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T.D.Palev,* O.Ts.Stoytchev*

TYPICAL REPRESENTATIONS OF THE LIE SUPERALGEBRA sl(1,n)

* Institute of Nuclear Research and Nuclear Energy, Boul. Lenin 72, Sofia, Bulgaria.

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The finite-dimensional irreducible representations of the classical Lie superalgebra (LS's) are nowadays fully classified [1]. In the physical applications, however, it is often important to have explicit formulae for the matrix elements of the generators in a certain basis of the representation space. Although some results in this direction are available [2-8], the problem as a whole remains unsolved. In the present paper we give also a partial answer to the same problem. We write down formulae for the representations of the special linear Lie superalgebra $s\ell(1,n)$ induced by finite-dimensional irreducible representations of the linear span of the even part $g\ell(n)$ and all positive root vectors. The corresponding induced $s\ell(1,n)$-modules are always finite-dimensional, but are not necessarily fully reducible. The representations of $s\ell(1,n)$ realized in the irreducible induced modules are the typical representations of the Lie superalgebra [1].

We were led to the present investigation from a study of a noncanonical quantization [9]. The position operators $q_1, \ldots, q_n$ and the momentum operators $p_1, \ldots, p_n$ span in this case a basis in the odd part of $s\ell(1,n)$ and generate it. The problem to determine the representations of all q_i, p_i is the same one as to construct the representations of $s\ell(1,n)$ The last problem is of independent mathematical interest; the answer to it may be relevant in several branches of the theoretical physics.

We consider $s\ell(1,n)$ as a superalgebra of the general linear LS $\ell(1,n)$. As a homogeneous basis in $\ell(1,n)$ we choose the generators e_{AB}, $A,B=0,1,\ldots,n$. In the defining representation e_{AB} is an $(n+1)\times(n+1)$ matrix with 1 in the A-th row and the B-th column and zero elsewhere. The odd and the even parts are $\ell_1(1,n)=$ $=\text{lin. env.}\{e_{0i}, e_{i0} \mid i=1,\ldots,n\}$ and $\ell_0(1,n)=\text{lin.env.}\{e_{00}, e_{ij} \mid i,j=1,\ldots,n\}$, respectively. As a subalgebra of $\ell(1,n)$ the LS $s\ell(1,n)$ reads: $s\ell(1,n) = \text{lin.env.}\{e_{00}+e_{ii}, e_{AB} \mid A \neq B=0,\ldots,n;\ i=1,\ldots,n\}$. Its even subalgebra $G_0 = \text{lin. env.}\{E_{ij} = \delta_{ij}e_{00} + e_{ij} \mid i,j=1,\ldots,n\}$ is isomorphic to the general linear Lie algebra $g\ell(n)$. Since $[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{\ell i}E_{kj}$, E_{ij} are the Weyl generators of $g\ell(n)$.

Let G_+ be the linear span of $e_{01}, \ldots, e_{0n}$. Here we compute the representations of $s\ell(1,n)$ induced by the subalgebra $P = G_0 + G_+$. The latter are defined as follows [1]. Let $V^0(L)$ be a finite-dimensional simple G_0-module with the highest weight L. Extend it to a P-module putting $G_+V^0(L)=0$. Denote by U and U_P

the universal enveloping algebras of $s\ell(1,n)$ and P, correspondingly. Then the induced $s\ell(1,n)$-module

$$\bar{V}(L) = \mathrm{Ind}_P^{s\ell(1,n)} V^\circ(L) \qquad (1)$$

is the factor-space

$$\bar{V}(L) = U \otimes V^\circ(L)/I \qquad (2)$$

of the tensor product of U and $V^\circ(L)$ with respect to the linear span I of all elements of the form $gp \otimes v - g \otimes p(v)$, $g \in U$, $p \in U_p$ and $v \in V^\circ(L)$. The space $\bar{V}(L)$ is equipped with a structure of an $s\ell(1,n)$-module in a natural way:

$g(u \otimes v) = gu \otimes v$, $g \in s\ell(1,n)$, $u \in U$, $v \in V^\circ(L)$.

As a basis in $\bar{V}(L)$ we choose the vectors

$$|\theta_1,\dots,\theta_n; m\rangle = (e_{10})^{\theta_1} \dots (e_{n0})^{\theta_n} \otimes m; \quad \theta_i = 0,1; \ m \in \Gamma(L). \qquad (3)$$

The restriction $\theta_i = 0$ or 1 is a consequence of the fact that $(e_{i0})^2 = 0$ in U. $\Gamma(L)$ is the set of all Gel'fand-Zetlin patterns for $g\ell(n)$ in $V^\circ(L)$; its elements $m \equiv (m_{jp})$ span a basis in $V^\circ(L)$; m_{jp} are in general complex numbers, such that $\mathrm{Re}(m_{jp} - m_{j,p-1})$ and $\mathrm{Re}(m_{j,p-1} - m_{j+1,p})$ are nonnegative integers and all m_{jp} have the same imaginary part. The highest weight L is determined uniquely by the first row $(m_{1n},\dots,m_{nn})$, which is the same for every pattern $m \in \Gamma(L)$. The representations corresponding to different n-tuples $(m_{1n},\dots,m_{nn}) \neq (m'_{1n},\dots,m'_{nn})$ are inequivalent*.

Since the generators $e_{A,A+1}$ and $e_{A+1,A}$ determine through the LS-product all other generators, here we write down the transformation properties of the basis vectors (3) only with respect to these generators. To this end we introduce first the following notation

$$B^k_{pq}(m) = \epsilon(p-q) \left| \frac{\prod_{j=1, j\neq q}^{k} (\ell_{p,k-1} - \ell_{j,k} - 1) \prod_{j=1, j\neq p}^{k-1} (\ell_{j,k-1} - \ell_{q,k})}{\prod_{j=1}^{k-1} (\ell_{p,k-1} - \ell_{j,k-1} - 1) \prod_{j=1, j\neq q}^{k} (\ell_{j,k} - \ell_{q,k})} \right|^{1/2}, \qquad (4)$$

* This is not the case for the representations of $s\ell(n)$. The $g\ell(n)$-modules $V^\circ(m_{1n},\dots,m_{nn})$ and $V^\circ(m'_{1n},\dots,m'_{nn})$ are also $s\ell(n)$-irreducible, however, they give the same representation of $s\ell(n)$ if $m_{in} - m_{i+1,n} = m'_{in} - m'_{i+1,n}$ for all $i = 1,\dots,n-1$.

$$b^k_i(m) = \left| \frac{\prod_{j=1}^{k-1} (\ell_{j,k-1} - \ell_{i,k})}{\prod_{j=1, j\neq i}^{k} (\ell_{j,k} - \ell_{i,k})} \right|^{1/2}, \qquad (5)$$

$$\tilde{b}^k_i(m) = \left| \frac{\prod_{j=1}^{k} (\ell_{i,k-1} - \ell_{j,k} - 1)}{\prod_{j=1}^{k-1} (\ell_{i,k-1} - \ell_{j,k-1} - 1)} \right|^{1/2}, \qquad (6)$$

$$d^k_i(m) = \left| \frac{\prod_{j=1}^{k-1} (\ell_{j,k-1} - \ell_{i,k} - 1)}{\prod_{j=1}^{k} (\ell_{j,k} - \ell_{i,k} - 1)} \right|^{1/2} \qquad (7)$$

$$\tilde{d}^k_i(m) = \left| \frac{\prod_{j=1}^{k} (\ell_{j,k} - \ell_{i,k} - 1)}{\prod_{j=1, j\neq i}^{k-1} (\ell_{j,k-1} - \ell_{i,k-1})} \right|^{1/2}, \qquad (8)$$

where $\ell_{ik} = m_{ik} - i$, $\epsilon(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ and it is understood that whenever some of the multiples in the above expressions are not defined (as, for instance, the numerator of $b^1_1(m)$) then they have to be replaced by 1; moreover, if the denominator of the rught-hand side of some of the above equalities (4)-(8) is zero, then the corresponding coefficient in the left-hand side has to be replaced by 0.

In terms of this notation the generators $e_{A,A+1}$ and $e_{A+1,A}$, $A = 0,\dots,n$, transform the basis vectors (3) as follows:

$$e_{10}|\theta_1,\dots,\theta_n;(m_{jp})\rangle = (1-\theta_1)|1,\theta_2,\dots,\theta_n;(m_{jp})\rangle,$$

$$e_{k+1,k}|\dots,\theta_k,\theta_{k+1},\dots;(m_{jp})\rangle =$$

$$= \sum_{i=1}^{k} b^k_i(m)\tilde{b}^{k+1}_i(m)|\dots,\theta_k,\theta_{k+1},\dots;(m_{jp} - \delta_{ji}\delta_{pk})\rangle +$$

$$+\theta_k(1-\theta_{k+1})|\ldots,\theta_k-1,\theta_{k+1}+1,\ldots;(m_{jp})\rangle ,$$

$$e_{01}|\theta_1,\ldots,\theta_n;(m_{jp})\rangle = \theta_1\Big(m_{11}-\sum_{j=2}^{n}\theta_j\Big)|0,\theta_2,\ldots,\theta_n;(m_{jp})\rangle +$$

$$+\sum_{s=1}^{n-1}\sum_{i_1=1}^{1}\sum_{i_2=1}^{2}\cdots\sum_{i_s=1}^{s}\theta_{s+1}(-1)^{\theta_1+\ldots+\theta_s}\, b^1_{i_1}(m)\,B^2_{i_1 i_2}(m)\times$$

$$\times B^3_{i_2 i_3}(m)\ldots B^s_{i_{s-1},i_s}(m)\tilde{b}^{s+1}_{i_s}(m)\times$$

$$\times|\theta_1,\ldots,\theta_s,0,\theta_{s+2},\ldots,\theta_n;\Big(m_{jp}-\sum_{r=1}^{s}\delta_{j i_r}\delta_{pr}\Big)\rangle ,$$

$$e_{k,k+1}|\ldots,\theta_k,\theta_{k+1},\ldots;(m_{jp})\rangle = \tag{9}$$

$$=\sum_{i=1}^{k} d^k_i(m)\tilde{d}^{k+1}_i(m)|\ldots,\theta_k,\theta_{k+1},\ldots;(m_{jp}+\delta_{ji}\delta_{pk})\rangle +$$

$$+\theta_{k+1}(1-\theta_k)|\ldots,\theta_k+1,\theta_{k+1}-1,\ldots;(m_{jp})\rangle .$$

We omit the derivation. The proof that the above relations give a representation of the LS $s\ell(1,n)$ can be carried out in a straightforward way.

Proposition. The induced $s\ell(1,n)$ module $\bar{V}(L)$ with a highest weight L corresponding to $(m_{1n},\ldots,m_{nn})$ is irreducible if and only if $m_{in}\neq i-1$ for all $i=1,\ldots,n$.

Proof. One can show in a straightforward way that all modules corresponding to $m_{in}\neq i-1$, $i=1,\ldots,n$ are irreducible. It is not simple, however, to prove the inverse. Therefore, we shall use a general criterion for irreducibility (Proposition 2.9 in Ref. /1/). First we introduce the notation and list the properties of $s\ell(1,n)$ we need (for more information see Ref. /8/).

Define by means of the diagonal matrix (g_{AB}) with $1=-g_{00}=g_{11}=\ldots g_{nn}$ a non-degenerate bilinear form of the Cartan subalgebra $H' = \mathrm{lin.env.}\{e_{AA}|A=0,\ldots,n\}$ of $\ell(1,n)$:

$$(e_{AA},e_{BB}) = 2(n-1)g_{AB} \tag{10}$$

On the Cartan subalgebra $H=\mathrm{lin.env.}\{E_{ii}|E_{ii}=e_{00}+e_{ii},\ i=1,\ldots,n\}$ of $s\ell(1,n)$ the form (10) coincides with the Killing form of $s\ell(1,n)$. Choose as an ordered basis in H' the vectors e_{00}, $e_{11},\ldots,e_{nn}$ and let $e^0, e^1,\ldots,e^n$ be the conjugate basis in the dual to H' space $\overset{*}{H}{}'$, i.e., $e^A(e_{BB})=\delta^A_B$. Then the bilinear form on $\overset{*}{H}{}'$, induced from (10) reads

$$(e^A,e^B) = \frac{g_{AB}}{2(n-1)}. \tag{11}$$

Since $[h,e_{AB}]=(e^A-e^B)(h)e_{AB}$, $h\in H$, the correspondence between the root vectors and their roots is $e_{AB}\to e^A-e^B$. Therefore, $\Delta^+_0=\{e^i-e^j\,|i<j,\ i,j=1,\ldots,n\}$ and $\Delta^+_1=\{e^0-e^i\,|i=1,\ldots,n\}$ are the even and the odd positive roots of $s\ell(1,n)$, correspondingly.

The induced $s\ell(1,n)$-module $\bar{V}(L)$ is irreducible if and only if /1/

$$(L+\rho,\ e^0-e^i)\neq 0 \text{ for all } i=1,\ldots,n, \tag{12}$$

where ρ is the half sum of the even positive roots minus the half sum of the odd positive roots,

$$\rho = -\frac{n}{2}e^0 + \frac{1}{2}\sum_{i=1}^{n}(n-2i+2)e^i \tag{13}$$

and L is the highest weight of the $g\ell(n)$-module $V^0(L)$. From the Gel'fand-Zetlin formulae for $g\ell(n)$ /10/ one derives that any Cartan element $h=\sum_{i=1}^{n}\xi^i E_{ii}$ acts on the highest weight vector $m_L\in V^0(L)$ as

$$hm_L = \sum_{i=1}^{n}\xi^i E_{ii}m_L = \sum_{i=1}^{n}\xi^i m_{in}m_L = \Big(\sum_{i=1}^{n}m_{in}e^i\Big)(h)m_L .$$

Therefore, as an element from $\overset{*}{H}{}'$ L reads

$$L = \sum_{i=1}^{n}m_{in}e^i . \tag{14}$$

Inserting (13) and (14) in (12) and using (11) one obtains *

$$(L+\rho,\ e^0-e^i) = \frac{m_{in}-i+1}{2(1-n)}. \tag{15}$$

The right-hand side differs from zero if and only if $m_{in}-i+1\neq 0$, which completes the proof.

By definition the representations of the LS $s\ell(1,n)$ realized in the irreducible modules $\bar{V}(L)$ are typical. Thus, the relations (9) give expressions for the generators of the typical representations. In order to classify them we recall /1/ that the finite-dimensional irreducible representations of $s\ell(1,n)$ are labelled with the eigenvalues $(a_0, a_1,\ldots,a_{n-1})$ of the Cartan elements $h_A=e_{AA}-g_{AA}e_{A+1,A+1}$, $A=0,1,\ldots,n-1$ on the highest weight vector $\bar{L}\in\bar{V}(L)$. From (9) one is easily convinced that the simple positive root vectors $e_{i-1,i}$ and hence all positive root vec-

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* We consider always the case $n>1$. Otherwise $s\ell(1,n)$ is not simple and its Killing form is degenerate.

tors $e_{ij}, i<j$, annihilate $|0,\ldots,0;L\rangle$. Therefore, $\bar{L}=|0,\ldots,0;L\rangle$ in the typical modules $\bar{V}(L)$. Since

$$h_0\bar{L} = m_{1n}\bar{L}, \quad h_i\bar{L} = (m_{in} - m_{i+1,n})\bar{L}, \quad i = 1,\ldots,n-1, \tag{16}$$

we conclude that the typical representations are characterized with the set of all n-tuples

$$(m_{1n}, m_{1n} - m_{2n}, \ldots, m_{n-1,n} - m_{nn}), m_{1n} \neq 0, \; m_{2n} \neq 1, \ldots, m_{nn} \neq n-1 \tag{17}$$

To construct the rest of the finite-dimensional irreducible representation, i.e., the non-typical representations, one has to overcome one essential difficulty, namely to determine the maximal invariant submodules $I(L)$ in $\bar{V}(L)$ and then write the relations (9) in the factor-modules $V(L) = \bar{V}(L)/I(L)$.

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Typical Representations of the Lie Superalgebra $s\ell(1,n)$

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The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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