t)) \quad \text{if} \quad p(0) = p, \ \vec{n}(0) = \vec{n};$$

$$T_t^{(2)}(\vec{p}) = \vec{p}(t) \quad \text{if} \quad \vec{p}(0) = \vec{p}. \tag{2.3}$$

The states of (2.1) are the vectors $\vec{p}(t)$. Since $\vec{p}(t) = p(t)\,\vec{n}(t)$, where $\vec{n}(t) \in S_3$, we have $\vec{p}(t) \in S_3(p(t))$, here $S_3(r)$ stands for the sphere of radius r. Put

$$\bar{S}_3 = \bigcup_{r \geq 0} S_3(r), \tag{2.4}$$

where the spheres $S_3(r)$ are considered as concentric. A state of (2.2) is a pair $(p(t), \vec{n}(t))$ belonging to the space

$$L = \boldsymbol{R}_+ \times S_3 \quad (\boldsymbol{R}_+ = (0, \infty)). \tag{2.5}$$

Let

$$\Phi: (p, \vec{n}) \in L \rightarrow p\vec{n} (= \vec{p}) \in \bar{S}_3. \tag{2.6}$$

Then the above reasoning results in the assertion that

$$\Phi[T_t^{(1)}(p, \vec{n})] = T_t^{(2)}[\Phi(p, \vec{n})]$$

is valid for any $t \in [0, T]$, $p \in \boldsymbol{R}_+$, and $\vec{n} \in S_3$, i.e.,

$$\Phi \circ T_t^{(1)} = T_t^{(2)} \circ \Phi. \tag{2.7}$$

Since it is clear that Φ is invertible, the flows $T^{(1)} = (T_t^{(1)};\ t \in [0, T])$ and $T^{(2)} = (T_t^{(2)};\ t \in [0, T])$ are isomorphic. Furthermore, we can introduce canonical topologies in spaces L and $\bar{S}_3$.

If k is the minimum of degrees of smoothness of right-hand sides of (2.1) and (2.2), then it is possible to show that Φ is a C^k-diffeomorphism /1/. Hence, the two dynamical systems are C^k-flow equivalent.

The main difference between the descriptions of a particle's motion in terms of equations (2.1), (2.2), and in terms of flows $T^{(1)}$ and $T^{(2)}$, respectively, is the following. The former description is local (instantaneous) while the latter one is global in the sense that it describes the total change up to a fixed time instant t. We pass to the global description because of memory effects in random factors mentioned above. But first let us introduce the local operators corresponding to non-random factors.

3. FIELD AND MEAN LOSS OPERATORS

As pointed out in Section 2, non-random factors admit for a local description. Consequently, we shall describe the influence of the magnetic field and of mean energy losses by instantaneous operators $A_{H,t}$ and $A_{ML,t}$, respectively, where

$$A_{H,t}(p(t), \vec{n}(t)) = (Q(t)p(t), \vec{e}(t)); \tag{3.1}$$

$$A_{ML,t}(p(t), \vec{n}(t)) = (\frac{dp}{dt}, \vec{n}(t)) = (\psi(t, X), \vec{n}(t)). \tag{3.2}$$

A summary of Ref.II (see also the preceding section) says that the action of the pair $A_{H,t}$, $A_{ML,t}$ is equivalent with the action of the operator

$$A_{HML,t}(p(t), \vec{n}(t)) = ([Q(t)^2 + S(t)^2]^{1/2}, \vec{\eta}(t)), \tag{3.3}$$

where $\vec{\eta}(t)$ stands for the unit vector directed as the derivative $d\vec{n}/dt$ (see (2.23) of Ref.II).

A formal integration of (2.1) and (2.2) within the intervals $[0, t]$, $t \leq T$, enables us to express the global flow operators $T_t^{(1)}$ and $T_t^{(2)}$ in terms of operators $A_{H,r}$, $A_{ML,r}$ and $A_{HML,r}$ for $r \in [0, t]$. Hence, the local operators uniquely determine the corresponding flows.

Let us make a remark concerning the magnetic field. An exact statistical model requires also that we take random fluctuations of the magnetic field into account. On the other hand, the majority of measurement devices work with averaged characteristics of the magnetic field so that we continue considering it as deterministic (and smooth enough for the solvability of the Cauchy problem for (2.1) of Ref.II).

4. OPERATOR EXPRESSIONS FOR RANDOM FACTORS

The basic aim of this section is to give a full description of the flow $(\tilde{T}_t \,;\, t \in [0,T])$ corresponding to the motion of a particle under influence of magnetic field, energy losses (including random radiation losses), and scattering processes, respectively. As far as concerns the latter, it seems necessary to investigate small and large angle scattering separately. In what follows we suppose that the small angle scattering constitutes the essential part of scattering processes. In this case we can give, based on the microscopic theory of the processes involved, the following phenomenological description of all forces which change the initial state $(p_0, \vec{n}_0)$ into the final state $(p(t), \vec{n}(t))$, $t \in [0, T]$. We make this transparent by means of the following scheme:

$$
\begin{array}{l}
(p_0, \vec{n}_0) \xrightarrow{H,\Pi} (p(r_1), \vec{n}(r_1)) \\
\qquad\qquad\qquad \big\downarrow \text{SC} \\
\qquad (p(r_1), \underbrace{\vec{n}(r_1) + \vec{\xi}_1}_{\vec{n}^{(2)}(r_1)}) \xrightarrow{H,\Pi} (p(r_2), \vec{n}^{(2)}(r_2)) \\
\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow \text{SC} \\
\qquad\qquad\qquad\qquad (p(r_2), \underbrace{\vec{n}^{(2)}(r_2) + \vec{\xi}_2}_{\vec{n}^{(3)}(r_2)}) \to \dots \xrightarrow{H,\Pi} (\vec{p}(t), \vec{n}(t)).
\end{array}
$$

However, a formal description of the process just depicted is very tedious matter because of necessity to introduce complicated normalizations preserving the interpretation of $\vec{n}^{(2)}(r_1)$, $\vec{n}^{(3)}(r_2)$, etc., as unit vectors. For this reason it is more convenient to pass to the frame of corresponding spatial angles.

So, let us fix an orthogonal coordinate system in S_3, say (Oxyz). Each $\vec{n} \in S_3$ then may be identified with an angle θ. The scattering process in our scheme is described as a continuous-time process which comes into effect at random discrete time instants (Markov moments) $0 < r_1 < r_2 < \dots < r_K \le t$. At the time instant r_i the result of scattering is described by a random rotation which transforms the direction (angle) of $\vec{n}(r_i)$ into the direction (angle) of the vector sum $\vec{n}(r_i) + \vec{\xi}_i$. The mechanism of scattering does not change in time, however, the distribution of the scattering angle depends on time through the total energy at a given time instant.

Unless the time instants r_i we suppose the motion is under influence of merely the processes H (magnetic field) and Π (energy losses). Following /3/, we suppose that the process of energy losses is expressed as a sum

$$\pi(t) = p(t) + \xi(t), \qquad t \in [0, T], \tag{4.1}$$

where $p(t)$ is the instantaneous value of the mean impulse modulus and $(\xi(t);\ t \ge 0)$ is a white noise process. We suppose it is independent of the random angles θ_t. Also, we suppose that $r_1, r_2, \dots, r_K$ are the Markov moments of a Poisson process with intensity function $t \to \lambda(t)$ /2/. In particular

$$\mathrm{Prob}[K(t) = m] = (m!)^{-1} \left(\int_0^t \lambda(r)\,dr \right)^m \exp\left(- \int_0^t \lambda(r)\,dr\right). \tag{4.2}$$

It follows that the Markov moments have exponential distributions. Using (4.1) and (4.2) we may express the operator $A^{(t)}_{\Pi,SC}$, corresponding to the random part of energy losses and to scattering as follows:

$$A^{(t)}_{\Pi,\,SC}(p_0, \theta_0) = \left(p_0 + \int_0^t d\xi(r),\ \theta_0 + \sum_{k=1}^{K(t)} \theta_{r_k}\right). \tag{4.3}$$

In which sense does (4.3) describe interactions between the two processes? To this end recall that the distribution of scattering angles includes the total energy as a parameter (this may be seen from the well-known Rutherford's formula). In the presence of the random process Π we see from (4.1) that the total energy at a given time instant depends also on the random summand $\xi(t)$. Consequently, the distributions of sums in (4.3) also depend on Π.

We may consider (4.1) also in another way. Replace the "mean" equation for p (cf. (2.1b)) by

$$\frac{d\pi}{dt} = \phi(t, X) + \xi(t); \qquad t \in [0, T]. \tag{4.4}$$

Let us formally integrate:

$$\pi(t) = p_0 + \int_0^t \phi(t, X)\,dr + \int_0^t d\xi(r). \tag{4.5}$$

Then we get instead of S(t) the quantities

$$S_\pi(t) = -\pi(t)^{-1}\,[\phi(t, X) + \xi(t)]. \tag{4.6}$$

Though the solution to (4.4) does not possess an ordinary meaning, the properties of white noise entail that $S_\pi(t)$ is a well-defined random variable for any $t \in [0, T]$ such that $\mathrm{Prob}[\pi(t) > 0] = 1$. Now we can formally describe the "stochastic" flow $\tilde{T} = (\tilde{T}_t;\ t \in [0,T])$ which corresponds to the process depicted on the above scheme.

We have

$$T_{r_1}^{-}(p_0, \theta_0) = (p(r_1), \tilde{\theta}(r_1)),$$

$$p(r_1) = p_0 + \int_0^{r_1} [Q(r)^2 + S_\pi(r)^2]^{1/2} dr; \qquad (4.7)$$

$$\tilde{\theta}(r_1) = \theta_0 + \int_0^{r_1} \frac{Q(r)}{[Q(r)^2 + S_\pi(r)^2]^{1/2}} a(r)\, dr + \int_0^{r_1} \frac{S_\pi(r)}{[Q(r)^2 + S_\pi(r)^2]^{1/2}} \beta(r)\, dr,$$

where $a(r)$ is the spatial angle of $\vec{e}(r)$ and $\beta(r)$ is that of $\vec{n}(r)$. At the time instant r_1 the scattering process is working and results in a change $T_{r_1}^{-}(p_0, \theta_0) \to T_{r_1}(p_0, \theta_0)$, where the latter is obtained from the former by adding the random angle θ_{r_1} to the second component of the state vector. Similarly,

$$T_{r_2}^{-}(p_0, \theta_0) = T_{r_2 - r_1}^{-}[T_{r_1}(p_0, \theta_0)], \qquad (4.8)$$

etc. If $r_{K(t)} = t$ then

$$\tilde{T}_t(p_0, \theta_0) = T_{r_{K(t)}}(p_0, \theta_0). \qquad (4.9)$$

If $r_{K(t)} < t$ then

$$\tilde{T}_t(p_0, \theta_0) = T_{t - r_{K(t)}}^{-}[T_{r_{K(t)}}(p_0, \theta_0)]. \qquad (4.10)$$

In this way we got a full description of the flow along "stochastic" trajectories. The formula (4.8) suggests there exists a property very similar to the requirement that the flow $\tilde{T}$ be a Markov semigroup. However, a detailed investigation of the statistical properties of the flow $\tilde{T}$ exceeds the frame of the present paper.

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Емельяненко Г.А., Шуян Ш. E5-83-695

Новый взгляд на обработку информации в экспериментальной физике. Геометрия случайных факторов

В настоящей работе вводится операторная запись общей системы уравнений, исследованной в работе /II/, и понятие потока, соответствующего уравнениям движения. Показана эквивалентность уравнения Лоренца и геометрического представления с точки зрения изоморфизма потоков. Определяются операторы, соответствующие случайным факторам, и дано описание движения под действием совокупности всех /не случайных и случайных/ факторов.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Emelyanenko G.A., Šujan Š. E5-83-695

A New Look upon Information Processing in Experimental Physics. Geometry of Random Factors

In this paper an operator form of a general system of equations, obtained in Ref.II, is introduced. The notion of a flow corresponding to equations of motion is defined. Operators are defined which correspond to random factors. A description of motion under the influence of all (non-random and random) factors is given.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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