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ON A BUSY PERIOD OF DISCRETIZED GI/GI/∞ QUEUE

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1. INTRODUCTION

The busy period, time when at least one customer is served, of the queueing system with infinitely many servers is known only in special cases. In ref./1/ for the $M/GI/\infty$ queue it is solved by a way of solving of the partial differential equation of the first order. The simpler method based on the recurrent events is used in ref./4/ for this queue. The limit distribution of the busy period in $GI/D/\infty$ and $M/GI/\infty$ queues is studied in refs./1,8/. The busy periods of order n of the $GI/D/\infty$ queue are investigated in /3,6/. The Laplace transform of a cycle, time between two neighbouring beginnings of the busy periods for the $GI/GI/\infty$ queue, can be found in ref./4/.

The busy period of the discretized queueing systems with infinitely many servers plays an important role, for example, in the blob length measurement in track chambers by the automatical scanning device in high energy physics /2,7/.

For the discretized queueing system with the geometric input the Laplace transform of the busy period is determined in ref./5/, and for a special case of a discretization of $M/GI/\infty$ queue the exact probability formulae of the busy period are given in ref./2/. We note, moreover, that in ref./2/ the discretized queueing system with a group arrivals arised.

In the present paper we deal with a general case of the discretized $GI/GI/\infty$ queue. We derive the recurrent probability formulae for the busy period. Also the formulae for the idle period (time when none customer is served) and the cycle of a queue are investigated. Finally, the particular case of the queue with a geometric input is treated and the new simple recurrent formulae, derived in a different way as one used above, are given.

2. NOTATIONS AND PRELIMINARY RESULTS

Let us suppose that the customers arrive at the epochs of a discretized renewal process with a step $h > 0$, and are served immediately upon arrival by one of an infinite number of servers. We denote by T_n the interarrival between the n-th and the $n+1$-th customers. The service times, C_n, of customers are assumed to be i.i.d. with the distribution function $H(t) = P(C_1 \leq 1)$ and they are multipliers of h. For our purposes it suffices to assume that the customers arrive by one, since a group arrivals may be changed by the single one. Indeed, let p_k be the probability that k customers, k=1,2,..., arrive to the queueing system. Then $\tilde{H}(t) = \sum_{k=1}^{\infty} p_k H^k(t)$ corresponds to the distribution of the longest service time in the group. Hence the busy period of a queue with the original group arrivals is the same as one in the modified queue with the same input.

The busy period, B, is defined as the time when at least one customer is served. The idle period, I, is the time when none customer is served. The sum of the busy and idle periods is the cycle U of the queue, i.e., $U=B+I$.

The busy and idle periods are dependent, in general. When we have the geometric input, then they are independent. It is clear that due to our assumption the busy (idle) periods are i.i.d.

We will suppose that the busy period begins at the moment $t=0$ and $h=1$.

Let us put for the searched probability

$$P(n) = P(B=n), \quad n = 1,2,\dots \quad . \tag{2.1}$$

Then these probabilities can be evaluted by the chain of the following recurrent formulae (2.2)-(2.15).

Let us denote

$$W(n,m) = P(B=n,\ I=m), \quad n,m = 1,2,\dots \quad . \tag{2.2}$$

Then

$$P(n) = \sum_{m=1}^{\infty} W(n,m), \quad n = 1,2,\dots \quad . \tag{2.3}$$

For $1 \le k \le n$ we define

$$W(n,k,m) = P(B=n,\ C_1 = k,\ U=m), \tag{2.4}$$

where C_1 denotes the service time of the first customer. Hence

$$W(n,m) = \sum_{k=1}^{n} W(n,k,m). \tag{2.5}$$

It is simple to prove that

$$W(1,1,m) = P(C_1 = 1,\ T_1 = m+1) = P(C_1 = 1)P(T_1 = m+1) = W(1,m), \tag{2.6}$$

and

$$P(1) = P(C_1 = 1)P(T_1 \ge 2). \tag{2.7}$$

Analogically for n=2 we have

$$\left.\begin{aligned} &W(2,1,m) = P(C_1 = 1)P(T_1 = 1)W(1,m), \\ &W(2,2,m) = P(C_1 = 2)(P(T_1 = m+2) + P(T_1 = 1)W(1,m)). \end{aligned}\right\} \tag{2.8}$$

For $n \ge 3$ we continue stepwisely

$$W(n,1,m) = P(C_1 = 1)P(T_1 = 1)W(n-1,m). \tag{2.9}$$

Let $2 \le k \le n-1$, then

$$W(n,k,m) = P(C_1 = k)(P(T_1 = k)W(n-k,m) + \sum_{i=1}^{k-1} P(T = k-i) \times \\ \times A(n,k,m,i)), \tag{2.10}$$

where

$$A(n,k,m,i) = \sum_{r=1}^{[\frac{k-i+r}{2}]} B(n,k,m,i,r). \tag{2.11}$$

and $B(n,k,m,i,r)$ is the probability that $B=n$, $C_1 = k$, $T_1 = i$, and from the service times of the second and the following by him customers, r cycles are created of the total length $n-i$ without account of the service time of the first customer, Therefore we have

$$B(n,k,m,i,r) = \sum W(j_1, k_1) \dots W(j_{r-1}, k_{r-1}) W(j_r, m), \tag{2.12}$$

where the summation runs over the set of the integers j_i, k_i, for which $1 \le j_1 \le k_1, \dots, 1 \le j_{r-1} \le k_{r-1}$, $n-k \le j_r$, $j_1 + k_1 + \dots + j_{r-1} + k_{r-1} + j_r = n-i$

If, finally, $k=n$, then

$$W(n,n,m) = P(C_1 = n)(P(T_1 = n+m) + \sum_{i=1}^{n-1} P(T_1 = n-i) \times \\ \times A(n,n,m,i)), \tag{2.13}$$

where

$$A(n,n,m,i) = \sum_{r=1}^{[\frac{i+1}{2}]} B(n,n,m,i,r), \tag{2.14}$$

and

$$B(n,n,m,i,r) = \sum W(j_1, k_n) \dots W(j_r, k_r).$$

Here the summation runs over the set of the integers j_i, k_i with $1 \le j_1 \le k_1, \dots, 1 \le j_{r-1} \le k_{r-1}$, $m \le k_r$, $j_1 + k_1 + \dots + j_r + k_r = n+m+i$.

3. THE BUSY PERIOD

For the practical employment of the formulae (2-1)-(2.12) we change the summation ad infinitum in (2.1) by the finite one according to the following simple way.

It is clear that

$$P(n) = \sum_{k=1}^{n} W(n,k,.)\,, \tag{3.1}$$

where

$$W(n,k,.) = \sum_{m=1}^{\infty} W(n,k,m)\,.$$

Then

$$W(1,1,\cdot) = P(C_1 = 1)P(T_1 \geq 2) = P(1)\,, \tag{3.2}$$

$$\left.\begin{aligned} &W(2,1,\cdot) = P(C_1 = 1)P(T_1 = 1)P(1)\,,\\ &W(2,2,\cdot) = P(C_1 = 2)(P(T_1 \geq 3) + (P(T_1 = 1)P(1)). \end{aligned}\right\} \tag{3.3}$$

If $n \geq 3$, then

$$W(n,1,\cdot) = P(C_1 = 1)P(T_1 = 1)P(n-1)\,, \tag{3.4}$$

and for $2 \leq k \leq n-1$ we have

$$W(n,k,\cdot) = P(C_1 = k)(P(T_1 = k)P(n-k) + \sum_{i=1}^{k-1} P(T_1 = k-i) \times \times A(n,k,\cdot,i))\,, \tag{3.5}$$

where

$$A(n,k,\cdot,i) = \sum_{r=1}^{\left[\frac{k-i+1}{2}\right]} B(n,k,\cdot,i,r)\,, \tag{3.6}$$

and

$$B(n,k,\cdot,i,r) = \sum W(j_1,k_1) \ldots W(j_{r-1},k_{r-1})P(j_r)\,. \tag{3.7}$$

Here the summation is the same as in (2.12), and the $W(j_i, k_i)$ are determined by (2.5).

For $k=n$ we put $W(n,n,\cdot) = \sum_{m=1}^{\infty} W(n,n,m)$, where $W(n,n,m)$ are evaluted by (2.13).

If, for example, the service times are bounded, then for sufficient large n_0 $W(n,n,m)=0$, $n \geq n_0$, $m=1,2,\ldots$.

Note. Knowing $W(n,m)$ we can determine the probability law of the idle period $P_I(m) = P(I=m)$

$$P_I(m) = \sum_{n=1}^{\infty} W(n,m)\,, \quad m = 1,2\ldots \,. \tag{3.8}$$

Analogically for the distribution of the cycle of a queueing system, $P_U(k) = P(U=k)$, we have

$$P_U(k) = \sum_{\substack{n+m=k\\ 1\leq m,n}} W(n,m)\,, \quad k = 2,3\ldots \,. \tag{3.9}$$

4. THE GEOMETRIC INPUT

When the interarrival times of customers are distributed according to the geometric law

$$P(T_1 = m) = (1-p)p^{m-1}\,, \quad m = 1,2,\ldots\,, \tag{4.1}$$

where $0 < p < 1$, then the busy and idle periods are independent, and the probability of the idle period has the same geometric distribution as T_1.

Due to this independence, we can, using the methods for a distribution law determination of the busy period in a special case of the queueing system from [2], simplify the formulae (2-1)-(2.13) and (3.1)-(3.7), respectively.

Let us denote by A an event that the busy period begins from $t=0$. Because of the geometric input, which can be enlarged to the whole time axis, we have that $P(A) = 1-p$. Let us put $P_0(k) = P(C_1 = k, A)$, $k = 1,2,\ldots$. Then

$$P_0(k) = P(C_1 = k)(1-p)\,, \quad k = 1,2,\ldots\,, \tag{4.2}$$

and for $k=0$ we define

$$P_0(0) = p\,. \tag{4.3}$$

We denote the conditional probability in question, $P(B=n/A)$, by $P(n)$ and the joint probability $P(B=n,A)$ by $PP(n)$. Clearly $P(n) = PP(n)/(1-p)$.

Let $\bar{W}(n,k) = P(B=n, C_1 = k, A)$, then

$$PP(n) = \sum_{k=1}^{n} \bar{W}(n,k)\,, \quad n = 1,2,\ldots\,, \tag{4.4}$$

and $W(n,k,\cdot) = \bar{W}(n,k)/(1-p)$, where $W(n,k,\cdot)$ is an expression from Section 3.

Applying the analogical reasonings from [2] we obtain the following recurrent formulae (4.5)-(4.11) for the busy period distribution law.

$$\left.\begin{aligned} \bar{W}(1,1) &= P_0(1)P(0), \\ PP(1) &= \bar{W}(1,1), \end{aligned}\right\} \qquad (4.5)$$

$$\left.\begin{aligned} \bar{W}(2,1) &= P_0(1)PP(1), \\ \bar{W}(2,2) &= P_0(2)(P_0(1)+P_0(2))P_0(0), \\ PP(2) &= \bar{W}(2,1)+\bar{W}(2,2). \end{aligned}\right\} \qquad (4.6)$$

Let us put

$$S(k) = \sum_{i=0}^{k} P_0(i), \quad k = 0,1,2,\dots,$$

$$SS(k) = \prod_{i=0}^{k} S(i), \quad k = 1,2,\dots .$$

For the general case $n \geq 3$ we define

$$\bar{W}(n,1) = P_0(1)PP(n-1). \qquad (4.7)$$

The detailed investigation of properties $\bar{W}(n,k)$ shows us that between them there are the following recurrent relation ships.

For $2 \leq k \leq n-1$ we introduce the next helpful notations

$$\bar{B}(n,k-1,1) = S(1)\,PP(n-k) + \sum_{i=2}^{n-k+1} \bar{W}(n-k+1,i), \qquad (4.8)$$

and for $2 \leq j \leq k-1$

$$\bar{B}(n,k-1,j) = S(j)\,\bar{B}(n,k-1,j-1) + \sum_{i=j+1}^{n-k+j} \bar{W}(n-k+j,i). \qquad (4.9)$$

Finally, we have

$$\left.\begin{aligned} \bar{W}(n,k) &= P_0(k)\bar{B}(n,k-1,k-1), \\ \bar{W}(n,n) &= P_0(n)SS(n-1), \end{aligned}\right\} \qquad (4.10)$$

and

$$P(n) = PP(n)/(1-p). \qquad (4.11)$$

For the distribution P_U of the cycle U of the queue with the geometric input we have

$$P_U(m) = \sum_{i=1}^{m-1} P(i)(1-p)p^{m-i-1}, \quad m = 2,3,\dots . \qquad (4.12)$$

The detailed comparison of the formula (4.11) with (2.3) shows us that they are the same. We note that the Laplace transforms and the expectation values of the busy period and the cycle of the queue with the geometric input are determined in [5], where the method of recurrent events was used to determine the number of customers served during the busy period.

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О периоде занятости дискретной системы массового обслуживания GI/GI/∞

В работе изучается задача определения распределения периода занятости /т.е. периода, когда обслуживается хотя бы один заказчик/ дискретной системы массового обслуживания с бесконечным числом каналов обслуживания. Кроме того, изучаются период простоя и цикл системы. Получены рекуррентные формулы, и в частном случае геометрического входа определены более простые рекуррентные формулы. Эти проблемы возникают при определении дискретной длины сгустков в трековых камерах в физике высоких энергий.

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On a Busy Period of Discretized GI/GI/∞ Queue

The problem of determining the distribution of the busy period, i.e., of the time when at least one customer is served, of the discretized queueing system with infinitely many servers is investigated. Moreover, the idle period and the cycle of a queue are studied. The recurrent formulae are determined and in particular case of a queue with the geometric input the simpler recurrent formulae are given. Those problems arise in the discrete blob length determination in track chambers in high energy physics.

The investigation has been performed at the Laboratory of Computing Technique and Automation, JINR.

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