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ASYMPTOTIC BEHAVIOUR OF THE SCATTERING PHASE FOR NON-TRAPPING METRICS

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1. INTRODUCTION

The aim of this article is to study the asymptotic behaviour of the scattering phase $s(\lambda)$ related to an elliptic second order formally self-adjoint operator H, defined either in R^n or in an unbounded domain Ω with Dirichlet or Neumann boundary conditions. Recently, this problem was investigated by many authors. In ref. [4] Buslaev announced a result about the asymptotic of $s(\lambda)$ as $\lambda \to \infty$ for differential operators in R^n, as well as in the obstacle case with Dirichlet boundary conditions. The perturbed operator H, considered in ref. [4], has a principal symbol with constant coefficients and 0 is not an eigenvalue of H. Later, the same problem was studied by A.Majda and J.Ralston [14]. They proved the existence of an asymptotic expansion and computed the first three coefficients when $s(\lambda)$ is the scattering phase of the Laplacian with Dirichlet boundary conditions on a convex obstacle, and when $s(\lambda)$ is the scattering phase of the Laplace-Beltrami operator for a non-trapping metric on R^n, which is Euclidean in a neighbourhood of ∞. The authors conjectured that the same asymptotic expansion holds for any non-trapping obstacle. This conjecture was proved in ref. [18] by V.Petcov and the author for the Laplacian with Dirichlet or Neumann boundary conditions.

For the Schrödinger operator $H = -\Delta + V$, $V \in C_0^\infty(R^n)$ and $n = 3$, the asymptotic behaviour of $s(\lambda)$ as $\lambda \to \infty$ was investigated by Colin de Verdiere [5], and for any n-odd by Guillopé [7]. Recently, an asymptotic expansion of $s(\lambda)$ related to a first or second order elliptic operator on a Hermitian bundle over an odd dimensional Riemannian manifold was announced by V.Ivrij and M.Shubin [9].

In this paper, both cases, n-even and n-odd, are considered, as well as the case when $\lambda = 0$ is an eigenvalue of the perturbed operator H. The asymptotic behaviour of the scattering phase $s(\lambda)$ as $\lambda \to \infty$ is investigated for arbitrary second order elliptic, formally self-adjoint differential operators H in a domain $\Omega \subset R^n$, satisfying a non-trapping condition and such that $H = -\Delta$ in a neighbourhood of ∞. The self-adjoint extension of H in $L^2(\Omega)$ with Dirichlet or Neumann boundary conditions on $\partial\Omega$ when $\Omega \neq R^n$ is bounded from below but it allows to have a finite number of non-positive eigenvalues in contrast to refs. [14,18]. Therefore there is not always a good rate of local decay for $H_{ac}^{-1/2} \sin(tH_{ac}^{1/2})$, $t \to \infty$ essentially used in ref. [18]

where H_{ac} is the absolutely continuous part of the operator H. In order to overcome this difficulty we study the asymptotic behaviour of the S -matrix at infinity.

Suppose K is a bounded domain in R^n with smooth boundary ∂K and $\Omega = R^n \setminus K$ or $\Omega = R^n$. Consider an elliptic, formally seld-adjoint second order differential operator P in Ω with Dirichlet or Neumann boundary conditions on ∂K, when $\Omega \neq R^n$ and $P = -\Delta$ outside the ball $B_R = \{x; |x| \le R\}$. Without loss of generality assume that P has the form $P = -\Delta_g + hD + V$, where $V \in C_{(0)}^{\infty}(R^n)$, $hD = \sum_{j=1}^{n} h_j(x) D_j$, $D_j = -i\partial/\partial x_j$ and Δ_g is the Laplace-Beltrami operator for a Riemannian metric g,

$$\Delta_g = \sum_{i,j=1}^{n} g^{-1/2}\, \partial/\partial x_i \,(g^{ij} g^{1/2})\, \partial/\partial x_j$$

$g^{ij} \in C^{\infty}(\bar{\Omega})$, $g = \det(g_{ij})$, $g_{ij} = (g^{ij})^{-1}$ and $g_{ij} = \delta_{ij}$ for $|x| > R$. The projections of the (generalized) bicharacteristics of P on $\bar{\Omega}$ are called (generalized) geodesics of g /16/.

Definition. The metric g is said to be non-trapping if there is $T > 0$ such that every (generalized) geodesics, beginning in B_R, leaves the ball B_R by the time T_R.

Let H_0 and H be the self-adjoint extension of the free Laplacian $-\Delta$ in $L^2(R^n)$ and of P in $L^2(\Omega)$ with Dirichlet or Neumann boundary conditions on ∂K when $\Omega \neq R^n$. These operators generate groups of unitary operators $\exp(itH_0)$ and $\exp(itH) \oplus 1$ in $L^2(R^n) = L^2(\Omega) \oplus L^2(K)$. The wave operators $W_{\pm}$ are defined as follows

$$W_{\pm} = s\text{-}\lim_{t \to \mp\infty} (e^{itH} \oplus 1)\, e^{-itH_0}.$$

It is well known /3/ that $W_{\pm}$ are isometrics on $L^2(R^n)$ and Rang (W_+) = Rang (W_-), so the scattering operator $S = W_+^* W_-$ exists as a unitary operator on $L^2(R^n)$. In the spectral representation of H_0 on $L^2(R^+, L^2(S^{n-1}))$ the scattering operator S can be considered as a function of unitary operators $S(\lambda)$ on $L^2(S^{n-1})$ which is called a scattering matrix. Moreover, $S(\lambda) = I + K(\lambda)$, where $K(\lambda)$ is a trace class operator for $\lambda > 0$. This enables us to define the function $\det S(\lambda) : R^+ \to S^1 = \{z \in \mathbf{C};\ |z| = 1\}$ as a product of the eigenvalues of $S(\lambda)$. It was proved in refs. /7,10,11/ that there exists a continuous (even analytic) in R^+ function $s(\lambda)$, satisfying the equality

$$\det S(\lambda) = \exp(2\pi i\, s(\lambda)), \quad \lambda > 0.$$

Such a function $s(\lambda)$ is called a scattering phase.

We shall prove the following results.

Theorem 1. Suppose the metric g is non-trapping in R^n. Then

$$s(\lambda) \sim \sum_{j=0}^{\infty} a_j \lambda^{n/2 - j} \quad \text{as } \lambda \to \infty. \qquad (1.1)$$

Moreover,

$$a_0 = (4\pi)^{-n/2} (\Gamma(n/2+1))^{-1} \{\mathrm{vol}_g(B_R) - \mathrm{vol}_e(B_R)\},$$

$$a_1 = (4\pi)^{-n/2} (\Gamma(n/2))^{-1} \int \left(\frac{K\sqrt{g}}{3} - \frac{|h|_g}{4} + V(x)\right) dx,$$

where $\mathrm{vol}_g(B_R)$ and $\mathrm{vol}_e(B_R)$ are the Riemannian and Euclidean volume of the ball B_R, $K(x)$ is the scalar curvature and $|h|_g = \sum_{i,j=1}^{n} g_{ij} h^i h^j$ is the Riemannian length of the vector h.

In the case of the Schrödinger operator we prove

Theorem 2. Let $H = -\Delta + V$, $V \in C^{\infty}(R^n)$, $n \geq 3$. Then $s(\lambda)$ has the form (1.1) when $\lambda \to \infty$, where

$$a_j = \int P_j^n(V, DV, \dots, D^{\alpha} V)\, dx, \qquad D^{\alpha} = D_1^{\alpha_1} \dots D_n^{\alpha_n}$$

and P_j^n are some universal polynomials. Moreover $P_0^n = 0$, $P_1^n(V) = (4\pi)^{-n/2} (\Gamma(n/2))^{-1} V$ and

$$P_j^n(\lambda V, \lambda^{3/2} DV, \dots, \lambda^{1+|\alpha|/2} D^{\alpha} V) = \lambda^j P_j^n(V, DV, \dots, D^{\alpha} V), \quad \lambda > 0.$$

In the obstacle case we prove

Theorem 3. Let $H = -\Delta_g + hD + V$ in $L^2(\Omega)$ with Dirichlet or Neumann boundary conditions on $\partial\Omega$ and suppose the metric g is non-trapping in $\Omega \subset R^n$, $n \geq 3$. Then

$$s(\lambda) \sim \sum_{j=0}^{\infty} a_j \lambda^{(n-j)/2} \quad \text{as } \lambda \to \infty$$

and

$$a_0 = (4\pi)^{-n/2} (\Gamma(n/2+1))^{-1} \{\mathrm{vol}_g(\Omega \cap B_R) - \mathrm{vol}_e(B_R)\},$$

$$a_1 = \pm \frac{1}{4} (4\pi)^{-(n-1)/2} \left(\Gamma\left(\frac{n+1}{2}\right)\right)^{-1} \mathrm{vol}_g(\partial\Omega),$$

where $+(-)$ sign is used in the case of Dirichlet (Neumann) boundary conditions and $vol_g(\partial\Omega)$ is the Riemannian volume of $\partial\Omega$.

The plan of the paper is as follows. In section 1 we prove that the point spectrum of the operator H is finite and investigate some properties of the scattering phase. In section 2 we study the behaviour of the scattering matrix at ∞ in order to find functions $s_1(\lambda)$ add $s_2(\lambda)$ such that $s(\lambda^2) = s_1(\lambda) + s_2(\lambda)$ and $s_1(t) \in \epsilon'(R^1)$, $\hat{s}_2(\lambda) = O(\lambda^{-N})$, $N \in \mathbf{Z}$, $\lambda \to \infty$ for any $N \in \mathbf{Z}$. In section 3 we investigate the distribution $\hat{s}_1(t)$ using suitable trace formulas and prove a similar to theorem 1 result in the case of matrices of first order differential operators.

2. THE SCATTERING PHASE AND THE SPECTRUM OF H

We begin to study the spectrum of H in $L^2(R^n)$. First we prove that the point spectrum of H is finite. Since $H = -\Delta$ outside the ball B_R the Rellich's theorem and the unique continuation property of second order elliptic operators yield the absence of the positive point spectrum of H. Moreover, $H \geq -\epsilon\Delta + V_1$ for some $\epsilon > 0$, $V_1 \in C_0^\infty(R^n)$ when $\Omega = R^n$ and since the negative point spectrum of $-\epsilon\Delta + V_1$ is finite, so is those of H. In the case $\Omega \neq R^n$ we use the inequality $H \geq H_1 \oplus H_2$ in $L^2(\Omega \cap B_R)$ $L^2(R^n \cap B_R)$, where $H_1 = H$ in $\Omega \cap B_R$, $H_2 = -\Delta$ in R^n B_R with Dirichlet boundary conditions on $\partial(\Omega \cap B_R)$ and ∂B_R respectively. Notice that both operators H_1 and H_2 have finite negative point spectrum. Moreover, the eigenvalue o has a finite multiplicity. Indeed, assume there exist infinitely many $\phi_j \in L^2(R^n)$, $(\phi_j, \phi_k) = \delta_{jk}$ such that $H\phi_j = 0$. Then $\Delta\phi_j = (\Delta - H)\phi_j$ and $\phi_j(x) = \int |x-y|^{-n+2}(\Delta-H)\phi_j(y)dy$. Let $n > 4$, $\chi \in C^\infty(R^n)$, $\chi \equiv 0$ on B_R and $\chi(x) = 1$ for $|x| > R+1$. Then $|\chi\phi_j(x)| \leq c|x|^{-n+2}$ and $\|(1+|x|^2)^\epsilon \phi_j\|_{H^2(R^n)} \leq C$ for some $\epsilon > 0$, $C > 0$. Now it is not hard to choose a Cauchy subsequence of ϕ_j in $L^2(R^n)$, which contradicts our assumption. When $n = 3$ or $n = 4$,

we have $\int(\Delta - H)\phi_j(y)\,dy = 0$ since $|\xi|^{-2}(\Delta-H)\phi_j(\xi) = \hat{\phi}_j(\xi) \in L^2(R^n)$. Then $\phi_j(x) = \int(\Delta-H)\phi_j(y)[|x-y|^{-n+2} - |x|^{-n+2}]\,dy$ and the arguments given in the case $n > 4$ can be repeated. Therefore the point spectrum $\sigma_p(H)$ of H is finite and non-positive. Moreover, the continuous spectrum of H is absolutely continuous and coincides with R^+.

In the rest of this section we study the scattering phase $s(\lambda)$ related to the pair H, H_0. First consider $K_0 = (a+H_0)^{-1}$, $K = (a+H)^{-1}$ which are bounded, self-adjoint operators for $a > \inf\{\sigma_p(H), 0\} = \lambda_1$. Moreover, the operator $K^p - K_0^p$ is a trace class one for $p > n$ (see ref. [2]). Then the scattering phase $s(\lambda; K^p, K_0^p)$ related to the pair K^p, K_0^p is defined as follows

$$s(\lambda; K^p, K_0^p) = \pi^{-1} \lim_{\epsilon \to +0} \arg\det[1 + (K^p - K_0^p)(K_0^p - \lambda - i\epsilon)^{-1}], \quad \lambda \in R^1 \qquad (2.1)$$

and has the properties (see refs. [3,7,10])

(i) $s(\lambda; K^p, K_0^p) \in L^1(R^n)$ and $\operatorname{supp} s \subset [0, (\lambda_1 + a)^{-1}]$,

(ii) For any $\Phi \in C^\infty(R^n)$ the operator $\Phi(K^p) - \Phi(K_0^p)$ is a trace class one and

$$\operatorname{Tr}\{\Phi(K^p) - \Phi(K_0^p)\} = \int \Phi'(\lambda) s(\lambda; K^p, K_0^p)\, d\lambda,$$

(iii) $\det S(\lambda; K^p, K_0^p) = \exp(-2\pi i s(\lambda; K^p, K_0^p))$, $\lambda > 0$,

where $S(\lambda; K^p, K_0^p)$ is the scattering matrix for the pair H, H_0. The function $s(\lambda) = s((a+\lambda)^{-1}; K^p, K_0^p)$ will ba called a scattering phase for the pair H, H_0. This notion is motivated by the property (iii) $\det S(\lambda) = \exp(-2\pi i s(\lambda))$ derived from (iii)' by the invariance principle. Using (i)' and (ii)' it is not hard to see, that (i) $(1+\lambda^2)^{-p} s(\lambda) \in L^2(R^1)$ for $p > n$ and $\operatorname{supp} s \subset [\lambda_1, \infty)$. (ii) For any $\Phi \in \mathcal{S}(R^1)$ the operator $\Phi(H) - \Phi(H_0)$ is a trace class one and

$$\operatorname{Tr}\{\Phi(H) - \Phi(H_0)\} = \int \Phi'(\lambda) s(\lambda)\, d\lambda.$$

Moreover, the function $s(\lambda)$ is analytic in R^+ since the operator H has no positive point spectrum (see refs. [10,18]).

Two special choices of the function Φ in (ii) are very useful for studying the asymptotics of $s(\lambda)$ at infinitely. Let $\Phi(\lambda) = e^{-t\lambda}\phi(\lambda)$, $\phi \in C^\infty(R^1)$, $\phi(\lambda) = 1$ for $\lambda \in [-a, \infty)$ and $\phi(\lambda) = 0$ for $\lambda \in (-\infty, -a-1)$. Then $\Phi \in \mathcal{S}(R^1)$ and

$$\operatorname{Tr}\{e^{-tH} \oplus 0 - e^{-tH_0}\} = -t\int_{-\infty}^{\infty} e^{-t\lambda} s(\lambda)\, d\lambda, \quad t > 0, \qquad (2.2)$$

where $e^{-tH} \oplus 0$ acts as e^{-tH} in $L^2(\Omega)$ and as 0 in $L^2(R^n \setminus \Omega)$. Denote $\Phi(\lambda) = \phi(\lambda)\int \cos(\sqrt{\lambda}\, t)\rho(t)\,dt$, $\rho \in C_0^\infty(R^1)$ and $B_0 = \sqrt{H_0}$, $B_1 = \sqrt{H_{ac}} \oplus i\sqrt{-H_p}$, where H_{ac} and H_p are respectively the absolutely continuous and discrete part of H. Then $\Phi \in \mathcal{S}(R^1)$ and it is not hard to see from (ii), that

$$\operatorname{Tr}\int_{-\infty}^{\infty} \rho(t)\{\cos B_1 t \oplus 0 - \cos B_0 t\}\,dt = 1/2 \int_\Gamma \frac{d}{d\mu}\hat{\rho}(\mu)\tilde{s}(\mu)\,d\mu, \qquad (2.3)$$

where $\Gamma = (-\infty, \infty) \cup (-ia, ia)$ and $\tilde{s}(\mu) = s(\mu^2)$ for $\mu \in (0, \infty)$ (ia,0), $\tilde{s}(\mu) = -s(\mu^2)$ for $\mu \in (-\infty, 0) \cup (0, ia)$.

<u>Remark.</u> It turns out that the function $\tilde{s}(\mu)$, $\mu \in R^1$ is the scattering phase for the wave equation in Lax-Phillips scattering theory (see ref. [18]). Moreover, using (2.1) one can obtain the equality (see refs. [5,7])

$$1/2 \int_{-ia}^{ia} \frac{d}{d\mu}\hat{\rho}(\mu)\tilde{s}(\mu)\,d\mu = \sum_{\lambda_j \in \sigma_p(H)\setminus\{0\}} \frac{\hat{\rho}(\sqrt{\lambda_j}) + \hat{\rho}(-\sqrt{\lambda_j})}{2} s(\lambda_j).$$

3. DECOMPOSITION OF $s(\lambda)$

In this section we construct functions $s_j(\lambda)$, $j=1,2$ with the properties

(i) $s(\lambda^2) = s_1(\lambda) + s_2(\lambda)$,

(ii) $|s_2(\lambda)| \le C_N(1+\lambda)^{-N}$ when $\lambda\to\infty$, $N \in \mathbf{Z}$,

(iii) The Fourier transform of $s_1(\lambda)$ is a compactly supported distribution.

To do this we use the equality

$$-\frac{d}{d\lambda} s(\lambda) = \mathrm{Tr}\{S(\lambda)\frac{d}{d\lambda}S^*(\lambda)\}, \quad \lambda > 0,$$

as well as an explicit form of the scattering matrix. We are going to obtain a representation formula for the S-matrix. By the invariance principle we have $S(\lambda) = S((a+\lambda)^{-1}; K^p, K_0^p)$, $\lambda>0$. Moreover, the stationary approach[1,11,12] can be applied to derive a representation formula for the S-matrix of the pair K, K_0. Denote by A the operator of multiplication by $(1+|x|^2)^{-\beta/2}$, $\beta>n$ and let C be the operator given by $K=K_0+ACA$. Since H coincides with H outside the ball B_R, the operator C is a compact one from H^{0,m_1} to H^{0,m_2} for every $m_1, m_2 \in R$. Hereafter $H^{s,m}$ will be the weighted Sobolev space with norm

$$||f||^2_{s,m} = \int (1+|\xi|^2)^s |\mathcal{F}[(1+|x|^2)^{m/2} f](\xi)|^2 d\xi$$

and $\mathcal{F}$ stands for the Fourier transform $\mathcal{F}(f)(\xi) = \int e^{ix\xi} f(x)dx$. The operator $Q_0(\zeta) = A(K_0-\zeta)^{-1}A$ has the norm-continuous boundary values $Q_0^\pm(\mu)$ for $\mu \in I=(0,a^{-1})$ as $\zeta\to\mu\pm i0$. Moreover, the compact operator $CQ_0^\pm(\mu)$ has no eigenvalue 1 in $L^2(R^4)$ since H has no positive point spectrum (see ref.[11] §7). Following Agmon, Kato, Kuroda[1,11,12] one can prove that $Q^\pm(\mu) = \lim A(K-\zeta)^{-1}A$ exists as a continuous function of operators bounded in $L^2(R^n)$ for $\mu\in I$. Moreover $1-CQ^\pm(\mu)=(1+CQ_0^\pm(\mu))^{-1}$ for $\mu \in I$. The S-matrix for the pair K, K_0 can be written in the form

$$S(\mu;K,K_0) = 1-2\pi i F_0(\mu)[1+CQ_0^+(\mu)]^{-1}F_0(\mu)^* = 1-2\pi i F_0(\mu)[1-CQ^+(\mu)]F_0(\mu)^*. \tag{3.2}$$

The operator $F_0(\mu): L^2(R^n)\to L^2(S^{n-1})$ is determined by the equality $F_0(\mu)F_0^*(\mu) = -(2\pi i)^{-1}[Q_0^+(\mu)-Q_0^-(\mu)]$. Denote by $\gamma(\lambda)$ the trace operator on the sphere with a radius λ, $(\gamma(\lambda)u)(\omega) = u(\lambda\omega)$, $\omega\in S^{n-1}$ for $u\in C^\infty(R^n)$, where polar coordinates $\xi=\rho\omega$ are used. The operator $\gamma(\lambda)$ extends to a Hölder continuity with respect to λ function of bounded operators from $H^{s,m}(R^n)$ to $L^2(S^{n-1})$ for any $s>1/2$, $m\in R^1$. Using the equality $(K_0-\zeta)^{-1} = -(1+z)-(1+z)^2(H_0-z)^{-1}$, $\zeta=(a+z)^{-1}$ and the Hölder continuity of $\gamma(\lambda)$ we obtain $F_0(\mu) = 2^{-1/2}(1+\lambda)\lambda^{(n-1)/4}\sigma(\lambda^{1/2})\mathcal{F}A$, $\mu=(a+\lambda)^{-1}$. Then (3.2) and the invariance principle yield

$$S(\lambda) = 1-\pi i(1+\lambda)^2\lambda^{(n-1)/2}G(\lambda)[1+(1+\lambda)V+(1+\lambda)^2VR(\lambda^2+i0)]VG^*(\lambda) \tag{3.3}$$

for $\lambda>0$, where $G(\lambda)=\gamma(\lambda^{1/2})\mathcal{F}$, $V=K-K_0$ and $R(z)=(H-z)^{-1}$.

<u>Remark 1</u>. In the case $\Omega=R^n$ a more simple formula than (3.3) is known

$$S(\lambda) = 1-\pi i\lambda^{(n-2)/2}G(\lambda)[V-VR(\lambda+i0)V]G^*(\lambda), \tag{3.4}$$

where $V=H-H_0$ (see ref.[13]). This formula is also valid when H and H_0 are matrices of differential operators and $G(\lambda)$ is suitably choosen.

<u>Lemma 1.</u> The S-matrix has the form $S(\lambda^2)=S_1(\lambda)+S_2(\lambda)$, where

(1) $(\frac{d}{d\lambda})^j S_2(\lambda)$ is a trace class operator with norm

$$||(\frac{d}{d\lambda})^j S_2(\lambda)||_{Tr} \le C_N(1+\lambda)^{-N}, \quad \lambda>0, \quad N\in\mathbf{Z}, \quad j<n-1,$$

(2) $\hat{S}_1(t) = \int_0^\infty e^{it\lambda} S_1(\lambda)d\lambda$ has a compact support with respect to t.

Obviously Lemma 1 and (3.1) give together the desired decomposition of the scattering phase. In order to prove Lemma 1 we need the following assertion.

<u>Lemma 2</u>. The operator $V=K-K_0$ has the form $V=V_1+V_2$ where the distribution kernel of V_1 is compactly supported and $V_2: H^{s,m_1}\to H^{s+N,m_2}$ is a bounded operator for each s, m_1, m_2, $N\in R^1$. Moreover $\mathrm{supp}\, V_2u \subset R^n \setminus B_R$ for any $u\in H^{s,m_1}$.

<u>Proof.</u> Let $\phi\in C^\infty(R^{n+1})$, $\phi(t,x)=1$ for $|x|<t+R$, $\phi(t,x)=0$ for $|x|>t+R+1$. Choose $\chi\in C_0^\infty(R^{n+1})$, $\chi=1$ on B_R and $\psi\in C_0^\infty(R^1)$, $\psi(t)=1$ for $|t|<1$, $\psi(t)=0$ for $|t|>2$. Using the finite propagation speed of $B_j^{-1}\sin tB_j$ we obtain

$$V=\int_0^\infty e^{-t}\psi(t)\phi\{B_1^{-1}\sin tB_1\oplus 0 - B_0^{-1}\sin tB_0\}\phi\,dt + \int_0^\infty e^{-t}(1-\psi)\chi\{B_1^{-1}\sin tB_1\oplus 0 - B_0^{-1}\sin tB_0\}dt + \int_0^\infty e^{-t}(1-\psi)(1-\chi)\phi\{B_1^{-1}\sin tB_1\oplus 0-B_0^{-1}\sin tB_0\}\phi\,dt.$$

Denote the third integral by V_2 and the sum of the first and second one by V_1. Obviously the distribution kernel of V_1 is compactly supported. Moreover $\text{supp}\, V_2 u \subset \text{supp}(1-\chi) \subset R^n \; B_R$. Integrating by parts in the third integral and taking into account the inequality $|x| \le t+R$ on supp ϕ we claim that $V_2 : H^{s,m_1} \to H^{s+N,m_2}$ is a bounded operator.

Lemma 3. Suppose that the operators $W_j \in \mathcal{L}(L^2)$ have compactly supported distribution kernels. Let the metric g be non-trapping in $\bar{\Omega}$. Then the operator $Q(\lambda) = W_1 R(\lambda^2+i0) W_2$ has the form $Q(\lambda) = Q_1(\lambda) + Q_2(\lambda)$, where

(i) $\|(d/d\lambda)^k Q_2(\lambda)\|_{\mathcal{L}(L^2(\Omega))} \le C_N (1+\lambda)^{-N}$, $\lambda \in R^+$, $N \in \mathbf{Z}$, $k < n-1$

(ii) $\hat{Q}_1(t) = \int_0^\infty e^{i\lambda t} Q_1(\lambda)\, d\lambda$ is compactly supported.

Proof. Consider the operators $P_j(t) = B_j^{-1} \sin t B_j$, $j = 0,1$. Obviously $P_0(t)$ and $P_1(t)$ solve the problems

$$(D_t^2 - D_0) P_0(t) = 0 \qquad (D_t^2 - H) P_1(t) = 0$$

$$P_0(0) = 0, \; P_{0t}(0) = I \qquad P_1(0) = 0, \; P_{1t}(0) = I$$

$$BP_1(t) = 0,$$

where $Bu = u/\partial_\Omega$ or $Bu = \frac{\partial u}{\partial n}/\partial_\Omega$ and n is the outward normal to $\partial\Omega$. Let $\chi \in C_0^\infty(R^n)$, $\chi \equiv 1$ on $\text{supp}_{x,y} W_j(x,y)$, $j = 1,2$ and $\chi(x) = 0$ for $x \notin B_{R_1}$ where $W_j(x,y)$ are the distribution kernels of W_j. Due to the non-trapping condition, there exist $T > 0$, such that every generalized null bicharacteristic of $D_t^2 - H$ passing over supp χ $\bar{\Omega}$ at $t = 0$ lies for $|t| > T$ completely over the set $R^n \; B_R$. Moreover the bicharacteristics of $D_t^2 - H$ are straight lines outside the ball B_R. The propagation of singularities for the distribution kernel $P_1(t,x,y)$ of $P_1(t)$ yield

$$\text{sign supp}\, P(t,x,y)\chi(y) \subset \{(t,x,y);\; ||x| - t| < T\}, \quad T > R_1. \qquad (3.5)$$

Choose a cut-off function $\xi \in C^\infty(R^{n+1})$ such that $\xi \equiv 1$ on a neighbourhood of $\{(t,x);\; ||x| - t| < T\}$, $\xi(t,x) = 0$ if $(t,x) \notin \{(t,x);\; ||x| - t| < T+1\}$ and suppose $_x\xi(t,x) = 0$ for $t \in R^1$ x

Consider the operators $P_{0\chi} = \chi P_0(t)\chi$, $P_{1\chi} = \chi P_1(t)\chi$, $E_0 = \xi P_1(t)\chi$, $R_\chi(\lambda) = \chi R(\lambda)\chi$. Then we have

$$W_1 R(\lambda^2 + i0) W_2 = W_1 \{\chi \hat{E}(\lambda) + [R_\chi(\lambda^2 + i0) - \chi \hat{E}_0(\lambda)]\} W_2 .$$

It is easy to see that the operator $\chi E_0(t)$ has a compact support with respect to t.

So we need the following estimate

$$\|D_\lambda^j [R(\lambda^2 + i0) - \chi \hat{E}(\lambda)]\|_{\mathcal{L}(L^2(\Omega))} \le C_N (1+\lambda)^{-N}, \; N \in \mathbf{Z}, \; j < n-1, \qquad (3.6)$$

where $\mathcal{L}(L^2(\Omega))$ is the space of bounded operators from $L^2(\Omega)$ to $L^2(\Omega)$. A similar to (3.6) estimate was obtained by Vainberg [21] and Rauch [20]. Our proof of (3.6) is close to that given in ref. [20] and we only shall sketch it.

Consider the operator $F(t) = [D_t^2 - H, \xi] P(t)\chi$, $F(t) \in \mathcal{L}(L^2(\Omega))$. It follows from (3.5) that the kernel $F(t,x,y)$ of $F(t)$ is a smooth function, $\text{supp}\, \tilde{F} \subset \{(t,x,y);\; T < ||x| - t| < T+1\}$ and $F^{(\ell)}(0) = 0$ for any $\ell \in \mathbf{Z}^+$, since $\xi \equiv 1$ on supp χ. Moreover

$$(D_t^2 - H) E_0(t) = F(t), \qquad (3.7)$$
$$E_0(0) = 0, \quad E_{0t}(0) = \chi, \quad BE_0(t) = 0,$$

where $Bu = u/\partial\Omega$ or $Bu = \frac{\partial u}{\partial n}/\partial\Omega$. Let $\tilde{F}(t,x,y)$ be a smooth function in $R^1 \times R^n \times \Omega$ such that $\tilde{F} = F$ for $x \in \Omega$ and $F(t,x,y) = 0$ for $x \in K$, $t > 2T$. Consider the problem

$$(D^2 - H_0) W(t) = \tilde{F}(t) \qquad (3.8)$$
$$W(0) = 0, \quad W_t(0) = 0.$$

Choose $\psi \in C^\infty(R^n)$, $\psi \equiv 1$ on supp χ, $\psi(x) = 1$ for $|x| > 2T$. From (3.7), (3.8) and Duhamel's formula we have

$$W(t) = E_0(t) \oplus 0 - P_0(t)\chi + \int_0^t P_0(t-s)(H_0 - H \oplus 0)(E_0(s) \oplus 0)\, ds \qquad (3.9)$$

in $L^2(R^n)$,

$$E_0(t) = \psi W(t) + P_1(t)\chi + \int_0^t P_1(t-s) Q(s)\, ds \qquad (3.10)$$

in $L^2(\Omega)$, where $Q(s) = (1-\psi)F(s) + [H, \psi]W(s)$. Since χE_0 has a compact support with respect to t, we can choose $T > 0$ so that

$$\chi W(t) = P_{0\chi}(t) + \int^T P_{0\chi}(t-s)(H_0 - H \oplus 0)(E_0(s) \oplus 0)\, ds .$$

The local energy decay of the operator $P_0(t)$, i.e.,

$$\|D_t^j P_{0\chi}(t)\|_{\mathcal{L}(H^{-s}, S^s)} \le C_{s,j} t^{-n}, \quad t > C$$

and the smoothness of the kernel of $W(t)$ yield the estimate

$$\|D_t^j \chi W(t)\|_{\mathcal{L}(H^{-s}, H^s)} \le C_{s,j} (1+t)^{-n} \quad \text{for } t \in \bar{R}^+ .$$

Using the equalities $\chi W^{(\ell)}(0) = 0$ for $\ell \in Z^+$, we obtain

$$\|D_t^j \chi \hat{W}(\lambda)\|_{\mathcal{L}(H^{-s}, H^s)} \le C_N(1+\lambda)^{-N} \quad \text{for } N \in Z^+,\ j < n-1.$$

Therefore

$$\|D_\lambda^j \hat{Q}(\lambda)\|_{\mathcal{L}(L^2(\Omega))} \le C_N(1+\lambda)^{-N} \quad \text{for } N \in Z^+,\ j < n-1. \tag{3.11}$$

Moreover, the function $\hat{Q}(\lambda) = \int_0^\infty e^{ikt} Q(t)dt$ is analytic on the half-plane $\mathrm{Im}\, k > 0$ with values in $\mathcal{L}(L^2(\Omega))$ and it has a C^{n-2} continuation on R. Multiplying (3.10) by χ and taking a Fourier-Laplace transform with respect to t we get

$$\chi \hat{E}_0(k) - R_\chi(k^2) = R_\chi(k^2)\hat{Q}(k)$$

for $\mathrm{Im}\, k \ge C$, C -sufficiently large. We can extend this equality in $\{k;\ \mathrm{Im}\, k > 0,\ \mathrm{Re}\, k > 0\}$ since the functions $R_\chi(k^2)$ and $\hat{Q}(k)$ are analytical in this region with values in $\mathcal{L}(L^2(\Omega))$. Using (3.11) we obtain

$$\|D_\lambda^j R_\chi(\lambda^2 + i0)\|_{\mathcal{L}(L^2(\Omega))} \le C\lambda^p, \quad \lambda \ge \lambda_0,\ j < n-1, \tag{3.12}$$

for some p, and prove the estimate (3.6). So we complete the proof of Lemma 3.

We are ready to prove Lemma 1. Using Lemma 2 and Lemma 3 with $W_j = V_1$ we can write $S(\lambda^2)$ in the form $S(\lambda^2) = S_1(\lambda) + S_2(\lambda)$, where

$$S_1(\lambda) = 1 + \pi i(1+\lambda^2)G(\lambda^2)\chi[V_1 + Q_1(\lambda)]\chi G^*(\lambda^2)\lambda^{n-2},$$

$$S_2(\lambda) = \pi i(1+\lambda^2)\lambda^{n-2}\{(1+\lambda^2)^2 G(\lambda^2)Q_2(\lambda)G^*(\lambda^2) +$$

$$+ G(\lambda^2)[1 + (1+\lambda^2)V_1 + (1+\lambda^2)^2 V_1 R(\lambda + i0) \oplus 0]V_2 G^*(\lambda^2) +$$

$$+ G(\lambda^2)V_2[1 + \lambda^2 + (1+\lambda^2)^2 R(\lambda + i0) \oplus 0]VG^*(\lambda^2)\}.$$

The operator $S_1(\lambda)$ satisfies the second condition of Lemma 1. Indeed, the operator $\hat{Q}_1(t)$ has a compact support with respect to t in view of Lemma 3 and so does $G(\lambda^2)\chi(t)$ with distribution kernel $\delta(t - x\omega)\chi(x)$, $\chi \in C_0^\infty(R^n)$, $\omega \in S^{n-1}$. In what follows we shall prove that $S_2(\lambda)$ satisfies the first condition of Lemma 1.

1. First consider the operator $I_1(\lambda) = G(\lambda^2)Q_1(\lambda)G^*(\lambda^2)$. The kernel of $G(\lambda^2)$ is equal to $e^{i\lambda\omega x}$, therefore $I_1(\lambda)$ is an operator with smooth kernel $I_1(\lambda, \omega, \theta)$ and

$$|I_1(\lambda, \omega, \theta)| = |\int e^{i\lambda\omega x}\chi(x)Q_2(\lambda)(e^{-i\lambda\theta y}\chi(\cdot y))dx| \le$$

$$\le C\|Q_2(\lambda)\|_{\mathcal{L}(L^2(\Omega))} \le C_N(1+\lambda)^{-N}.$$

Therefore $I_1(\lambda)$ is a trace class operator and

$$\|I_1(\lambda)\|_{Tr} \le C_N(1+\lambda)^{-N}.$$

2. In order to estimate the other terms of $S_2(\lambda)$ we use the inequality

$$\|R(\lambda^2 + i0)\|_{\mathcal{L}(H^{0,n}, H^{0,-n})} \le C(1+\lambda)^p \quad \text{for some p.} \tag{3.13}$$

This estimate was proved for $\chi R(\lambda^2 + i0)\chi$ (see (3.12)). To derive it for $R(\lambda^2 + i0)\chi$ consider the resolvent equation $R(\lambda^2 + i0)\chi = R_0(\lambda^2 + i0)\chi - R_0(\lambda^2 + i0)(H - H_0)\chi R(\lambda^2 + i0)\chi$. Using the inequality $\|D_x^\alpha R_0(\lambda^2 + i0)\|_{\mathcal{L}(H^{0,n}, H^{0,-n})} \le C\lambda$ for $|\alpha| \le 2$ we obtain (3.13) for $R(\lambda^2 + i0)\chi$ and repeating this argument we prove (3.13).

Consider the operator $I_2(\lambda) = G(\lambda)V_1 R(\lambda^2 + i0)V_2 G^*(\lambda)$. This operator has a smooth kernel

$$I_2(\lambda, \omega, \theta) = (i\lambda)^{-2N}\int e^{i\lambda x\omega} V_1 R(\lambda^2 + i0)(V_2\Delta^N(e^{-i\lambda y\theta}))dx$$

and Lemma 2 yields $|I_2(\lambda, \omega, \theta)| \le C\lambda^{-2N}$. The other terms of $S_2(\lambda)$ can be estimate in a similar way.

4. PROOF OF THE THEOREMS

In this section we show that the scattering phase has an asymptotic development at infinity and compute the coefficients. Denote by σ the distribution

$$\langle\sigma, \rho\rangle = \mathrm{Tr}\int \rho(t)\{\cos B_1 t \oplus 0 - \cos B_0 t\}dt, \quad \rho \in C_0^\infty(R^1).$$

Using the trace formula (2.2) we have

$$\hat{\rho}\sigma(\lambda) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{d\mu}\hat{\rho}(\lambda-\mu)\tilde{s}(\mu)d\mu + \frac{1}{2}\int_{-ia}^{ia}\frac{d}{d\mu}\hat{\rho}(\lambda-\mu)\tilde{s}(\mu)d\mu$$

and the second integral is $O(\lambda^{-N})$ for any N as $\lambda \to \infty$ The decomposition of $s(\lambda^2)$ obtained in Sect.2 yields

$$\hat{\rho} * \frac{d}{d\lambda}s_3(\lambda) = -2\hat{\rho}\sigma(\lambda) + O(\lambda^{-N})$$

with $s_3(\lambda) = s_1(\lambda)$ for $\lambda > 0$ and $s_3(\lambda) = -s_1(\lambda)$ for $\lambda < 0$. For $\rho \in C_0^\infty(R^1)$ and $\rho \equiv 1$ on $\mathrm{supp}\, \hat{s}_1(t)$ the last equality leads to

$$\frac{d}{d\lambda}s(\lambda^2) = -2\hat{\rho}\sigma(\lambda) + O(\lambda^{-N}), \quad N \in Z^+, \quad \lambda > 0. \tag{4.1}$$

First consider the case $\Omega = R^n$. To study the right-hand side of (4.1), introduce the distribution kernels $v_0(t,x,y)$ and $v_1(t,x,y)$ of the operators $\cos B_0 t$ and $\cos B_1 t$. Obviously v_0 and v_1 are solutions of the problems

$$(D_t^2 - H_0)v_0 = 0 \qquad (D_t^2 - H)v_1 = 0$$

$$v_0/_{t=0} = \delta(x-y),\ v_{0t}/_{t=0} = 0 \qquad v_1/_{t=0} = \delta(x-y),\ v_{1t}/_{t=0} = 0$$

and $\sigma(t)$ is equal to the distribution $\int [v_1(t,x,x) - v_0(t,x,x)]\,dx$. Repeating the arguments in the proof of Corollary 1.2 in ref. /6/, one can prove that sing supp $\sigma \subset \{T_j, T_j$ is a period of a periodic geodesic of $g\}$. Since the non-trapping condition sing supp $\sigma = \{0\}$. Then (4.1) holds for any $\rho \in C_0^L(R^1)$, $\rho \equiv 1$ on a neighbourhood of $t = 0$. Using the finite speed of propagation and applying a finite partition of unity, one can reduce the problem to the investigation of the functions

$$I_j(\lambda) = \iint e^{-i\lambda t}\rho(t)\phi(x)v_j(t,x,x)\,dx\,dt, \quad j = 0,1,$$

with $\phi \in C_0^\infty(R^n)$. It turns out that for $|t| < \delta$ and δ sufficiently small, the distributions v_1 and v_0 are sums of oscillating integrals

$$v_\pm(t,x,y) = \int e^{i\Phi_\pm(t,x,y,\theta)} a^\pm(t,x,y,\theta)\,d\theta, \tag{4.2}$$

where $a^\pm$ are classical amplitudes, $a^\pm \sim \sum_{j=0}^\infty c_j^\pm$, $c_j^\pm$ – homogeneous of order $-j$ with respect to θ. The phase functions $\Phi_\pm$ have the form $\Phi_\pm = \psi(x,y,\theta) \pm tg(y,\theta)$ (see /6/), q^2 is the principle symbol of H and ψ is a local solution of $q(x, d\psi(x,y,\theta)) = q(y,\theta)$, $\psi(x,y,\theta) = 0$ when $\langle x-y,\theta\rangle = 0$ and $d_x\psi(x,y,\theta) = \theta$ for $x=y$. Then the integral $I_1(\lambda)$ became

$$I_1(\lambda) = \lambda^n \int e^{i\lambda t(1-q(y,\theta))}\phi(x)\rho(t)a^t(t,x,x;\theta)\,d\theta\,dx + O(\lambda^{-N}).$$

Substituting

$$\theta = r\omega \quad q(x,\omega) = 1, \quad r > 0, \quad S = \{\omega : q(x,\omega) = 1\}$$

we have

$$I(\lambda) = \lambda^n \iiint_{0\ S}^{\infty} e^{i\lambda t(1-r)} \rho(t)\phi(x)a^+(t,x,x,\lambda r\omega)|\nabla q|^{-1}\,dS\,dr\,dx\,dt,$$

and applying the method of stationary phase we obtain

$$I_1(\lambda) \sim (2\pi)^{-n}\sum_{j=0}^\infty\sum_{k=0}^\infty \frac{\lambda^{n-j-k-1}}{k!}\iint_S (i^{-1}\frac{\partial^2}{\partial t\partial r})[r^{n-1-j}c_j^+(t,x,x,\omega)]_{\substack{t=0\\r=1}} \phi(x)\frac{dS}{|\nabla q|}dx.$$

This formula leads to (1.1) with

$$a_j = (2\pi)^{-n}\iint [c_j^+(0,x,x,\omega) + \sum_{k=1}^{j}(n-j-1)\ldots(n-j-k)(k!)^{-1}(i^{-1}\partial/\partial t)^k c_{j-k}^+(0,x,x,\omega)\,\frac{dS\,dx}{|\nabla q|}. \tag{4.3}$$

In order to compute the coefficients in the case of the Schrödinger operator $H = -\Delta + V$, observe that $\Phi_\pm = \langle x-y,\theta\rangle \pm t|\theta|\nabla c_1^+ = 1/2$, and c_ℓ, $\ell > 0$ solves the transport equation

$$\partial c_\ell^+ - \langle\theta, \nabla_x\rangle c_\ell^+ - i/2(\partial_t^2 - \Delta + V)C_{\ell-1}^+ = 0.$$

$$c_\ell^+|_{t=0} = 0.$$

Using (4.3) we prove inductively that a_j has the form prescribed in theorem 2.

The investigation of the asymptotic behaviour of $\hat{\rho\sigma}(\lambda)$ as $\lambda \to \infty$ in the obstacle case $\Omega = R^n$ for the Laplace operator with Dirichlet or Neumann boundary conditions was done in refs. /8,18/. It was proved, that $\hat{\rho\sigma}(\lambda) \sim \sum_{j=0}^\infty a_j\lambda^{(n-j-2)/2}$ and the first three coefficients a_j, $j = 0,1,2$ were obtained in the case $a^{ij} = \delta_{ij}$. The method used in refs. /8,18/ by Ivrij, can be applied to the investigation of $\hat{\rho\sigma}(\lambda)$ for arbitrary second order differential operators in Ω with Dirichlet or Neumann boundary conditions. In order to compute the first two coefficient of $s(\lambda)$ one can use the trace formula (2.2) as well as the asymptotics of the right-hand side of (2.2) as $t \to +0$ given by Mc Keen and Singer (see /15/ §4 and §5, formula (2)). Comparing the coefficients of the two sides of (2.2) as $t \to +0$ we get a_0 and a_1.

The method used in the previous sections can be applied without change to study the asymptotics of the scattering phase related to systems of first order differential operators. Let $H_0 = \sum_{j=1}^n A_j^0 D_j$, $H_1 = \sum_{j=1}^n A_j^1(x)D_j + B(x)$ be self-adjoint operators in $L^2(\mathbf{R}^n; \mathbf{C}^{2m})$, A_j^0 be constant $2m \times 2m$ matrices, $A_j^1 \in C^\infty(\mathbf{R}^n; \mathbf{R}^{4m^2})$, $A^k = \sum_{j=1}^n A_j^k\xi_j$, $k = 0,1$; $H_1 = H_0$ outside the ball B_R. Assume, that the eigenvalues $\lambda_j(x,\xi)$ of $A^1(x,\xi)$ are simple and

$$\lambda_1(x,\xi) < \ldots < \lambda_m(x,\xi) < 0 < \lambda_{m+1}(x,\xi) < \ldots < \lambda_{2m}(x,\xi). \tag{4.4}$$

Then the spectrum of H_0 is absolutely continuous and $\sigma(H_0) = R^1$. The eigenfunctions of H_1 in L^2 corresponding to a non-zero eigenvalue are smooth and supported in B_R and so they are finitely many. Moreover, the eigenvalue $\lambda = 0$ has a finite multiplicity. Thus $\sigma(H_1) = \sigma_p(H_1)$ $\sigma_{ac}(H_1)$ and $\sigma_p(H_1)$ is finite, $\sigma_{ac}(H_1) = R^1$.

Consider the scattering phase $s(\lambda)$ related to the pair H_1, H_0. The function $s(\lambda)$ has the properties (i)-(iii) described in Sect.2. Choosing $\Phi(\lambda) = \hat{\rho}(\lambda)$, $\rho \in C_0^\infty(R^1)$ in (iii) we obtain the following trace formula

$$\mathrm{Tr} \int \rho(t) \{ e^{itH} - e^{itH_0} \} dt = \int \frac{d}{d\lambda} \hat{\rho}(\lambda) s(\lambda) d\lambda. \tag{4.5}$$

Denote by $\rho_j(t)$, $1 \le j \le 2m$ the projections of the bicharacteristics of $\lambda_j(x,\xi)$ on the x-space. We shall use the following non-trapping condition. There exists $T>0$, such that

$$\rho_j(t) \notin B_R \quad \text{for } t>T \quad \text{if} \quad \rho_j(0) \in B_R . \tag{4.6}$$

Theorem 4. Suppose that (4.4) and (4.6) are valid. Then

$$s(\lambda) \sim \sum_{j=0}^{\infty} a_j^{\pm} \lambda^{n-j} \quad \text{as} \quad \lambda \to \pm\infty$$

and

$$a_0^{\pm} = (4\pi)^{-n/2} (\Gamma(n/2+1))^{-1} \int \mathrm{Tr}(\pi_1^{\pm} A^1(x,\xi) - \pi_0^{\pm} A^\circ(\xi))\, dx\, d\xi,$$

where $\pi_j^+(\pi_j^-)$ is the projection on the positive (negative) eigenspace of A. The proof of theorem 4 is similar to that of theorem 1 and we shall only sketch it. In order to decompose $s(\lambda)$ as a sum of functions $s_j(\lambda)$, $j=1,2$ with the properties (ii) and (iii) described in Sect.3, we use the formula

$$S(\lambda) = 1 - 2\pi i G(\lambda) [V + VR(\lambda + io)V] G^*(\lambda),$$

where $\lambda \in R^1 \setminus (\sigma_p(H_1) \cup 0)$, $V = H_1 - H_0$. Here $G(\lambda)$ are bounded operators from $H^{\rho,s}$, $s>1/2$, $\rho \in R^1$ to an auxiliary space $\mathcal{H}$. Denote by $\pi_j(\xi)$ the orthogonal projection onto the eigenspace of $A^\sigma(\xi)$ corresponding to $\lambda_j(\xi)$. Then $\pi_1(\xi)$ is a smooth, homogeneous function of order one in $R^n \setminus 0$. Let $S_{j,\lambda} = \{\xi \in R^n; \lambda_j(\xi) = \lambda\}$ and $d\mu_j(\omega) = |\nabla \lambda_j(\xi)|^{-1} dS_j$, where dS_j is the usual Lebesque measure on $S_{j,\lambda}$. Consider the trace operators $\gamma_j(\lambda)$ on $S_{j,\lambda}$ defined by $(\gamma_j(\lambda) u)(\omega) = u(\lambda\omega)$, , $u \in C^\infty(R^n)$, where polar coordinates $\xi = \lambda\omega$, $\omega \in S_{j,1}$ are used. Denote $\gamma_\lambda = \sum_{j=1}^{2m} \gamma_j(\lambda) \pi_j(\xi)$ and $\mathcal{H} = \sum_{j=1}^{2m} \pi_j(\lambda\omega) L^2(S_{j,\lambda}; d\mu_j; C^{2m})$. It turns out that $G_\lambda = \gamma_\lambda \mathcal{F}$. Moreover, the operator $\int e^{i\lambda t} G_\lambda V d_\lambda$ has a compactly supprted distribution kernel and using an analogy of Lemma 2 we find the functions $s_j(\lambda)$ $j=1,2$. From (4,5) we obtain $\frac{d}{d\lambda} s(\lambda) = -\hat{\rho}\sigma(\lambda) + O(\lambda^{-N})$, where $\sigma(t) = \int [u_1(t,x,x) - u_0(t,x,x)] dx$ and u_j are the fundamental solutions of the Cauchy problem for $D_t - H_1$ and $D_t - H_0$ respectively. Using a microlocal parametrix for the Cauchy problem and the method of the stationary phase we complete the proof of theorem 4.

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Попов Г.С. E5-82-669

Асимптотическое поведение фазы рассеяния в незахватывающих метриках

Рассмотрено асимптотическое поведение фазы рассеяния на бесконечности для эллиптического самосопряженного дифференциального оператора H либо в R^n, либо в области $\Omega \subset R^n$ с краевыми условиями Дирихле или Неймана. Оператор H имеет вид $H = -\Delta_g + hD + V(x)$, где Δ_g - оператор Лапласа-Бельтрами римановой метрики g в Ω, $hD = \sum_{j=1}^{n} h_j D_j$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Предполагается, что H равен оператору Лапласа Δ в окрестности бесконечности и что метрика g не является ловушкой для лучей, т.е. все геодезические метрики уходят с любого компакта в $\bar{\Omega}$ через некоторое время, зависящее только от компакта. При этих ограничениях получено полное асимптотическое разложение фазы рассеяния $s(\lambda)$ для $\lambda \to \infty$. Найдены первые члены этого разложения.

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Popov G.S. E5-82-669

Asymptotic Behaviour of the Scattering Phase for Non-Trapping Metrics

The asymptotic behaviour of the scattering phase is considered at infinity for an elliptic, self-adjoint, second order differential operator H, defined either in R^n or in an unbounded domain $\Omega \subset R^n$ with Dirichlet or Neumann boundary conditions. The operator H has the form $H = -\Delta_g + hD + V$, where Δ_g is the Laplace-Beltrami operator related to a Riemannian metric g in $\bar{\Omega}$, $hD = \sum_{j=1}^{n} h_j D_j$, $D_j = i^{-1}\partial/\partial x_j$. Provided a non-trapping hypothesis is fulfilled and H coincides with the Laplace operator Δ in a neighbourhood of infinite, an asymptotic development of the scattering phase $s(\lambda)$ is obtained as $\lambda \to \infty$. The first coefficients in this development are found.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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