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LAGRANGIAN DISTRIBUTIONS AND THEIR SINGULARITIES.

II. CALCULUS OF SINGULARITIES AND APPLICATIONS

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1. THE CASE WHEN THE RANK OF THE ξ-PROJECTION OF Λ IS CONSTANT*

Let $u \in \mathcal{S}'(R^n)$ be a Lagrangian distribution, singspec $u \subset \Lambda$ and let the ξ-projection of Λ has a constant rank, but is not necessarily one-to-one. Such is, for instance, the Schwartz kernel $K(x,y)$ of a pseudodifferential operator, where $(x,y) \in R^{2n}$. In this case Λ is the normal bundle of the diagonal $x=y$, so that the rank of the projection is not $2n$ but n, hence the projection is not one-to-one. It is possible to carry out the procedure of the previous section for defining the symbol of u, but it is simpler in that case to use another symbol which also characterizes, the singularities, and which for the example of Schwartz kernel coincides with the usual symbol of pseudodifferential operators.

First of all let us prove that Λ may be represented by

$$\begin{aligned} x_I &= F_1(x_{\bar I}, \xi_I), \\ \xi_{\bar I} &= F_2(\xi_I), \end{aligned} \qquad (1.1)$$

where $I, \bar I$ are subsets of $\{1,2,\dots,n\}$, $I \cap \bar I = \emptyset$, $I \cup \bar I = \{1,\dots,n\}$. Indeed there obviously exist representations of Λ of the form

$$\begin{aligned} x_I &= \Phi_1(x_{\bar I}, \xi_I), \\ \xi_{\bar I} &= \Phi_2(x_{\bar I}, \xi_I). \end{aligned} \qquad (1.2)$$

Between all the representations of Λ of the form (1.2) we choose one for which the number of elements of I, i.e., the number of ξ-variables in the right-hand part, is maximal. In that case Φ_2 doesn't depend on $x_{\bar I}$, i.e., $\partial\Phi_2/\partial x_{\bar I}=0$, because, if for instance, $j \in \bar I$, $\xi_j = \Phi_{2,j}(x_{\bar I}, \xi_I)$ and $\dfrac{\partial \Phi_{2,j}}{\partial x_i} \neq 0$ for some $i \in \bar I$, then, by the implicit function theorem, x_i will be represented as a function of ξ_j, ξ_I, $x_{\bar I \setminus \{i\}}$ and substituting x_i in (1.2) we get a representation of Λ of the form (1.2), in which the right-hand parts depend on $x_{\bar I \setminus \{i\}}, \xi_j, \xi_I$. This contradicts the assumption of maximality of the number of ξ-variables in (1.2).

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* This paper is the continuation of ref. [9] and (k)[9] means formula (k) from ref. [9].

It is easy to see that the number of elements of I in the representation (1.1) equals the rank r of the ξ-projection of Λ.

Now consider the trace of u on the subspace $x_{\bar I}=x^{\circ}_{\bar I}$. Such trace exists since (see, ref. /5/) the conormal bundle of the subspace $x_{\bar I}=x^{\circ}_{\bar I}$ does not intersect Λ, as is easily seen from (1.1). We denote this trace by $u(\circ, x^{\circ}_{\bar I})$ The singular spectrum of $u(\circ, x^{\circ}_{\bar I})$ is contained in the Lagrangian manifold $\Lambda' \subset R^{2r}_{x_I,\xi_I}$ determined by the equations

$$x_I = F_1(x^{\circ}_{\bar I}, \xi_I). \tag{1.3}$$

Thus the ξ_I-projection of Λ' is one-to-one and $u(\circ, x^{\circ}_{\bar I})$ is a primitive Lagrangian distribution. Take the symbol $\mathcal{G}[u(\circ, x^{\circ}_{\bar I})](x_I,\xi_I)$ as defined in section 2 /9/. Here $(x_I,\xi_I)\in\Lambda'$ and consequently (1.3) holds. We denote

$$\mathcal{G}'[u](x_{\bar I},\xi_I) = [u(\circ, x_{\bar I})](F_1(x_{\bar I},\xi_I),\xi_I)$$

and call it the modified symbol of u.

If u is the kernel $K(x,y)$ of a pseudodifferential operator

$$Av(x) = \int e^{i(x-y)\xi}\, a(x,\xi)\, v(y)\, d\xi\, dy, \qquad v\in C_0^{\infty},$$

then we can take $x_I=y$, $x_{\bar I}=x$, $\xi_I=\xi$, $\xi_{\bar I}=\eta$. The equations (1.1) are

$$y=x, \quad \eta=-\xi$$

and it is easy to calculate that

$$\mathcal{G}'[K](x,\xi)=a(x,\xi),$$

i.e., in this case the modified symbol of K is the usual symbol of the pseudidifferential operator.

2. CALCULUS OF SINGULARITIES

For a distribution $u\in\mathcal{L}_0$ it is possible to calculate $AF^{loc}[u]$ provided singspec u and $\mathcal{G}[u]$ are given.

Theorem 2.1. If $u\in\mathcal{L}_0$, then the following formula holds.

$$\operatorname{asym} F[u\phi](\lambda\xi) = e^{-i\lambda g(\xi)} \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\left(\frac{i}{\lambda}\right)^{|\alpha|} \partial_\xi^\alpha \mathcal{G}[u](g'(\xi),\xi)\,\phi^{(\alpha)}(g'(\xi)), \quad \phi\in C_0^\infty. \tag{2.1}$$

Proof Let $(x^\circ,\xi^\circ)\in\Lambda$. By the Taylor formula we have

$$\phi(x)=\sum_{|\alpha|\le N}\frac{1}{\alpha!}(x-x^\circ)^\alpha\phi^{(\alpha)}(x^\circ)+\sum_{|\beta|=N+1}\frac{1}{\beta!}(x-x^\circ)^\beta\phi^{(\beta)}(x^\circ+(x-x^\circ)\theta(x)).$$

Thus we get for $\phi\in 1_0^\infty(x^\circ)$

$$F[u\phi](\lambda\xi^\circ)=F[\psi u\phi](\lambda\xi^\circ)=\sum_{|\alpha|\le N}\frac{1}{\alpha!}\phi^{(\alpha)}(x^\circ)F[(x-x^\circ)^\alpha\psi u](\lambda\xi^\circ)+$$

$$+\sum_{|\alpha|=N+1}\frac{1}{\alpha!}F[(x-x^\circ)^\alpha\psi u\phi^{(\alpha)}(x^\circ+(x-x^\circ)\theta(x))](\lambda\xi^\circ).$$

The last term may be represented in the form

$$F[(x-x^\circ)^\alpha\psi u\phi^{(\alpha)}(x^\circ+(x-x^\circ)\theta(x))](\lambda\xi^\circ) =$$

$$=F[(x-x^\circ)^\alpha\psi u]^{\xi} * F[\phi^{(\alpha)}(x^\circ+(x-x^\circ)\theta(x))](\lambda\xi^\circ) = \int\Phi_\alpha(x^\circ,\lambda\xi^\circ-\xi)\Psi_\alpha(x^\circ,\xi)d\xi, \tag{2.2}$$

where

$$\Phi_\alpha(x^\circ,\xi)=F[\phi^{(\alpha)}(x^\circ+(x-x^\circ)\theta(x))](\xi),$$

$$\Psi_\alpha(x^\circ,\xi)=F[(x-x^\circ)^\alpha\psi u](\xi).$$

Using (2.6)[9] and standard considerations it may be proved that (2.2) is an asymptotic expansion indeed. According to (2.3)[9]

$$e^{i\lambda g(\xi^\circ)}F[(x-x^\circ)^\alpha\psi u](\lambda\xi^\circ)=\left(\frac{i}{\lambda}\right)^{|\alpha|}\partial_{\xi^\circ}(e^{i\lambda g(\xi^\circ)}F[u\psi](\lambda\xi^\circ))\in$$

$$\in\left(\frac{i}{\lambda}\right)^{|\alpha|}\partial^\alpha_{\xi^\circ}\mathcal{G}[u](g'(\xi^\circ),\xi^\circ). \tag{2.3}$$

From (2.2) and (2.3) we get (2.1) and this completes the proof.

Let u be a distribution from the class $\mathcal{L}_0$ of which we know the singular spectrum and the symbol. How can we calculate the symbol of Au, where A is a pseudodifferential operator? In order to answer this question we resort to a somewhat formal reasoning.

Let A be a pseudodifferential operator from the class $\Psi^m_{\rho,\delta}$ (see ref. /5/), $1-\rho\le\delta<\rho$ with symbol $a(x,\xi)$. Take $(x^\circ,\xi^\circ)\in\Lambda$, where Λ is a Lagrangian manifold, $\Lambda\supset$ sing spec u and let $\psi\in 1_0^\infty(x^\circ)$. Then

$$F[\psi Au](\lambda\xi^\circ)=\int e^{-i\lambda x\xi^\circ}\psi(x)Au(x)\,dx=$$

$$=\int e^{-i\lambda x.\xi^\circ}\psi(x)\left[\iint e^{i(x-y)\cdot\eta}a(x,\eta)u(y)\,dy\,d\eta\right]dx \approx \tag{2.4}$$

$$\approx\int\left[\sum_\alpha\frac{1}{\alpha!}\iint e^{-i(y-x)\cdot\eta}\partial_x^\alpha a(x^\circ,\eta)(x-x^\circ)^\alpha e^{-i\lambda x\xi^\circ}\psi(x)\,dx\,d\eta\right]u(y)\,dy=$$

$$=(-1)^n\sum_\alpha\frac{1}{\alpha!}\int A^\circ_\alpha\left[e^{-i\lambda x\xi^\circ}(x-x^\circ)\psi(x)\right](y)\,u(y)\,dy,$$

where A°_α is a pseudodifferential operator, whose symbol $\sigma[A^\circ_\alpha](y,\eta)=\partial^\alpha_x a(x^\circ,\eta)$ doesn't depend on y and belongs to class $S_{\rho,0}$ (see, ref. /5/). We apply a well-known asymptotic expansion formula for pseudodifferential operators (see ref. /5/) and obtain

$$A^{o}_{\alpha}[e^{-\lambda x\xi^{o}}(x-x^{o})\psi(x)](y) \approx e^{-i\lambda y\xi^{o}} \sum_{\beta}\frac{1}{\beta!}\partial^{\beta}_{\eta}\partial^{\alpha}_{x}a(x^{o},-\lambda\xi^{o})\, D^{\beta}_{z}[(z-k^{o})^{\alpha}\psi(z)]|_{z=y} \,. \tag{2.5}$$

By the Leibniz formula

$$D^{\beta}_{z}[(z-x^{o})^{\alpha}\psi(z)] = \sum_{\gamma\leq\beta}\binom{\beta}{\gamma} D^{\beta-\gamma}_{z}(z-x^{o})^{\alpha} D^{\gamma}_{z}\psi(z) = \tag{2.6}$$

$$= \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\frac{\alpha!}{(\alpha-\beta+\gamma)!}(z-x^{o})^{\alpha-\beta+\gamma}\; D^{\gamma}_{z}\psi(z).$$

We substitute (2.6) in (2.5) and then (2.5) in (2.4)

$$F[\psi A u](\lambda\xi^{o}) \approx (-1)^{n}\sum_{\alpha}\frac{1}{\alpha!}\sum_{\beta}\frac{1}{\beta!}\partial^{\beta}_{\eta}\partial^{\alpha}_{x}a(x^{o},-\lambda\xi^{o}) \;\times \tag{2.7}$$

$$\times \sum_{\gamma\leq\beta}\frac{\alpha!\,\beta!}{(\alpha-\beta+\gamma)!\,(\beta-\gamma)!\,\gamma!}\int e^{-i\lambda y\xi^{o}}(y-x^{o})^{\alpha-\beta+\gamma}\; D^{\gamma}_{y}\psi(y).u(y)\,dy =$$

$$=(-1)^{n}\sum_{\alpha}\sum_{\beta}\sum_{\gamma\leq\beta}\frac{1}{(\alpha-\beta+\gamma)!\,(\beta-\gamma)!\,\gamma!}\partial^{\beta}_{\eta}\partial^{\alpha}_{x}a(x^{o},-\lambda\xi^{o}) \;\times$$

$$\times \int e^{-i\lambda y\xi^{o}}(y-x^{o})^{\alpha-\beta+\gamma}\; u(y)D^{\gamma}_{y}\psi(y)\,dy.$$

For $\gamma>0$ the last integral equals zero asymptotically when $\lambda\to\infty$ since $D^{\gamma}_{y}\psi(y)=0$ in a neighbourhood of x^{o}. For $\gamma=0$ we get in view of (2.4)[9]

$$\int e^{-i\lambda y\xi^{o}}(y-x^{o})^{\alpha-\beta}\; u(y)\psi(y)\,dy \approx e^{-i\lambda x^{o}\xi^{o}}\; s[(y-x^{o})^{\alpha-\beta}\; u(y)](x^{o},\xi^{o};\lambda)\approx$$

$$\approx \left(\frac{i}{\lambda}\right)^{|\alpha-\beta|}\partial^{\alpha-\beta}_{\xi^{o}}\, s[u](g'(\xi^{o}),\xi^{o};\lambda). \tag{2.8}$$

Substituting (2.8) into (2.7) we get

$$s[Au](x^{o},\xi^{o};\lambda) \approx e^{i\lambda x^{o},\,\xi^{o}}\; F[\psi Au](\lambda\xi^{o}) \approx \tag{2.9}$$

$$\approx (-1)^{n}\sum_{\alpha}\sum_{\beta}\frac{1}{(\alpha-\beta)!\;\beta!}\left(\frac{i}{\lambda}\right)^{|\alpha-\beta|}\partial^{\beta}_{\eta}\partial^{\alpha}_{x}a(x^{o},-\lambda\xi^{o})\,\partial^{\alpha-\beta}_{\xi^{o}}s[u](g'(\xi^{o}),\xi^{o};\lambda).$$

We won't justify the above reasoning, but only state the final result.

Theorem 2.2. Let be a distribution from the class $\mathcal{L}_0$ and A be a pseudodifferential operator from the class $\Psi_{\rho,\delta}, 1-\rho\leq\delta<\rho$ with symbol $a(x,\xi)$. Then the symbol of Au is given by formula (2.9)

3. HINTS ON APPLICATIONS

We will not consider any of the sophisticated and important applications of "microlocal" methods in different fields of mathematics and physics, which are the subject of an overwhelming torrent of papers during the last years. In this section we only want to give a rough idea of how the previous concepts might be used for exploring the singularities of solutions to partial differential equations.

A most important problem in analysis is to determine the singularities of the kernels of different operators connected with differential equation problems - first of all the kernels of the differential operators themselves and of their inverses, i.e., the singularities of their fundamental solutions. We already saw that the singularities of the kernel of a pseudodifferential operator are characterized by the symbol of the operator. What about the inverse operators? Consider an example.

Let P be a hyperbolic differential operator of second order. We are interested in the singularities of the Green function of the Cauchy problem for P. If we denote $x=(x_0,x_1,\dots,x_n)= (x_0, x')$, then the Green function $G(x;y')$ satisfies

$$\begin{aligned} &P_x G(x;y') = 0,\\ &G(x;y')|_{x_0=0} = 0,\\ &\frac{\partial G}{\partial x_0}(x;y')|_{x_0=0} = \delta(x'-y'). \end{aligned} \tag{3.1}$$

We want to determine the singularity of $G(x;y')$ considered as a distribution with respect to y' depending on the parameter x.

Suffice it to calculate the local Fourier asymptotic $\operatorname{asym} F_{y'}[G(x;y')\phi(y')](\lambda\eta')$. We multiply by $\phi(y')$ in (3.1) and carry out the Fourier transform with respect to y'

$$\begin{aligned} &P_x F_{y'}[G\phi](\lambda\eta') = 0,\\ &F_{y'}[G\phi](\lambda\eta')|_{x_0=0} = 0,\\ &\frac{\partial}{\partial x_0}F_{y'}[G\phi](\lambda\eta')|_{x_0=0} = e^{-i\lambda x'\eta'}\,\phi(x'). \end{aligned} \tag{3.2}$$

If $\operatorname{asym} F_{y'}[G\phi](\lambda\eta')$ may be differentiated with respect to x then (3.2)' implies

$$P_x \operatorname{asym} F_{y'}[G\phi](\lambda\eta') \approx 0,$$

$$\operatorname{asym} F_{y'}[G\phi](\lambda\eta')|_{x_0=0} \approx 0, \qquad (3.3)$$

$$\frac{\partial}{\partial x^0} \operatorname{asym} F_{y'}[G\phi](\lambda\eta')|_{x_0=0} \approx e^{-i\lambda x'\eta'} \phi(x').$$

But (3.3) means that $\operatorname{asym} F_{y'}[G\phi](\lambda\eta')$ is an asymptotic solution of the oscillatory Cauchy problem. We state this result in a slightly more general form.

Theorem 3.1. Let P be a differential operator and $G(x,y)$ be the Green function of the Cauchy problem for P. If $\operatorname{asym} F_y [G(x;y)\phi(y)](\lambda\eta')$ is differentiable with respect to x then it is an asymptotic solution of an oscillatory Cauchy problem.

This is the reason why asymptotic solutions characterize the singularities of the Green function. Obviously if we know the singularities of the Green function we can deduce information about the singularities of the solutions. Using the results of the previous section we can develop a technique for calculation of the singularities of solutions.

Let P be a differential operator and let $u \in \mathcal{L}_0$ be a solution of the $Pu=0$. What relations will satisfy the singularities of u, i.e., the singular spectrum and the symbol of u?

Theorem 3.2. Let $P = \sum_{|\alpha|<m} a_\alpha(x) D^\alpha$ be a differential operator. Denote

$$P_0(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \quad P_1(x,\xi) = \sum_{|\alpha|=m+1} a_\alpha(x)\xi^\alpha.$$

Let $u \in \mathcal{L}_0$, singspec $u \subset \Lambda = \{(x,\xi): x = g'(\xi)\}$. Suppose that the symbol of u has the asymptotic expansion

$$s[u](x,\xi;\lambda) \approx \frac{1}{\lambda^\nu} \sum_{k=0}^{\infty} \frac{r_k(x,\xi)}{\lambda^k}, \qquad (x,\xi) \in \Lambda.$$

If $Pu = 0$ and $s[u] \neq 0$ then

1. $P_0(g'(\xi),\xi) = 0$, i.e., Λ is contained in the characteristic set $P_0(x,\xi) = 0$.

2. $r_0, r_1, \ldots$ satisfy the following relations

$$H_{P_0} r_0(x,\xi) + \left(P_1 - \frac{1}{2i}\frac{\partial^2 P_0}{\partial x_j \partial \xi_j}\right) r_0(x,\xi) = 0, \qquad (3.4)$$

$$H_{P_0} r_1(x,\xi) + \left(P - \frac{1}{2i}\frac{\partial^2 P_0}{\partial x_j \partial \xi_j}\right) r_1(x,\xi) + B r_2(x,\xi) = 0,$$

where H_{P_0} is the Hamiltonian field of P_0 and B is a differential operator of second order.

The proof of this theorem is a direct application of the results of the previous section.

APPENDIX 1

ASYMPTOTICS

We recall here some well known notions and facts about asymptotics. The proofs may be found in refs.[4,7].

Let f and g be two functions $\mathbf{R}_+ \to V$, where V is a topological vector space. We say that f and g are V-asymptotically equal at infinity (and note $f \underset{\lambda\to\infty}{\overset{V}{\approx}} g$) iff for every $N>0$ we have

$$\lim \lambda^N (f(\lambda) - g(\lambda)) = 0.$$

This is obviously an equivalence relation in the set of functions $\mathbf{R}_+ \to V$ and we call the equivalence classes V-asymptotics (or V-asymptotic classes). Denote by $\mathfrak{a}(V)$ the set of all V-asymptotics. We note by

$$\underset{\lambda\to\infty}{\overset{V}{\operatorname{asym}}} f$$

the class to which f belongs and call it V-asymptotic of f.

Let the topology of V is determined by a set of functionals $\{F_i\}_{i\in I}$ in the sense that

$$u_n \xrightarrow{V} u \iff F_i(u_n - u) \xrightarrow{R^1} 0, \quad i \in I.$$

$f: \mathbf{R}_+ \to V$, $g: \mathbf{R}_+ \to V$ we say that

$$\operatorname{asym} f \prec \operatorname{asym} g$$

iff

$$F_i(f(\lambda)) \underset{\lambda\to\infty}{=} o(F_i(g(\lambda))), \quad i \in I.$$

We define operations with asymptotics: $\alpha_1+\alpha_2$, $\alpha_1\alpha_2$, etc.

A sequence of asymptotics $\{\alpha_n = \operatorname{asym}\phi_n\}$, $n=1,2,\ldots$ is called asymptotic scale iff $\alpha_{n+1} \prec \alpha_n$, $n = 1,2,\ldots$ – and for each N there is an n such that $\alpha_n \prec \operatorname{asym}(\lambda^{-N})$

Example: $a_1 = \mathrm{asym}\lambda^m$, $a_2 = \mathrm{asym}\lambda^{m-\rho}$, $a_3 = \mathrm{asym}\lambda^{m-2\rho}, \ldots$

If $\{a_n\}$ is an asymptotic scale, then

$a_1 + a_2 + \ldots$

is called an asymptotic series.

Let $\{a_n\}$ be an asymptotic scale and a be an asymptotic such that

$$a - \sum_{n=1}^{N} a_n \prec a_N \qquad \text{for each } N.$$

In that case we say that a is the sum of the asymptotic series $a_1 + a_2 + \ldots$ or that $a_1 + a_2 + \ldots$ is an asymptotic expansion of a and note

$$a \approx a_1 + a_2 + \ldots .$$

Theorem 1. Any asymptotic series has a sum and this sum is unique.

Theorem 2. Let $a_1, a_2, \ldots$ be an asymptotic scale. An asymptotic a may have not more than one asymptotic expansion of the form

$$a \approx c_1 a_1 + c_2 a_2 + \ldots ,$$

where $c_1, c_2, \ldots$ are numbers.

APPENDIX 2

LAGRANGIAN MANIFOLDS

Some elementary facts concerning Lagrangian manifolds will be enumerated. The proofs may be found in refs./1,3,4,5,8/.

Consider $R^{2n} = R^n_x \times R^n_\xi$ provided with the differential 2-form

$$\omega = d\xi_1 \wedge dx_1 + \ldots + d\xi_n \wedge dx_*,$$

which is non-degenerated.

An n-dimensional manifold $\Lambda \subset R^{2n}$ is called Lagrangian manifold iff $\omega|_\Lambda = 0$.

A Lagrangian manifold Λ may be represented locally by a single function of n variables. For instance if Λ projects one-to-one on the ξ-space, then there exist a function $g(\xi_1, \ldots, \xi_n)$ so that Λ is represented by the equations

$$x_1 = \frac{\partial g}{\partial \xi_1}, \quad \ldots \quad x_n = \frac{\partial g}{\partial \xi_n}. \qquad (1)$$

In the general case the following theorem holds:

Theorem 1. Let Λ be a Lagrangian manifold and $(x^\circ, \xi^\circ) \in \Lambda$. There exists a neighbourhood $U \subset R^{2n}$ of (x°, ξ°) and a function $g(\xi_{i_1}, \ldots, \xi_{i_k}, x_{j_1}, \ldots, x_{j_{n-k}})$ so that $\Lambda \cap U$ is represented by the equations

$$x_{i_1} = \frac{\partial g}{\partial \xi_{i_1}}, \quad \ldots \quad x_{i_k} = \frac{\partial g}{\partial \xi_{i_k}}, \quad \xi_{j_1} = -\frac{\partial g}{\partial x_{j_1}}, \quad \ldots \quad \xi_{j_{n-k}} = -\frac{\partial g}{\partial x_{j_{n-k}}}, \qquad (2)$$

where $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, 2, \ldots, n)$.

The next theorem enables us to reduce the general case (1) to the special case (1) by a suitable change of local coordinates.

Theorem 2. Let Λ be a Lagrangian manifold and $(x^\circ, \xi^\circ) \in \Lambda$. There exists a neighbourhood $U \subset R^{2n}$ of (x°, ξ°) and a change of the x-coordinates $x = \kappa(y)$, so that if we perform in U the canonical change

$$x = \kappa(y),$$

$$D\kappa^* \xi = \eta,$$

then $\Lambda \cap U$ has a one-to-one projection on the η-space.

There is a close connection between Lagrangian manifolds and Hamiltonian systems in classical mechanics. Roughly speaking any Lagrangian manifold may be foliated into Hamiltonian paths.

REFERENCES

1. Hörmander L. Fourier Integral Operators I, Acta Math., 1971, 127, p. 79-183.
2. Маслов В.П. Теория возмущений и асимптотические методы. Изд-во МГУ, М., 1965.
3. Маслов В.П., Федоркк М.В. Квазиклассическое приближение для уравнений квантовой механики. "Наука", М., 1976.

4. Sternberg S., Guillemin V. Geometric Asymptotics. Providence, AMS, 1977.
5. Treves F. Introduction to Pseudodifferential and Fourier Integral Operators. Plenum Press, New York and London, 1980.
6. Брычков Ю.А., Прудников А.П. Интегральные преобразования обобщенных функций. "Наука", М., 1977.
7. Эрдейки А.Асимптотические разложения. Физматгиз, М., 1962.
8. Арнольд В. Математические методы классической механики. "Наука", М., 1974.
9. Denchev R. Lagrangian Distributions and Their Singularities. I. Microlocal Analysis. ОИЯИ, Е5-81-174, Дубна, 1981.

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