объединенный
институт
ядерных
исследований
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G.Nenciu

ADIABATIC THEOREM AND SPECTRAL CONCENTRATION.

I. Arbitrary Order Spectral Concentration for the Stark Effect in Atomic Physics

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1. INTRODUCTION

The Stark effect in atomic physics has played an important role in the development of the theory of spectral concentration for self-adjoint operators. After the pioneering work by Titchmarch on the Stark effect in hydrogen a general theory of spectral concentration has been built up (see [1], notes to Chap. XII.5 for details). When applied to the Stark Hamiltonian, this general theory readily gives the first order spectral concentration(ref.[1] Chap.XII.5).Unfortunately,the situation is not the same for higher orders. The fact that the Stark Hamiltonian meets the conditions of the abstract theory, implying spectral concentration of arbitrary order, has been verified by Riddell[2] and by Conely and Rejto [3] for the hydrogen atom and by Rejto[4,5] for the helium atom, but to the best of our knowledge, no similar results exist for more complex situations.

On the other hand, recently, the complex and powerful machineries of dilatation analyticity,translation analyticity and complex scaling have been used to obtain a remarkable detailed description of the Stark effect in hydrogen[6-9]. Moreover similar results for arbitrary atoms are announced [9].The price one has to pay is that the proofs are far from being simple and depend on some peculiar (and remarkable) properties of the concrete Hamiltonians involved (e.g., the fact that $-\frac{d^2}{dx^2}+\epsilon x$ has empty spectrum for $\mathrm{Im}\,\epsilon \neq 0$ [7]).

In this paper which is the first in a series we shall take a somewhat complementary point of view:the proof of the existence of pseudo-eigenvalues and pseudo-eigenvectors of arbitrary order (see [1,2] and Def.1 below) should not depend on the very concrete form of the Hamiltonians involved. For, one must look for an abstract theorem, powerful enough to readily give the spectral concentration of any order, when applied to the Stark Hamiltonians atomic physics: atoms and molecules, impurity states in solids, relativistic hydrogen atom, etc. Such a result is provided by the Theorem 1 in Section 2. A slight modification in the proof of Theorem 1 gives also a refined form of the adiabatic theorem of quantum mechanics generalizing a recent result of the author[10], as well as some results of Lenard [11].

On the technical side, in the proofs of Theorems 1 and 2, there are no essentially new ideas besides those already appeared in [10,12]. We shall give here the proofs in an abstract form to emphasize their simplicity and the fact that both the spectral concentration phenomena and the adiabatic theorem, are facets of a new way of performing the perturbation theory. Moreover, for the sake of simplicity we shall not state and prove the results in the most general possible case. Some of the simple extensions are pointed out in Remarks.

Section 3 contains applications of the general theory to the Stark effect and to the barrier penetration phenomena.

2. THE GENERAL THEORY

We shall start with the following definition.

Definition 1. Let H_ϵ, P_ϵ, $\epsilon \geq 0$ be families of self-adjoint operators and orthogonal projections, respectively, in a Hilbert space, $\mathcal{H}$, satisfying the conditions:

i. $\lim_{\epsilon \to 0} \| P_\epsilon - P_0 \| = 0.$ (2.1)

ii. Let p be a positive integer. There exist $c_p < \infty$, $\epsilon_p > 0$ and bounded self-adjoint operators B_ϵ defined for $\epsilon \in [0, \epsilon_p]$ such that

$$\| B_\epsilon \| \leq c_p \, \epsilon^{p+1} \tag{2.2}$$

and $P_\epsilon \mathcal{H}$ are invariant subspaces of $H_\epsilon + B_\epsilon$. Then the family $P_\epsilon \mathcal{H}$ of subspaces is said to be an asymptotically invariant family of subspaces of order p for H_ϵ.

Remarks

1. The definition requires that $P_0 \mathcal{H}$ is an invariant subspace of H_0.

2. For ϵ sufficiently small, $\dim P_\epsilon = \dim P_0$. The case $\dim P_0 = \infty$ appears naturally in some problems of solid state physics [12].

The connection of the above definitions with the spectral concentration is given by the following proposition (for definitions see [2]).

Proposition 1. Suppose that

i. H_ϵ has an asymptotically invariant family of subspaces of order p, with P_0 corresponding to an isolated finitely degenerated eigenvalue, λ.

ii. $H_\epsilon \to H_0$ in the strong resolvent sense as $\epsilon \to 0$. Then in every isolating interval for λ the spectrum of H_ϵ is concentrated to order p.

Proof. For ϵ small enough $\dim P_\epsilon = \dim P_0 < \infty$ and then $\exp(-i(H_\epsilon + B_\epsilon)t) P_\epsilon \mathcal{H} = P_\epsilon \mathcal{H}$ implies that there exist $\lambda_j(\epsilon)$, $\phi_j(\epsilon)$, $j = 1,2,\dots,\dim P_0$, $(\phi_i(\epsilon), \phi_j(\epsilon)) = \delta_{ij}$ $\phi_j(\epsilon) \in \mathfrak{D}(H_\epsilon)$, $\{\phi_j(\epsilon)\}_1^{\dim P_0}$ is a basis in $P_\epsilon \mathcal{H}$, and

$$(H_\epsilon + B_\epsilon)\phi_j(\epsilon) = \lambda_j(\epsilon)\phi_j(\epsilon). \tag{2.3}$$

Then (2.1), (2.2) imply that $\{\phi_j(\epsilon)\}_1^{\dim P_0}$ is an asymptotic basis of order p (see [2] Definition 2.5) and the spectral concentration is implied by the easy part of the main result of Riddell (ref. [2] Th.2.7).

As expected, $P_\epsilon \mathcal{H}$ are almost invariant under the evolution given by H_ϵ.

<u>Proposition 2.</u> Suppose H_ϵ has an asymptotically invariant family of subspaces, $P_\epsilon \mathcal{H}$, of order p. Then

$$\|(1-P_\epsilon)\exp(-iH_\epsilon t)P_\epsilon\| \le c_p \epsilon^{p+1} t, \qquad \epsilon \in [0, \epsilon_p]. \tag{2.4}$$

The proof is easy and we shall omit it. In particular if P_ϵ is one-dimensional, then due to (2.4), ϕ_ϵ has, for small ϵ, a rather long life-time. This, together with the fact that $\lim_{\epsilon\to 0} \|P_\epsilon - P_0\| = 0$, says, in the language of physicists, that ϕ_ϵ describes a metastable state.

Suppose now that H_ϵ is of the form $H_\epsilon = H_0 + \epsilon X_0$, where H_0, X_0 are self-adjoint operators in $\mathcal{H}$. The problem is to find conditions on the pair H_0, X_0 under which one can prove the existence of asymptotically invariant subspaces for H_ϵ. The following heuristic discussion gives a hint. Let $\tau_t(X_0;\cdot)$ be the automorphism of $\mathcal{B}(\mathcal{H})$ (the Banach algebra of bounded operators in $\mathcal{H}$) given by

$$\tau_t(X_0;A) = e^{iX_0 t} A e^{-iX_0 t} \tag{2.5}$$

and $\operatorname{ad} X_0$ its generator. Suppose that $H_0 \in \ker \operatorname{ad} X_0$ in the sense that $(H_0 - z)^{-1} \in \ker \operatorname{ad} X_0$ for all $z \in \rho(H_0)$. Then all the invariant subspaces of H_0 are invariant subspaces of H_ϵ. On the other hand, if X_0 is bounded, i.e., the domain of $\operatorname{ad} X_0$ is the whole $\mathcal{B}(\mathcal{H})$, then for an arbitrary H_0, the usual perturbation theory provides convergent sequences of asymptotically invariant subspaces of H_ϵ. By some rearrangements of the perturbation series one can see that objects like $(\operatorname{ad} X_0)^p (H_0 - z)^{-1}$ appear. The above extreme situations suggest that, when X_0 is unbounded, one may still hope that some sort of perturbation theory can be performed if $(H_0-z)^{-1} \in \mathfrak{D}((\mathrm{ad}X_0)^p)$, $p=1,2,\dots$. That this is indeed the case says Theorem 1 below. Before stating the theorem let us remark that $(H_0-z)^{-1} \in \mathfrak{D}((\mathrm{ad}X_0)^p)$ is equivalent with the fact that $\tau_t(X_0;(H_0-z)^{-1})$ is p times norm differentiable with respect to t.

Theorem 1. Suppose that:

i. $H_\epsilon = H_0 + \epsilon X_0$ is essentially self-adjoint on $\mathfrak{D}(H_0) \cap \mathfrak{D}(X_0)$.

ii. $\tau_t(X_0;(H_0 \pm i)^{-1})$ is p+1 times norm differentiable.

iii. There exist, $-\infty < \lambda_1 < \lambda_2 < \infty$, such that the spectrum σ_0 of H_0 has the properties: $\sigma_0 = \sigma_0^1 \cup \sigma_0^2$ $\sigma_0^1 \subset [\lambda_1, \lambda_2]$, $\mathrm{dist}(\sigma_0^1, \sigma_0^2) = d > 0$.

Let P_0 be the spectral projection of H_0 corresponding to σ_0^1. Then H_ϵ has asymptotically invariant families of subspaces, of order q, $P_\epsilon^q \mathcal{H}$, $q = 0,1,\dots,p$, with $P_0^q = P_0$.

Proof. For simplicity, and having in mind the examples in Section 3, we shall consider the case $p=\infty$. The proof is by construction, and is divided in a series of steps.

1 . We shall start with the following, almost obvious lemma.

Lemma 1. Let $H_0(t)$ be defined by $H_0(t) = e^{i\epsilon X_0 t} H_0 e^{-i\epsilon X_0 t}$, $R_0(t;z) = \tau_{\epsilon t}(X_0;(H_0 - z)^{-1})$ be its resolvent and $P_0(t) = \tau_{\epsilon t}(X_0;P_0)$ its spectral projection corresponding to σ_0^1. Then $R_0(t;z)$; $z \in \rho(H_0)$, $P_0(t)$ are indefinitely norm differentiable, and there exist finite constants $b_{0,m}(z)$, $c_{0,m}$; $m = 1,2,\dots$ such that

$$\left\| \frac{d^m}{dt^m} R_0(t;z) \right\| = \left\| \left[\frac{d^m}{dt^m} R_0(t;z) \right]_{t=0} \right\| \le b_{0,m}(z)\, \epsilon^m \tag{2.6}$$

$$\left\| \frac{d^m}{dt^m} P_0(t;z) \right\| = \left\| \left[\frac{d^m}{dt^m} R_0(t;z) \right]_{t=0} \right\| \le c_{0,m}\, \epsilon^m . \tag{2.7}$$

Proof. For $z = \pm i$ (2.6) holds by hypothesis. For arbitrary $z \in \rho(H_0)$, one has to use the identity

$$R_0(t;z) = R_0(t;z_0)[1 + (z - z_0) R_0(t;z_0)]^{-1} . \tag{2.8}$$

Finally, (2.7) follows from (2.6) and the usual formula relating the resolvent and spectral projections.

2 . We shall use the following construction, which has been given by Kato [13,14].

Lemma 2. Let $P(t)$ be a norm differentiable family of orthogonal projections, with norm continuous derivative.

i. If $K(t)$ is defined by

$$K(t) = i(1-2P(t))\frac{d}{dt}P(t) \tag{2.9}$$

then $K(t)$ is self-adjoint.

ii. The equation

$$i\frac{d}{dt}A(t) = K(t)A(t); \quad A(0) = 1 \tag{2.10}$$

has a unique solution satisfying $A^{-1}(t) = A^*(t)$ and

$$P(t) = A(t)P(0)A^*(t). \tag{2.11}$$

3 . Let $K_0(t), A_0(t)$ be given by Lemma 2 applied to $P_0(t)$ and

$$B_0 = \epsilon^{-1}K_0(0). \tag{2.12}$$

Note that $\|B_0\| \le c_{0,1}$. Consider now the self-adjoint operator

$$X_1 = X_0 + B_0; \quad \mathfrak{D}(X_1) = \mathfrak{D}(X_0). \tag{2.13}$$

By the Stone theorem, for all $f \in \mathfrak{D}(X_0)$

$$i\frac{d}{dt}(e^{i\epsilon X_0 t}e^{-i\epsilon X_1 t}) = K_0(t)e^{i\epsilon X_0 t}e^{-i\epsilon X_1 t},$$

which together with Lemma 2 implies

$$A_0(t) = e^{i\epsilon X_0 t}e^{-i\epsilon X_1 t}. \tag{2.14}$$

From (2.11) and (2.14) one has

$$P_0 = e^{i\epsilon X_1 t}P_0 e^{-i\epsilon X_1 t}. \tag{2.15}$$

which implies that for $f \in \mathfrak{D}(X_0) \cap \mathfrak{D}(H_0)$

$$P_0(H_0 + \epsilon X_1)f - (H_0 + \epsilon X_1)P_0 f = 0, \tag{2.16}$$

Since $H_0 + \epsilon X_1 = H_\epsilon + \epsilon B_0$ is essentially self-adjoint on $\mathfrak{D}(X_0) \cap \mathfrak{D}(H_0)$ it follows that

$$[P_0, \exp(-i(H_\epsilon + \epsilon B_0)t)] = 0, \tag{2.17}$$

which says that $P_\epsilon^0 = P_0$ is asymptotically invariant of order zero for H_ϵ.

4. Consider now $H_1(t)$ given by

$$H_1(t) = A_0^*(t)[H_0(t) - K_0(t)]A_0(t). \tag{2.18}$$

Note that

$$H_1(t) = \exp(i\epsilon X_1 t) H_1 \exp(-i\epsilon X_1 t); \quad H_1 - H_0 - \epsilon B_0. \tag{2.19}$$

From the identity (valid for dist $(z, \sigma_0) > \epsilon \|B_0\|$)

$$R_1(t;z) = A_0^*(t) R_0(t;z)[1 - K_0(t) R_0(t;z)]^{-1} A_0(t) \tag{2.20}$$

and Lemma 1 it follows that $R_1(t;z)$ is indefinitely norm differentiable.

5 . For $\epsilon < \epsilon_0 = d/2\|B_0\|$ the spectrum of H_1 is still separated and we can perform again the construction from the 3rd point. Obviously, one can continue this process indefinitely, the values of ϵ for which the $q+1$ step can be done, being

$$\epsilon < \epsilon_q = d/2 \sum_{j=0}^{q} \|B_j\|. \tag{2.21}$$

At the step q, if P_ϵ^q is the spectral projection of

$$H_q = H_0 - \epsilon \sum_{j=0}^{q-1} B_j$$

corresponding to the part of the spectrum which coincides with σ_0^1 in the limit $\epsilon \to 0$, then

$$[P_\epsilon^q, \exp(-i(H_\epsilon + \epsilon B_q)t)] = 0 \tag{2.22}$$

so the only thing we have to do, in order to finish the proof of the theorem is to obtain bounds on $\|B_q\|$.

6 . The needed bounds are consequences of the following Lemma which is the main (and only) technical point of our paper.

Lemma 3. Let Γ be a contour (of finite length) surrounding σ_0^1, satisfying dist$(\Gamma, \sigma_0) = d/2$. Then there exist constants $b_{p,m}$, $c_{p,m}$, $p = 0,1,\ldots$; $m = 1,2,\ldots$ such that for $\epsilon < \epsilon_{p-1}$ (by definition $\epsilon_{-1} = \infty$) and $z \in \Gamma$

$$\left\|\frac{d^m}{dt^m} R_p(t;z)\right\| \leq b_{p,m}\, \epsilon^m \tag{2.23}$$

$$\left\|\frac{d^m}{dt^m} P_p(t)\right\| \leq c_{p,m}\, \epsilon^{p+m}. \tag{2.24}$$

Proof. The proof is by induction over p. The case $p=0$ is contained in Lemma 1. Suppose (2.23), (2.24) be true for $p-1$. Then (2.23) for p follows from a formula similar to (2.20) relating $R_p(t;z)$ and $R_{p-1}(t;z)$ and the induction hypothesis. For (2.24) the following observation [10] is crucial. From

$$P_{p-1}(t) = A_{p-1}(t) P_{p-1}(0) A^*_{p-1}(t)$$

it follows that $P_{p-1}(0)$ is the spectral projection of $A^*_{p-1}(t) H_{p-1}(t) A_{p-1}$ corresponding to σ^1_{p-1}, for all $t \in \mathbf{R}$. Then one can write

$$P_p(t) - P_{p-1}(0) = (2\pi i)^{-1} A^*_{p-1}(t) \times$$
$$\times \left[\int_{\Gamma} (H_{p-1}(t) - K_{p-1}(t) - z)^{-1} K_{p-1}(t) R_{p-1}(t;z)\, dz \right] A_{p-1}(t). \tag{2.25}$$

Now, (2.25) and the induction hypothesis implies (2,24) for p to be true, and the proof of the lemma is finished.

7 . From the definition of $K_p(t)$ and (2.24) for $m=1$ it follows

$$\|B_p\| \le c_{p,1}\, \epsilon^p, \qquad \epsilon < \epsilon_{p-1} \tag{2.26}$$

which finishes the proof of the Theorem 1.

Remarks

3. One can relax the condition that σ^1_0 be bounded, but then one needs that $\|\frac{d^m}{dt^m} R_0(t;z)\|$ have sufficiently rapid decrease for $\mathrm{dist}(z,\sigma_0) \to \infty$ in order to assure the convergence of integrals appearing in $\frac{d^m}{dt^m} P_0(t)$.

4. The whole proof works for H_ϵ of the type

$$H_\epsilon = H_0 + X_0(\epsilon)$$

as far as $R_0(t;z) = r_t(X_0(\epsilon);(H_0-z)^{-1})$ is indefinitely norm differentiable and satisfies (2.6).

5. The assumption ii of Theorem 1 already implies that $\mathfrak{D}(H_0) \cap \mathfrak{D}(X_0)$ is dense in $\mathcal{H}$. In fact we suspect that it implies assumption i. The assumption i. has been used to obtain (2.17) from (2.16). If $H_\epsilon = H_0 + \epsilon X_0$ has several self-adjoint extensions and $\dim P_0 < \infty$ then (2.16) implies (2.17) for any self-adjoint extension of $H_0 + \epsilon X_0$.

Formally, the recurrent construction in the proof of Theorem 1 is the following

$$B_q = (1 - 2P_\epsilon^q)[P_\epsilon^q, X_q]; \quad P_\epsilon^0 \equiv P_0$$

$$X_{q+1} = X_q + B_q; \quad H_{q+1} = H_q - \epsilon B_q. \tag{2.27}$$

The observation in (2.25) is nothing but

$$[P_\epsilon^q, X_q] = [P_\epsilon^q - P_\epsilon^{q-1}, X_q]. \tag{2.28}$$

If X_0 is H_0-bounded, one expects that the recurrent construction (2.27) converges.

Proposition 3. Suppose that

i. X_0 is H_0-bounded.

ii. H_0 satisfies the spectrum condition iii. of Theorem 1.

Let Γ be the contour in lemma 3, $b = \frac{d}{2} \sup_{z \in \Gamma} \|X_0(H_0 - z)^{-1}\|$

$k = \frac{1}{2\pi} \int_\Gamma |dz|, \quad a_0 = 4bk/d.$

Then for

$$\epsilon \le d^2/2^8 k a_0 = \epsilon_c, \tag{2.29}$$

$$\|B_n\| \le (\epsilon/2\epsilon_c)^n a_0. \tag{2.30}$$

Proof. The proof is by induction. Note that $\|B_0\| \le a_0$.

Denoting $a_n = \|B_n\|$, using $R_n = R_0[1 - \epsilon(\sum_{i=0}^{n-1} B_i)R_0]^{-1}$ and the fact that $b < a_0$, we have from (2.28)

$$a_n \le 8\epsilon k d^{-2} a_{n-1} \left(1 - 2\epsilon d^{-1} \sum_{i=0}^{n-1} a_i\right)^{-2} \sum_{i=0}^{n-1} a_i; \quad n = 1, 2, \ldots \tag{2.31}$$

as far as

$$2\epsilon d^{-1} \sum_{i=0}^{n-1} a_i < 1. \tag{2.32}$$

Then (2.30) follows from (2.29) and (2.31) by induction.

Remarks

6. Proposition 3 shows that, for regular perturbations, the construction in Theorem 1 is nothing but a different way to perform the perturbation theory. Moreover, in the general case, Theorem 1 implies that the formal perturbation theory for isolated finitely degenerated eigenvalues is finite to any order, and coincides with the Taylor expansions of ϕ_ϵ, λ_ϵ (see Proposition 2).

7. Using (2.27) and (2.28) one can give a "time independent" proof of Theorem 1. We preferred the above proof since

with few modifications it gives also a rather general form of the adiabatic theorem in quantum mechanics,which in some sense is the generalization of Theorem 1 to time-dependent Hamiltonians(see Theorem 2 below).Here we shall state and prove the adiabatic theorem only for bounded Hamiltonians,in order not to obscure the simplicity of the proof. In the second paper of this series we shall consider the general case of unbounded time-dependent Hamiltonians, where some technical point related to the possible nondifferentiability of the unitary propagators arise (ref.[15] Chap.X.12).

8. Under the conditions of Theorem 1, one cannot expect to obtain bounds on $c_{p,1}$ in (2.26), as a function of p. In the third paper of this series we shall explore the consequences of replacing the condition ii. of Theorem 1, by the following stronger one: $R_0(t;z)$ is, as a function of t, analytic in the strip $|\mathrm{Im}\, t| < a$ for some $a > 0$, or in other words $(H_0-z)^{-1}$ is an analytic vector for $(\mathrm{ad}\, X_0)$.

Theorem 2. Let $H(s)$, $s \in I = [0,S]$ be a norm continuous family of bounded self-adjoint operators satisfying the conditions

i. $\sigma(H(s)) = \sigma_1(s) \cup \sigma_2(s)$

$$\inf_{s \in I} \mathrm{dist}(\sigma_1(s), \sigma_2(s)) = d > 0 \tag{2.33}$$

ii. $R(s; \pm i) = (H(s) \mp i)^{-1}$ are indefinitely norm differentiable.

Let $U_\epsilon(s)$ be the unique solution of the Schrödinger equation

$$i\epsilon \frac{dU_\epsilon(s)}{ds} = H(s)\, U_\epsilon(s); \quad U_\epsilon(0) = 1 \tag{2.34}$$

and $P_0(s)$ be the spectral projection of $H(s)$ corresponding to $\sigma_1(s)$.

Then, for every positive integer q, there exist $\epsilon_q > 0$, $a_q < \infty$ and orthogonal projections $P_q^\epsilon(s)$ defined for $0 < \epsilon \le \epsilon_q$ such that

$$\lim_{\epsilon \to 0} \| P_q^\epsilon(s) - P_0(s) \| = 0 \tag{2.35}$$

$$\| U_\epsilon(s) P_q^\epsilon(0) - P_q^\epsilon(s) U_\epsilon(s) \| \le a_q \epsilon^q s; \quad s \in I. \tag{2.36}$$

Proof. Let $H_0(t)$ be defined by $H_0(t) = H(\epsilon t)$. The construction in the proof of Theorem 1 gives $H_q(t)$, $P_q(t)$, $K_q(t)$, $A_q(t)$ and the existence of a_q, ϵ_q such that

$$||K_q(t)|| \le a_q \epsilon^{q+1}; \quad t \in [0, \epsilon^{-1} S]; \quad 0 < \epsilon \le \epsilon_q; \quad q = 1,2,\dots . \tag{2.37}$$

Denote $Z_q(t) = \prod_{i=0}^{q-1} A_i(t), \quad q = 1,2,\dots$

$$B_0(t) = K_0(t); \quad B_q(t) = Z_q(t) K_q(t) Z_q^*(t) \tag{2.38}$$

and

$$H^q(t) = H_0(t) + \sum_{i=0}^{q-1} B_i(t) . \tag{2.39}$$

By construction

$$H_q(t) = Z_q^*(t) H^q(t) Z_q(t) . \tag{2.40}$$

Let $P_\epsilon^q(t)$ be the spectral projection of $H^q(t)$ corresponding to the part of the spectrum which coincides with $\sigma_1(t)$ in the limit $\epsilon \to 0$. Obviously

$$P_\epsilon^q(t) = Z_q(t) P_q(t) Z_q^*(t) . \tag{2.41}$$

Let $U(t)$, $V_q(t)$, $W_q(t)$ be defined by

$$U(t) = U_\epsilon(\epsilon t), \qquad t \in [0, \epsilon^{-1} S] . \tag{2.42}$$

$$i \frac{d}{dt} V_q(t) = A_q^*(t) H_q(t) A_q(t) V_q(t); \quad V_q(0) = 1, \tag{2.43}$$

$$U(t) = Z_q(t) A_q(t) V_q(t) W_q(t) . \tag{2.44}$$

By construction, since $P_\epsilon^q(0) = P_q(0)$

$[A_q^*(t) H_q(t) A_q(t), P_\epsilon^q(0)] = 0$

wherefrom

$$[V_q(t), P_\epsilon^q(0)] = 0. \tag{2.45}$$

By construction

$i \frac{d}{dt} W_q(t) = -V_q^*(t) A_q^*(t) K_q(t) A_q(t) V_q(t) W_q(t)$

which together with (2.37) gives

$$||W_q(t) - 1|| \le t a_q \epsilon^{q+1} . \tag{2.46}$$

On the other hand from (2.41) and (2.45)

$$Z_q(t)\, A_q(t)\, V_q(t)\, P^q_\epsilon(0) = P^q_\epsilon(t)\, Z_q(t)\, A_q(t)\, V_q(t)$$

which together with (2.44) and (2,46) implies

$$\| P^q_\epsilon(t)\, U(t) - U(t)\, P^q_\epsilon(0) \| \le t\, a_q\, \epsilon^{q+1} \qquad (2.47)$$

which is nothing but (2.36) with the identifications (2.42) and $P^\epsilon_q(s) = P^q_\epsilon(\epsilon t)$.

Remarks.

9. Suppose that $H(s)$ is constant in some neighbourhoods of 0 and S. Then $P_0(0) = P^p_\epsilon(0)$, $P_0(S) = P^p_\epsilon(S)$ for all p and in this case (2.36) for $s = S$ reduces to an infinite-dimensional generalization of Lenard's results (see also ref. [18] for related results in classical mechanics).

3. APPLICATIONS

1. Let M be a positive integer and $\alpha = \{\alpha_{ij}\}^M_{i,j=1}$ be a real, strictly positive $M \times M$ matrix. Consider in the Hilbert space $L^2(\mathbb{R}^M)$ the operators T, V, X_0 defined by

$$T = \sum_{i,j=1}^{M} \alpha_{ij}\, P_i\, P_j\, ; \quad P_i = -i\,\partial/\partial x_i\, ; \quad i = 1, \ldots, M \qquad (3.1)$$

$$(Vf)(\vec{x}) = V(\vec{x})\, f(\vec{x}) \qquad \vec{x} \in \mathbb{R}^M, \qquad (3.2)$$

$$(X_0 f)(\vec{x}) = \Big(\sum_{j=1}^{M} c_j x_j \Big) f(\vec{x})\, ; \quad \vec{x} = (\vec{x}_1, \ldots, x_M),\ C_j \in \mathbb{R} \qquad (3.3)$$

on their natural domains. Suppose that V is T-bounded with relative bound less than one, so that $T + V$ is self-adjoint on $\mathcal{D}(T)$ ((ref. [15] Chap.X.2).

Proposition 4. The operators $H_0 = T + V$ and X_0 defined by (3.1)-(3.3) satisfy the conditions i, ii of Theorem 1.

Proof. For condition i see (ref. [15] Th. X.38). For condition ii. remark that

$$H_0(t) = \exp(i\epsilon X_0 t) H_0 \exp(-i\epsilon X_0 t) = \sum_{i,j=1}^{M} \alpha_{ij} (P_i + \epsilon c_i t)(P_j + \epsilon c_j t) + V \qquad (3.4)$$

wherefrom the verification is straightforward. Obviously, this example covers the Stark effect in arbitrary atoms and molecules (see for example the form of the Hamiltonian in Zhislin's theorem (ref. [1] Th.XIII.7)). For $M = 3$, $a_{ij} = \frac{1}{2m}\delta_{ij}$, $V(\vec{x}) = V_1(\vec{x}) + V_2(\vec{x})$, where V_1 is periodic and locally L^2 (see ref. [1] Th.XIII.96) and $V_2 \in L^2(R^3) + L^p(R^3)$ $2 \le p < \infty$ the above example describes the Stark effect for impurity states in solid state physics.

2. (The Dirac Eq.). The Hilbert space of the problem is $(L^2(R^3))^4$,

$$T = \sum_{i=1}^{3} a_i P_i + \beta m, \tag{3.5}$$

where a_i, β are the Dirac 4x4 constant matrices,

$$(V\psi)_i(\vec{x}) = \sum_{j=1}^{4} V_{ij}(\vec{x})\psi_j(\vec{x})\ ;\ V_{ij}(\vec{x}) = V_{ji}(\vec{x}) = \overline{V_{ij}(\vec{x})} \tag{3.6}$$

and

$$(X_0\psi)_i(\vec{x}) = (\sum_{j=1}^{3} c_j x_j)\psi_i(\vec{x}), \quad \vec{x} = (x_1, x_2, x_3), \quad c_i \in R. \tag{3.7}$$

Again we shall suppose that V is T-bounded with relative bound less than one so that $T + V - H_0$ is self-adjoint on $\mathcal{D}(T)$.

Proposition 5. The operators H_0, X_0 defined by (3.5)-(3.7) satisfy the conditions i,ii of Theorem 1.

Proof. For i see ref. [16] For ii see the proof of proposition 4.

3. (Barrier penetration (for details see ref. [17])).

Consider in $L^2(R^3)$ the operators

$$H_\epsilon = -\Delta + V(\vec{x}) + X_0(\epsilon) = H_0 + X_0(\epsilon) \tag{3.8}$$

with $V \in L^2(R^3)$ and

$$(X_0(\epsilon)f)(\vec{x}) = K(\exp(-\epsilon|\vec{x}|) - 1) f(\vec{x})\ ;\ K > 0,\ \epsilon > 0. \tag{3.9}$$

Suppose that H_0 has eigenvalues in $(-K, 0)$ For all $\epsilon > 0$, $(-K, 0)$ is contained in the continuum spectrum of H_ϵ. As $\epsilon \to 0$ the spectrum of H_ϵ contained in $(-K, 0)$ shows arbitrary order spectral concentration. In this case the self-adjointness problem is trivial. Concerning the condition ii in Theorem 1 see Remark 4.

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