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REMOVING CUT-OFFS FROM SINGULAR PERTURBATIONS: AN ABSTRACT RESULT

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In this note we shall consider the addition problem for pairs of self-adjoint operators in Hilbert spaces, when these operators are mutually singular. The aim was to understand at the abstract level the following interesting phenomenon appearing in the study of one-dimensional Schrödinger operators[1,2]. Suppose $V_0(x)$, $x \in R \setminus \{0\}$ is negative and sufficiently singular near origin that the quadratic form of $-\frac{d^2}{dx^2} + V_0(x)$ is unbounded from below, but that $H_D = -(\frac{d^2}{dx^2})_D + V_0(x)$ is self-adjoint and bounded from below on the domain of $-(\frac{d^2}{dx^2})_D$, where D refers to the Dirichlet boundary conditions at origin. Suppose $V_a(x) \to V_0(x)$ pointwise and $H(a) = -\frac{d^2}{dx^2} + V_a(x)$ be self-adjoint and bounded from below on $D(-\frac{d^2}{dx^2})$. It is proved in[2] under some additional assumptions on V_a that $H(a) \to H_D$ in the norm resolvent sense.

In what follows H_0 and V are self-adjoint operators in a Hilbert space $\mathcal{H}$, $H_0 \geq 0$. A sequence $\{V_n\}_1^\infty$ of self-adjoint operators is said to be a regularising sequence for the pair (H_0, V) if the following conditions are met:

a. $H_0 + V_n$ are self-adjoint and bounded from below on $\mathfrak{D}(H_0)$.

b. $\mathfrak{D}(V_n) \supset \mathfrak{D}(V)$, $V_n f \to Vf$ all $f \in \mathfrak{D}(V)$.

The problem is to find conditions under which $H_0 + V_n$ converge in some sense and to identify the limit. The problem is well understood if $V = U + W$, where $U > 0$ and W is form bounded with respect to H_0 (see[3–6] and references therein). As it is clear from the example above we are interested in the case when $H_0 + V_n$ are not uniformly bounded from below, e.g., $V \leq 0$ and sufficiently singular with respect to H_0. Our result is contained in Theorem 1 below.

<u>Theorem 1</u>. Let H_0, V be self-adjoint operators, $H_0 \geq 0$ and $\{V_n\}_1^\infty$ be a regularising sequence for the pair H_0, V. Suppose that

i. $V_n \geq V$

ii. There exists $\mathfrak{D} \subset \mathfrak{D}(H_0) \cap \mathfrak{D}(V)$, $\bar{\mathfrak{D}} = \mathcal{H}$ such that

$$\|Vf\| \leq a\|H_0 f\| + b\|f\| \;; \quad a < 1,\ b < \infty \quad \text{all } f \in \mathfrak{D}, \tag{1}$$

$$\|(V - V_n)f\| \leq a_n\|H_0 f\| + b_n\|f\| \;; \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0, \quad \text{all } f \in \mathfrak{D}, \tag{2}$$

$$(H_0 + V)\restriction_{\mathfrak{D}} \text{ has deficiency indices } (m, m),\ m < \infty; \tag{3}$$

iii. There exists $\{c_n\}_1^\infty$, $\lim c_n = \infty$ such that the spectrum of $H_0 + V_n$ contained in $(-\infty, -c_n)$ consists in at least m eigenvalues (counting multiplicities).

Then $(H_0+V)\restriction_{\mathfrak{D}(H_0)\cap\mathfrak{D}(V)}$ is bounded from below and H_0+V_n converges in the norm resolvent sense to the Friedrichs extension $(H_0+V)_F$ of $(H_0+V)\restriction_{\mathfrak{D}(H_0)\cap\mathfrak{D}(V)}$.

The proof of Theorem 1 is based on the following result proved in /7/.

Theorem 2. Let A be a densely defined, closed symmetric operator with deficiency indices (m,m), $m<\infty$ and satisfying $(f,Af)\geq \|f\|^2$, $f\in\mathfrak{D}(A)$.

i. Let A_F be the Friedrichs extension of A and P be the orthogonal projection on $(A\mathfrak{D}(A))^\perp$. Then for $\lambda\in(-\infty,1)$, $P\lambda A_F(A_F-\lambda)^{-1}P: P\mathcal{H}\to P\mathcal{H}$ is a strictly increasing function of λ and there exists $-\infty<\alpha(A_F;\lambda)<\infty$ such that

$$\sigma(P\lambda A_F(A_F-\lambda)^{-1}P)\subset(-\infty,-\alpha(A_F;\lambda)), \tag{4}$$

$$\lim_{\lambda\to-\infty}\alpha(A_F;\lambda)=\infty . \tag{5}$$

ii. Let A_q be a sequence of self-adjoint extensions of A with the property that there exists $\{a_q\}_1^\infty$, $a_q>1$ $\lim_{q\to\infty} a_q=\infty$ such that the spectrum of A_q contained in $(-\infty,-a_q)$ consists in m eigenvalues (counting multiplicities). Then $0\in\rho(A_q)$ and

$$0\geq A_q^{-1}-A_F^{-1}\geq -P(Pa_q(A+a_q)^{-1}P)^{-1}P. \tag{6}$$

Proof of Theorem 1. Without loss of generality one can take $c_n\geq 1$, $H_0\geq 1$, $b=b_n=0$ and $H_0\restriction\mathfrak{D}$ to be closed. During the proof some technical points are stated as lemmas and proved at the end.

Let $\mathfrak{R}=(H_0+V)\mathfrak{D}$, $\mathfrak{R}_n=(H_0+V_n)\mathfrak{D}$. Due to (1), (2) and the fact that $H_0\geq 1$, $\mathfrak{R}$ and for sufficiently large n, $\mathfrak{R}_n$, are closed subspaces. Let Q and Q_n be the orthogonal projections on $\mathfrak{R}^\perp$ and $\mathfrak{R}_n^\perp$, respectively.

Lemma 1.

$$\lim_{n\to\infty}\|Q_n-Q\|=0. \tag{7}$$

From the von Neumann theory of symmetric extensions it follows that all symmetric extensions of a symmetric operator bounded from below and with finite deficiency indices are bounded from below [8.Ch.8]. Hence $(H_0+V)\restriction_{\mathfrak{D}(H_0)\cap\mathfrak{D}(V)}$ is bounded from below.

Suppose now $\mathfrak{D}\subsetneq\mathfrak{D}(H_0)\cap\mathfrak{D}(V)$. Since by Lemma 1, for sufficiently large n, $(H_0+V_n)\restriction\mathfrak{D}$ has deficiency indices (m,m) it follows that $(H_0+V_n)\restriction_{\mathfrak{D}(H_0)\cap\mathfrak{D}(V)}$ has deficiency indices (n,n),

$n<m$. On the other hand since $V_n\geq V$, $(H_0+V_n)\restriction_{\mathfrak{D}(H_0)\cap\mathfrak{D}(V)}$ are uniformly bounded from below and therefore (H_0+V_n) can have at most n eigenvalues going to $-\infty$ as $n\to\infty$ [8,§107]. This contradicts iii and hence $\mathfrak{D}=\mathfrak{D}(H_0)\cap\mathfrak{D}(V)$.

From (1), $V_n\geq V$ and $H_0\geq 1$, it follows that for all $f\in\mathfrak{D}$

$$(f,(H_0+V_n)f)\geq(f,(H_0+V)f)\geq(1-\sqrt{a})\|f\|^2. \tag{8}$$

Let $(H_0+V)_F$, $(H_0+V_n)_F$ be the Friedrichs extensions of $(H_0+V)\restriction\mathfrak{D}$ and $(H_0+V_n)\restriction\mathfrak{D}$, respectively.

From (8) it follows that $0\in\rho((H_0+V)_F)\cap\rho((H_0+V_n)_F)$.

Lemma 2.

$$\lim_{n\to\infty}\|(H_0+V_n)_F^{-1}-(H_0+V)_F^{-1}\|=0. \tag{9}$$

Now H_0+V_n is a self-adjoint extension of $(H_0+V_n)\restriction\mathfrak{D}$. Since by the general theory of self-adjoint extensions [8 § 107] the spectrum of (H_0+V_n) in the interval $(-\infty, \inf_{f\in\mathfrak{D},\|f\|=1}(f,(H_0+V_n)f))$ consists in at most m eigenvalues it follows that $0\in\rho(H_0+V_n)$.

From $\|(H_0+V)_F^{-1}-(H_0+V_n)^{-1}\|\leq$

$$\leq\|(H_0+V)_F^{-1}-(H_0+V_n)_F^{-1}\|+\|(H_0+V_n)_F^{-1}-(H_0+V_n)^{-1}\|$$

due to Theorem 2 ii and Lemma 2 the only thing we have to prove is that

$$\lim_{n\to\infty}\|(Q_n c_n(H_0+V_n)_F((H_0+V_n)+c_n)^{-1}Q_n)^{-1}\|=0.$$

For, let us remark first that from (8) it follows $(H_0+V)_F\leq(H_0+V_n)_F$ wherefrom for $\lambda<0$

$$0\geq\lambda(H_0+V)_F((H_0+V)_F-\lambda)^{-1}\geq\lambda(H_0+V_n)_F((H_0+V_n)_F-\lambda)^{-1}. \tag{10}$$

Consider, for sufficiently large n, the operator

$$U_n=(1-(Q_n-Q)^2)^{-1/2}(Q_nQ+(1-Q_n)(1-Q)).$$

Then [9, II 4.2] U_n is unitary and

$$U_nQ=Q_nU_n. \tag{11}$$

Moreover from the definition and Lemma 1

$$\lim_{n\to\infty}\|U_n-1\|=0. \tag{12}$$

From definition, for $\lambda<0$

$$\| \lambda (H_0+V)_F((H_0+V)_F-\lambda)^{-1}\| \le |\lambda| . \qquad (13)$$

Let $\epsilon_n = 2\|U_n-1\|+\|U_n-1\|^2$ and

$$b_n = \min\{c_n, \epsilon_n^{-1}\}. \qquad (14)$$

Then using (10), Theorem 2i, (13) and (14) one has

$$Q_n c_n (H_0+V_n)_F((H_0+V_n)_F+c_n)^{-1}Q_n \ge$$

$$\ge Q_n b_n(H_0+V)_F((H_0+V)_F+b_n)^{-1}Q_n \ge \qquad (15)$$

$$\ge Q_n U_n b_n (H_0+V)_F((H_0+V)_F+b_n)^{-1}U_n^* Q_n - Q_n .$$

Due to (11)

$$Q_nU_nb_n(H_0+V)_F((H_0+V)_F+b_n)^{-1}U_n^*Q_n =$$

$$= Q_nU_nQb_n(H_0+V)_F((H_0+V)_F+b_n)^{-1}QU_n^*Q_n,$$

wherefrom

$$\sigma(Q_nU_nb_n(H_0+V)_F((H_0+V)_F+b_n)^{-1}U_n^*Q_n) \subset (a((H_0+V)_F;b_n),\infty). \qquad (16)$$

From (15) and (16)

$$\sigma(Q_nc_n(H_0+V_n)_F((H_0+V_n)_F+c_n)^{-1}Q_n)\subset(a((H_0+V)_F;b_n)-1,\infty)$$

which together with (5) proves the theorem.

Proof of Lemma 1. Let $g\in\mathfrak{R}$, $\|g\|=1$, $g=(H_0+V)f$. From (1) and (2) one obtains

$$\|(H_0+V_n)f-(H_0+V)f\|\le a_n(1-a)^{-1},$$

wherefrom

$$D[g,\mathfrak{R}_n] = \inf_{h_n\in\mathfrak{R}_n}\|g-h_n\|\le a_n(1-a)^{-1}. \qquad (17)$$

In a similar way if $g_n\in\mathfrak{R}_n$, $\|g_n\|=1$ then for sufficiently large n

$$D[g_n,\mathfrak{R}]\le a_n(1-a-a_n)^{-1}. \qquad (18)$$

From (17) and (18) it follows that for sufficiently large n [8 § 39]

$$\|Q-Q_n\|\le a_n(1-a-a_n)^{-1}$$

and the proof of Lemma 1 is complete.

Proof of Lemma 2. Let $f\in\mathfrak{R}$, $\|f\|=1$. Then

$$\|(H_0+V_n)_F^{-1}-(H_0+V)_F^{-1}f\|\le$$

$$\le\|(H_0+V_n)_F^{-1}[f-(H_0+V_n)(H_0+V)^{-1}f]\|\le \qquad (19)$$

$$\le\|(H_0+V_n)_F^{-1}\|\,\|(V_n-V)(H_0+V)^{-1}f\|\le$$

$$\le a_n(1-a)^{-1}\|(H_0+V_n)_F^{-1}\|.$$

From Theorem 7.9 in /3/, $(H_0+V_n)_F\to(H_0+V)_F$ in the sense of strong resolvent convergence. On the other hand, from (19) the convergence is unifrom on $\mathfrak{R}$ which finishes the proof of Lemma 2 since $\dim\mathfrak{R} = m<\infty$.

Remarks

1. From (1) it follows that

$$H_{0,F}+V=(H_0+V)_F ,$$

where $H_{0,F}$ is the Friedrichs extension of $H_0\restriction\mathfrak{D}$, and $H_{0,F}+V$ is the from sum of $H_{0,F}$ and V.

2. Theorem 1 implies results of the sort given in /1,2/. The following is an example.

Corollary 1. Let $-\frac{d^2}{dx^2}$, $(-\frac{d^2}{dx^2})_D$, $x\in \mathbf{R}$ be the Laplacian, and the Laplacian with Dirichlet boundary conditions at 0, respectively. Let $V(x)$, $V_n(x)$, $x\in \mathbf{R}$, $n=1,2,\ldots$ be real functions satisfying: $V(x)\le V_n(x)\le 0$; $|x|^\gamma V(x)\in L^\infty$ for some $\gamma, 0\le\gamma<3/2$; $\int_{-1}^{1} V(x)\,dx = -\infty$; $V_n(x)\in L^\infty$; $\lim_{n\to\infty}V_n(x)=V(x)$ a.e. Then $-\frac{d^2}{dx^2}+V_n$ converge in the norm resolvent sense to $(-\frac{d^2}{dx^2})_D+V$.

Proof. Let $\mathfrak{D}=\mathfrak{D}_1\oplus\mathfrak{D}_2$, where

$$\mathfrak{D}_1=\{f\in\mathfrak{D}(-\frac{d^2}{dx^2})\,|\,f(x)=-f(-x)\},$$

$\mathfrak{D}_2 = \{ f \in C_0^\infty(R\setminus\{0\}) \,|\, f(x) = f(-x) \}.$

Then using the Hardy inequality in two of its variants

$$\int_0^\infty |x^{-\beta}\int_0^x f(y)\,dy|^2 dx \le (\beta - 1/2)^{-2}\int_0^\infty |x^{-\beta+1} f(x)|^2 dx, \quad \beta > 1/2,$$

$$\int_0^\infty |x^{-\beta}\int_x^\infty f(y)\,dy|^2 dx \le (\beta - 1/2)^{-2}\int_0^\infty |x^{-\beta+1} f(x)|^2 dx, \quad \beta < 1/2$$

one can easily verify that the conditions of Theorem 1 with $H_0 = -\frac{d^2}{dx^2}$, V, V_n, $\mathfrak{D}$ are fulfilled.

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Ненчу Г. E5-82-863

Снятие обрезания сингулярных возмущений: абстрактный результат

Пусть $H_0 \geq 0$, V - самосопряженные операторы в гильбертовом пространстве $\mathcal{H}$, причем квадратичная форма $H_0 + V$ не ограничена снизу. Пусть V_n - последовательность самосопряженных операторов, такая, что $V_n \to V$ в некотором смысле, причем операторы $H_0 + V_n$ самосопряжены и ограничены снизу. Формулируются условия, при которых, хотя операторы $H_0 + V_n$ неравномерно ограничены снизу, предел $\lim_{n\to\infty} (H_0 + V_n)$ существует в смысле равномерной резольвентной сходимости и является полуограниченным снизу самосопряженным оператором.

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Removing Cut-Offs from Singular Perturbations: An Abstract Result

Let $H_0 \geq 0$, V be the self-adjoint operators in a Hilbert space $\mathcal{H}$, and suppose the quadratic form of $H_0 + V$ to be unbounded from below. Consider a sequence, V_n, of self-adjoint operators, $V_n \to V$ in some sense, such that $H_0 + V_n$ are self-adjoint and bounded from below on $\mathfrak{D}(H_0)$. Under appropriate conditions, in spite of the fact that the spectra of $H_0 + V_n$ are not uniformly bounded from below, it is proved that $H_0 + V_n$ converge in the norm resolvent sense and the limit is identified.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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