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LAGRANGIAN DISTRIBUTIONS AND THEIR SINGULARITIES.

I. MICROLOCAL ANALYSIS

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INTRODUCTION

This paper is a slightly modified record of the lectures on "Microlocal analysis" given by the author at a seminar on mathematical problems of physics at the University of Sofia. Our approach is somewhat different from the approaches in refs. [1,2,4]. A new concept is introduced - that of the (total) symbol of a distribution and a slightly different calculus of singularities is developed. Before starting the formal exposition we briefly sketch the main ideas and consider several simple but suggestive examples.

The purpose we pursue is to introduce appropriate mathematical objects characterizing the singularities of distributions and to develop a relevant calculus so that we could, for instance, explore the singularities of solutions to partial differential equation problems, when the singularities of the data are given. Let us recall that we consider C^∞-singularities, that means we say that the distribution has a singularity in a point if there is no neighbourhood of that point in which the distribution coincides with a C^∞-function. The well known connection between the regularity of a function and the growth at infinity of its Fourier transform will play a fundamental role. To characterize the singularities of a distribution $u \in \mathcal{D}'(\boldsymbol{R}^n)$ locally, i.e., in the neighbourhood of a point $x \in \boldsymbol{R}^n$, we consider the asymptotic at ∞ of the "localized Fourier transform", i.e., (see Appendix)

$$\operatorname*{asym}_{\lambda \to \infty} F[u\phi](\lambda \xi), \tag{0.1}$$

where F is the Fourier transform, $\phi \in C_0^\infty(\boldsymbol{R}^n)$ is supported in a small neighbourhood of x, $\lambda \in \boldsymbol{R}^1_+$ and $\xi \neq 0$. Let us consider several examples of distributions u for which the asymptotics (0.1) may be easily calculated.

Example 1 $u = \delta(x_2 - x_1^2)$.

Here $n = 2$ and the stationary phase method gives

$$F[u\phi](\lambda\xi)\approx$$

$$\approx\begin{cases}\sqrt{\dfrac{\pi}{\lambda|\xi_2|}}\, e^{i\frac{\pi}{4}\operatorname{sgn}\xi_2}\, e^{i\lambda\frac{\xi_1^2}{4\xi_2}} \displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i}{16\lambda\xi_2^2}\right)^k (2\xi_2\partial_1-\xi_1\partial_2)^{2k}\phi\left(-\frac{\xi_1}{2\xi_2},\frac{\xi_1^2}{4\xi_2^2}\right), & \text{for } \xi_2\neq 0\\ 0, & \text{for } \xi_2=0,\ \xi_1\neq 0\end{cases}$$

$$=\begin{cases}\sqrt{\dfrac{\pi}{\lambda|\xi_2|}}\, e^{i\frac{\pi}{4}\operatorname{sgn}\xi_2}\Big\langle e^{i\lambda\xi x}\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{-i}{16\lambda\xi_2^2}\Big)^k(2\xi_2\partial_1-\xi_1\partial_2)^{2k}\delta\Big(x_1+\frac{\xi_1}{2\xi_2},x_2-\frac{\xi_1^2}{4\xi_2^2}\Big),\ \phi(x_1,x_2)\Big\rangle, & \text{for } \xi_2\neq 0\\ 0, & \text{for } \xi_2=0,\ \xi_1\neq 0,\end{cases}$$

where ∂_1 and ∂_2 denote derivatives with respect to the first and the second argument. We see that the asymptotic may be considered as depending on two variables x and ξ (as a distribution on x, depending on the parameter ξ). It is vanishing everywhere except the set of points

$$\Lambda=\{(x,\xi)\in R_x^2\times R_\xi^2:\ x_1=-\frac{\xi_1}{2\xi_2},\quad x_2=\frac{\xi_1^2}{4\xi_2},\quad \xi_2\neq 0\}. \qquad (0.2)$$

This set ("support" of the asymptotic (0.1)) is a two dimensional manifold in the four-dimensional space $R_x^2\times R_\xi^2$. Its projection on the x-plane is the parabola $x_2-x_1^2=0$, which represents exactly the singular support of u. If (x_1,x_2) is fixed on the parabola, then the set of points (ξ_1,ξ_2) such that $(x,\xi)\in\Lambda$ is a straight line, normal to the parabola at the point (x_1,x_2). For any (ξ_1,ξ_2), $\xi_2\neq 0$ there is a single x such that $(x,\xi)\in\Lambda$ and the projection of Λ on the ξ-plane is a diffeomorphism. We agree to present Λ graphically in the following way.

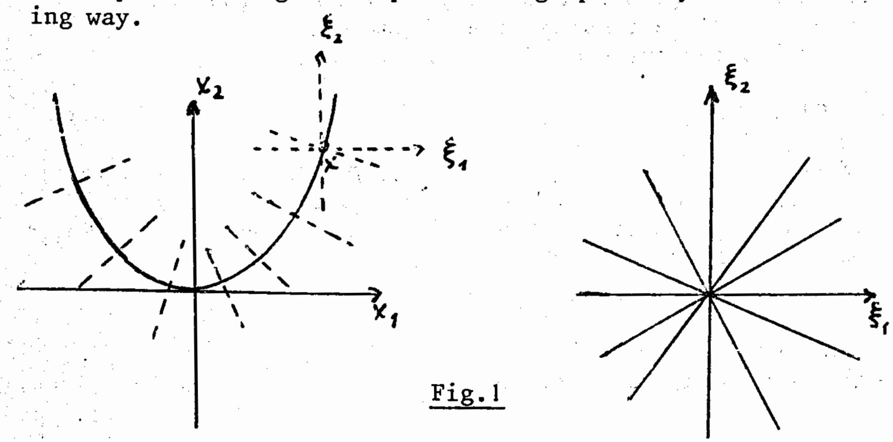

Fig.1

We draw the projection of Λ on the x-plane and then through any point x of that projection we draw a dotted line consisting of all ξ such that $(x,\xi)\in\Lambda$, provided that the ξ-plane has been laid on the x-plane with the origin at x and the ξ-axes parallel to the x-axes.

Example 2 $\quad u=\delta(x_2-x_1^3)$

$$F[u\phi](\lambda\xi)\approx\begin{cases}\dfrac{\sqrt{\pi}}{3^{1/4}}\dfrac{1}{\lambda^{1/2}(-\xi_1\xi_2)^{1/4}}\Big[e^{i\frac{\pi}{4}}\exp\Big(i\frac{2}{3}\lambda\xi_1\sqrt{\frac{-\xi_1}{3\xi_2}}\Big)\cdot\phi\Big(\Big(\frac{-\xi_1}{3\xi_2}\Big)^{1/2},\Big(\frac{-\xi_1}{3\xi_2}\Big)^{3/2}\Big)+\\ \quad+e^{-\frac{i\pi}{4}}\exp\Big(-i\frac{2}{3}\lambda\xi_1\sqrt{\frac{-\xi_1}{3\xi_2}}\Big)\phi\Big(-\Big(\frac{-\xi_1}{3\xi_2}\Big)^{1/2},-\Big(\frac{-\xi_1}{3\xi_2}\Big)^{3/2}\Big)+\\ \quad+\ldots\text{ term containing derivatives of }\phi\Big], & \text{for } \xi_1\xi_2<0\\ \displaystyle\sum_{\nu=0}^{\infty}\frac{1}{(\lambda\xi_2)^{\frac{2\nu+1}{3}}}\cdot\frac{2}{3}\Gamma\Big(\frac{2\nu+1}{3}\Big)e^{i\frac{\pi(2\nu+1)}{6}}\partial_1^{2\nu}\phi(0,0), & \text{for } \xi_1=0\\ 0, & \text{for } \xi_1\neq 0,\ \xi_1\xi_2\geq 0\end{cases}$$

$$=\begin{cases}\dfrac{\sqrt{\pi}}{3^{1/4}}\dfrac{1}{\lambda^{1/2}(-\xi_1\xi_2)^{1/4}}\Big\langle e^{i\lambda x\xi}\Big[e^{i\frac{\pi}{4}}\delta\Big(x_1-\Big(\frac{-\xi_1}{3\xi_2}\Big)^{1/2},x_2-\Big(\frac{-\xi_1}{3\xi_2}\Big)^{3/2}\Big)+\\ \quad+e^{-i\frac{\pi}{4}}\delta\Big(x_1+\Big(\frac{-\xi_1}{3\xi_2}\Big)^{1/2},x_2+\Big(\frac{-\xi_1}{3\xi_2}\Big)^{3/2}\Big)\Big],\ \phi(x_1,x_2)\Big\rangle+\ldots & \text{for } \xi_1\xi_2<0\\ \dfrac{2}{3}\Gamma\Big(\dfrac{1}{3}\Big)e^{i\frac{\pi}{6}}\dfrac{1}{(\lambda\xi_2)^{1/3}}\langle\delta(x_1,x_2),\phi(x_1,x_2)\rangle+\ldots & \text{for } \xi_1=0\\ 0 & \text{for } \xi_1\neq 0,\ \xi_1\xi_2\geq 0\end{cases}$$

where the dots represent the further terms of the asymptotic expansions, containing derivatives of δ. We see that in this example the "support" Λ of the asymptotic is again a two-dimensional manifold in $R_x^2\times R_\xi^2$, given by the relations

$$x_1=\pm\Big(\frac{-\xi_1}{3\xi_2}\Big)^{1/2},\quad x_2=\pm\Big(\frac{-\xi_1}{3\xi_2}\Big)^{3/2},\quad \xi_1\xi_2\leq 0,\quad \xi_2\neq 0. \qquad (0.3)$$

Its projection on the x-plane is $x_2-x_1^3=0$, i.e., the singular support of u and if (x_1,x_2) is fixed, (ξ_1,ξ_2) runs along the normal at (x_1,x_2).

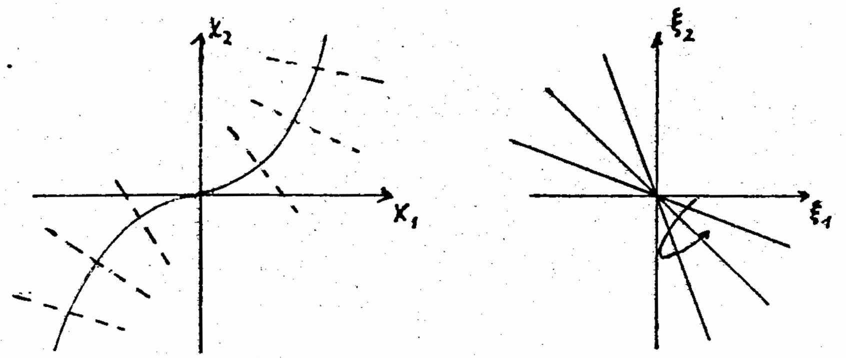

Fig.2

However there is an essential difference between examples 1 and 2. The formulas (0.3) do not determine (x_1, x_2) uniquely when given $(\xi_1 . \xi_2)$. When $\xi_1 \neq 0$ there are two points on Λ with the same projection on the ξ-plane. For $\xi_1 = 0$ these two points coincide but in no vicinity of such a point the projection of Λ on the ξ-plane is biunique. When a point (x_1, x_2) runs along the curve $x_2 - x_1^3 = 0$, the normal at this point turns and covers twice the shaded angle on Fig.2. The rank of the projection of Λ on the ξ-plane changes being 2 at the points with $\xi_1 \neq 0$ and 1 when $\xi_1 = 0$. In addition the asymptotic is given by two different expressions for $\xi_1 \neq 0$ and for $\xi_1 = 0$.

Example 3.

Another example are the Schwartz kernels of the differential operators. Take for instance the ordinary differential operator

$$L = a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x), \qquad x \in R^1.$$

Its kernel

$$K(x,y) = a_0(x)\delta''(x-y) + a_1(x)\delta'(x-y) + a_2(x)\delta(x-y), \; y \in R^1.$$

It is easy to calculate that

$$F[K(x,y)\phi(x,y)](\lambda\xi, \lambda\eta) \approx$$

$$\approx \begin{cases} <[a_0(x)(i\lambda\xi)^2 + a_1(x)(i\lambda\xi) + a_2(x)]\delta(x-y) + [2a_0(x)(i\lambda\xi) + a_1(x)(i\lambda\xi)]\delta'(x-y) + \\ \qquad + a_0(x)\delta''(x-y), \phi> \quad \text{for} \quad \xi + \eta = 0 \\ 0 \quad \text{for} \quad \xi + \eta \neq 0. \end{cases}$$

The "support" of this asymptotic is

$$\Lambda = \{(x,y;\xi,\eta) : x - y = 0, \quad \xi + \eta = 0\}.$$

Thus Λ is the direct product of the diagonal $x-y=0$ in $R^2_{(x,y)}$ (the singular support of $K(x,y)$) and the "other" diagonal $\xi + \eta = 0$ in $R^2_{(\xi,\eta)}$.

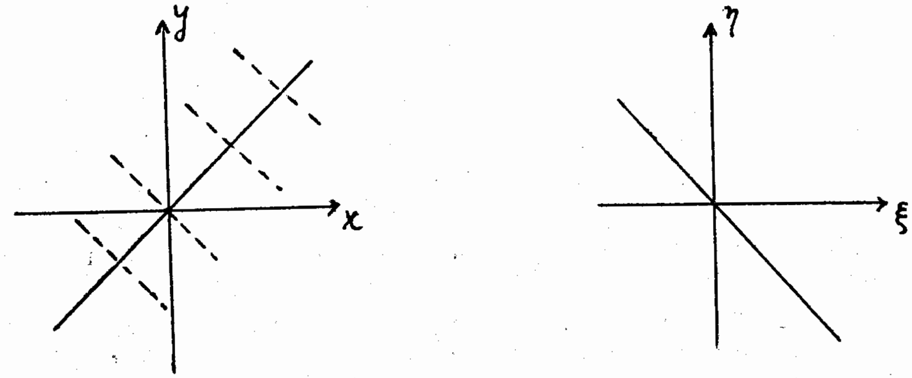

Fig.3

The rank of the projection on the (ξ,η)-plane at each point of Λ is 1, i.e., constant but less than 2. We remark (and this is important for our approach) that if in the asymptotic expansion we put together all the terms which do not contain derivatives of ϕ we get as a coefficient (the coefficient before $\delta(x-y)$) exactly the symbol of the operator L.

Example 4

Finally consider the distribution $u = \delta(x-a)$ where $x \in R^2, a \in R^2$. We immediately see that

$$F[u\phi](\lambda\xi) \approx e^{i\lambda\xi \cdot a} \cdot \phi(a) = < e^{i\lambda x \cdot \xi} \delta(x-a), \phi(x)>.$$

This asymptotic vanishes everywhere except $x = a$ and arbitrary ξ, thus the "support" is

$$\Lambda = \{(x,\xi) : x = a, \; \xi \in R^2_\xi\}.$$

It is again a two-dimensional set in the four-dimensional space $R^2_x \times R^2_\xi$ and is represented graphically by

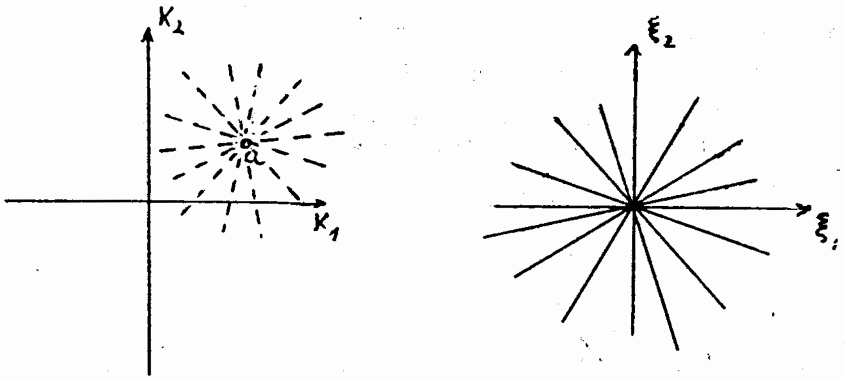

Fig.4

The examples we considered suggest the following conclusions:

1. The asymptotic (0.1) may be considered as depending on two variables x and ξ in the 2n-dimensional space $R^n_x \times R^n_\xi$, but it does'nt vanish only on an n-dimensional manifold Λ, the projection of Λ, on the x-space being the singular support of u. That fact plays a fundamental role in all further considerations. Working in the (x,ξ)-space instead of the x-space is a most important feature of the methods we are studying.

2. The rank of the projection of Λ on the ξ-space may change and accordingly changes the expression for the asymptotic.

3. If singsupp u is a manifold, then Λ is the normal bundle, hence the relations

$$\sum_{k=1}^{n} \xi_k \, dx_k|_\Lambda = 0,$$

consequently

$$\sum_{k=1}^{n} d\xi_k \wedge dx_k|_\Lambda = 0$$

are fulfilled. Thus Λ is a Lagrangian manifold (see Appendix 2).

4. The asymptotic (0.1) is a sum of derivatives of ϕ multiplied by some coefficients depending on ξ. The coefficient of ϕ is expected to play an important role, being in example 3, the symbol of the differential operator.

In the rest of the paper we consider a class of distributions generalizing the examples above. We define for them appropriate "support of the asymptotic" Λ and study first the case when the ξ-projection of Λ is one-to-one. In this case the notion of symbol of a distribution is introduced, as suggested by example 3. The general case of variable rank of the ξ-projection of Λ is, so to say, reduced to the case of one-to-one projection by a standard geometric construction. According to a well known property of Lagrangian manifolds there is a covering $\{U_\alpha\}$ of Λ and local coordinates $(x^{(\alpha)},\xi^{(\alpha)})$, $x^{(\alpha)}=\kappa_\alpha(x)$ so that in these coordinates $\Lambda \cap U_\alpha$ has a one-to-one $\xi^{(\alpha)}$-projection. Thus we can define the symbol of the local representation $u_\alpha = u \circ \kappa_\alpha$ of u and then find out how the symbol changes when local coordinates are changed. We get some transition formulas which define a fibre bundle over Λ. The symbol of u is then defined as a section of this bundle whose representations in local coordinates are the symbols of u_α.

The manifold Λ and the symbol characterize the singularities of the distribution and an appropriate calculus is developed. Finally some applications are considered.

1. SINGULAR SPECTRUM OF A DISTRIBUTION

We start with a precise definition of the asymptotic (0.1) of the introduction.

Let X be an open in R^n and u be a distribution from $\mathcal{D}'(X)$. Consider the distribution valued function $e^{-i\lambda x\cdot\xi}u$ of the parameters $\lambda \in R_+$ and $\xi \in R^n|0$

Definition 1.1. *The* $\mathcal{D}'(x)$ *asymptotic (see Appendix 1)*

$$\underset{\lambda\to\infty}{\overset{\mathcal{D}'(X)}{\mathrm{asym}}}\, e^{-i\lambda x\xi}\cdot u \in \mathfrak{A}(\mathcal{D}'(x))$$

is called local Fourier asymptotic of u *and is denoted by* $AF^{loc}[u](\xi)$*

According to Appendix 1, $AF^{loc}[u](\xi)$ is a class of distribution valued functions $a(x;\xi,\lambda)$ of ξ and λ and

$$a(x;\xi,\lambda) \in AF^{loc}[u](\xi) = \overset{\mathcal{D}'(X)}{\mathrm{asym}}\, e^{-i\lambda x\xi} \Leftrightarrow$$

$$\Leftrightarrow \langle a(x;\xi,\lambda),\phi(x)\rangle \sim \langle e^{-i\lambda x\cdot\xi}\cdot u,\phi\rangle = F[u\phi](\lambda\xi), \ \forall\, \phi\in C_0^\infty \Leftrightarrow$$

$$\Leftrightarrow \langle a(x;\xi,\lambda),\phi(x)\rangle \in \mathrm{asym}\, F[u\phi](\lambda\xi), \ \forall\ \phi \in C_0^\infty$$

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*Such asymptotic has been considered by Brychkov Yu.A. (see ref. /6/).

Thus $AF^{loc}[u](\xi)$ is determined by $\overset{R^1}{\text{asym}}\, F[u\phi](\lambda\xi)$, $\phi \in C_0^\infty$.

Now we shall define precisely the "support" of the asymptotic (0.1) mentioned in the introduction.

Definition 1.2. *The singular spectrum of* u, *denoted by* singspec u *(or wave front of* u, *denoted by* WF(u) *) is the set of points of* $X \times (R^n \setminus 0)$ *with the following property: a point* $(x^\circ, \xi^\circ) \in X \times R^n \setminus 0$ *does not belong to* singspec u *iff there exist a neighbourhood* $U_{x^\circ} \subset X$ *of* x° *and a conic neighbourhood* Γ_{ξ° *of* ξ° *such that* $\text{asym}\, F[u\phi](\lambda\xi) = 0$ *for all* $\xi \in \Gamma_{\xi^\circ}$ *and all* $\phi \in C_0^\infty(U_{x^\circ})$.

Lemma 1.1. *If* $\phi \in C_0^\infty$ *and* $(x, \xi^\circ) \notin$ singspec u *for any* $x \in \text{supp}\,\phi$ *then* $\text{asym}\, F[u\phi](\lambda\xi^\circ) = 0$.

The proof is left to the reader.

The examples in the introduction suggest to consider distributions whose singular spectrum is contained in a homogeneous Lagrangian manifold Λ. Such distributions will be called *Lagrangian distributions*. Moreover we shall suppose at first that Λ has the form

$$\Lambda = \{(x, \xi) \in R^n_x \times R^n_\xi \ : \ x = g'(\xi),\ \xi \in \Omega_\xi\}, \tag{1.1}$$

where g is a homogeneous function of degree 1 and Ω_ξ is an open cone in R^n_ξ. Thus Λ has a one-to-one projection on the ξ-space. Lagrangian distributions, for which (1.1) is fulfilled, will be calles *primitive Lagrangian distributions*.

2. SYMBOL OF A PRIMITIVE LAGRANGIAN DISTRIBUTION

Let u be a primitive distribution, i.e., singspec $u \subset \Lambda$ and Λ is given by (1.1). If $(x^\circ, \xi^\circ) \in \Lambda$ then, because of (1.1), $x \neq x^\circ$ implies $(x, \xi^\circ) \notin \Lambda$. Thus, if $\phi \not\ni x^\circ$, then $(x, \xi^\circ) \notin \Lambda$ (and consequently $(x, \xi^\circ) \notin$ singspec u) for any $x \in \text{supp}\,\phi$. Hence, according to lemma 1.1 $\text{asym}\, F[u\phi](\lambda\xi^\circ) = 0$. It follows that $\text{asym}\, F[u\psi](\lambda\xi^\circ)$ for $\psi \in C_0^\infty$ depends only on the term of ψ in x°. Thus if $\psi \in C_0^\infty$ and $\psi(x) = 1$ for x in some neighbourhood of x° (we shall denote the set of such functions by $1_0^\infty(x^\circ)$), then $\text{asym}\, F[u\psi](\lambda\xi)$ does not depend on ψ. The fact warrants the following definition.

Definition 2.1. *Let* u *be a primitive Lagrangian distribution and* singspec $u \subset \Lambda$, *where* Λ *is given by (1.1). The function* $\mathcal{G}[u] : \Lambda \to \mathcal{A}(R^1)$

$$\mathcal{G}[u](x^\circ, \xi^\circ) = e^{i\lambda x^\circ \xi^\circ}\ \text{asym}\, F[u\psi](\lambda\xi^\circ), \quad (x^\circ, \xi^\circ) \in \Lambda$$

where $\psi \in 1_0^\infty(x^\circ)$ *is called the total symbol of* u.

We shall denote by $s(x, \xi; \lambda)$ the elements of the asymptotic class $\mathcal{G}[u](x, \xi)$ and we introduce also the function

$$\sigma(x, \xi) = s(x, \frac{\xi}{|\xi|}; |\xi|)$$

which doesn't depend on λ. Obviously

$$\sigma(x, \xi) \underset{|\xi| \to \infty}{\approx} e^{ix\xi}\, F[u\psi](\xi)$$

and

$$s(x, \xi; \lambda) = \sigma(x, \lambda\xi). \tag{2.1}$$

The singular spectrum and the symbol characterize completely the singularities of the distributions, i.e.,

Theorem 2.1. *If* u *and* v *are primitive distributions and* singspec u = singspec v, $\mathcal{G}[u] = \mathcal{G}[v]$ *then* $u - v \in C^\infty$.

Proof. It follows from the conditions of the theorem that singsupp u = singsupp v. Let $x^\circ \in$ singsupp u. Take $\psi \in 1_0^\infty(x^\circ)$. The condition $\mathcal{G}[u] = \mathcal{G}[v]$ implies

$$\text{asym}\, F[(u - v)\psi](\lambda\xi) = 0 \tag{2.2}$$

for all points $\xi \in R^n_\xi \setminus 0$ for which $(x^\circ, \xi) \in$ singspec u. If $(x^\circ, \xi) \in$ singspec u then (2.2) is true in virtue of the definition of singspec u. Thus (2.2) is true for any $\xi \in R^n \setminus 0$ and consequently $(u - v)\psi$ is C^∞. That means $u - v$ is C^∞ in some neighbourhood of any point $x^\circ \in$ sing supp u = sing Obviously $u - v$ is C^∞ in the neighbourhood of all points $x^\circ \notin$ sing supp u and the proof is finished.

Now we are going to impose some additional restrictions on the distributuons we consider, which guarantee that their singularities are not very "exotic". If $x^\circ \in$ sing supp u then obviously $(x - x^\circ)^\alpha u(x)$, $|\alpha| > 0$ (α - multiindex) is not "more singular" than u at the point x°. We shall require, it to be "less singular". Let u be a primitive Lagrangian distribution and let us study the singularities of $(x_j - x_j^\circ)u(x)$ at x°, provided the singularity of u is given. Take $\psi \in 1_0^\infty(x^\circ)$. If $(x^\circ, \xi) \notin$ sing spec u then, of course, $(x^\circ, \xi) \notin$ sing spec$[(x_j - x_j^\circ)u]$ neither. Let $(x^\circ, \xi^\circ) \notin$ sing spec $u \subset \Lambda$. Then $x^\circ = g'(\xi^\circ)$ and since g is homogeneous of degree 1, $\xi \cdot g'(\xi) = g(\xi)$. Thus

$$e^{i\lambda \xi^\circ \cdot x^\circ} F[(x_j - x_j^\circ)u\psi](\lambda\xi^\circ) = e^{i\lambda g(\xi^\circ)} \{(i\frac{\partial}{\partial \eta_j} - g'(\xi^\circ)) F[u\psi](\eta)\}_{\eta = \lambda\xi^\circ} = \tag{2.3}$$

$$= \frac{i}{\lambda} \frac{\partial}{\partial \xi_j^\circ} (e^{i\lambda g(\xi^\circ)} F[u\psi](\lambda\xi^\circ)).$$

Let us suppose that $\mathcal{G}[u](g'(\xi),\xi) = \text{asym}\, e^{i\lambda g(\xi)} F[u\psi](\lambda\xi)$ may be differentiated with respect to ξ, i.e., that if $s_1[u](x,\xi;\lambda)$ and $s_2[u](x,\xi;\lambda)$ are two functions from the same asymptotic class $\mathcal{G}[u](x,\xi)$, where $(x,\xi)\in\Lambda$, then $\frac{\partial}{\partial\xi_j} s_1[u](g'(\xi),\xi;\lambda) \underset{\lambda\to\infty}{\approx} \frac{\partial}{\partial\xi_j} s_2[u](g'(\xi),\xi;\lambda)$. Under this assumption it follows from (2.3) that

$$\mathcal{G}[(x_j - x_j^{\circ})u](x^{\circ},\xi^{\circ}) = \frac{i}{\lambda}\frac{\partial}{\partial\xi_j^{\circ}}\mathcal{G}[u](g'(\xi^{\circ}),\xi^{\circ}). \tag{2.4}$$

This condition may also be written in the form

$$\sigma[(x_j - x_j^{\circ})u](g'(\xi),\xi) \underset{|\xi|\to\infty}{\sim} i\frac{\partial}{\partial\xi_j}\sigma[u](g'(\xi),\xi) \tag{2.5}$$

setting $\xi = \lambda\xi^{\circ}$ and taking into account (2.1) and the homogeneity of g.

As we already mentioned we shall require $(x-x^{\circ})^{\alpha}u(x)$, to be "less singular" than u at x° and thus following relations to be fulfilled

$$F[(x-x^{\circ})\ u\psi](\lambda\xi^{\circ}) = o(F[u\psi](\lambda\xi^{\circ})), \tag{2.6}$$

i.e.,

$$\text{asym}\, F[(x-x^{\circ})^{\alpha}u\psi](\lambda\xi^{\circ}) \prec \text{asym}\, F[u\psi](\lambda\xi^{\circ})$$

and consequently

$$\mathcal{G}[(x-x^{\circ})^{\alpha}u](x^{\circ},\xi^{\circ}) \prec \mathcal{G}[u](x^{\circ},\xi^{\circ}).$$

In view of (2.3) the last relation may be written in the form

$$\left(\frac{i}{\lambda}\right)^{|\alpha|}\partial^{\alpha}_{\xi^{\circ}}\mathcal{G}[u](g'(\xi^{\circ}),\xi^{\circ}) \prec \mathcal{G}[u](g'(\xi^{\circ}),\xi^{\circ})$$

or

$$\partial^{\alpha}_{\xi}\sigma[u](g'(\xi),\xi) = o(\sigma[u](g'(\xi),\xi)),\ |\xi|\to\infty.$$

This condition will be certainly fulfilled if we suppose that there exist $m\in R^1$ and $\rho>0$ such that

$$\partial^{\alpha}_{\xi}\sigma[u](g'(\xi),\xi) = O(|\xi|^{m-\rho|\alpha|}),\ |\xi|\to\infty,\ \xi\in\Omega_{\xi} \tag{2.7}$$

for any multiindex α. Thus we came up to the following definition.

<u>Definition 2.2.</u> *The class $\mathcal{L}_0$ consists of all distributions u with the following properties:*

1) u is a primitive Lagrangian distribution, i.e., sing spec u $\subset\Lambda$

$\Lambda = \{(x,\xi): x = g'(\xi),\ \xi\in\Omega\}$

where g is homogeneous of degree 1 and Ω is a conic open in R^n_{ξ}.

2) $\mathcal{G}[u](g'(\xi),\xi)$ may be differentiated with respect to ξ and there exist $m\in R^1$ and $\rho>0$ such that for any multiindex α.

$$\partial^{\alpha}_{\xi}\sigma[u](g'(\xi),\xi) = O(|\xi|^{m-\rho|\alpha|}),\ |\xi|\to\infty,\ \xi\in\Omega,$$

where $\sigma[u](x,\xi)$ is any function such that $\sigma[u](x,\lambda\xi)\in\mathcal{G}[u](x,\xi)$.

3. CONSTRUCTION OF DISTRIBUTION WITH GIVEN SINGULAR SPECTRUM AND SYMBOL

In the previous section we saw that the singular spectrum and the symbol determine the singularities of a distribution. Now the question arises how to construct a distribution when its singular spectrum and symbol are given.

Let Λ be a Lagrangian manifold and $\mathcal{G}(x,\xi)$ be an associated symbol satisfying the conditions of definition 2.2. Take $\sigma(x,\xi)$ such that $\sigma(x,\lambda\xi)\in\mathcal{G}(x,\xi)$. We want to find a distribution u such that

$$e^{i\lambda x\cdot\xi}F[u\psi](\lambda\xi) \underset{\lambda\to\infty}{\approx} \sigma(x,\lambda\xi) \tag{3.1}$$

for $\psi\in 1^{\infty}_0(x)$, $(x,\xi)\in\Lambda$. But $(x,\xi)\in\Lambda$ means $x = g'(\xi)$, thus (3.1) turns to be

$$F[u\psi](\lambda\xi) \approx e^{-i\lambda\xi\cdot g'(\xi)}\sigma(g'(\xi),\lambda\xi) = e^{-i\lambda g(\xi)}\sigma(g'(\xi),\lambda\xi) \tag{3.2}$$

because of the homogeneity of g. Let us write $\eta = \lambda\xi$ and replace the sign $\approx$ by $=$. Then instead of (3.2) we get

$$F[u\psi](\eta) = e^{-ig(\eta)}\sigma(g'(\eta),\eta). \tag{3.3}$$

If u satisfies (3.3), then of course, it will satisfy (3.2). Now (3.3) implies

$$u\psi(y) = \int e^{iy\cdot\eta}\, e^{-ig(\eta)}\sigma(g'(\eta),\eta)\,d\eta.$$

Taking $y = x$ we get (since $\psi(x) = 1$)

$$u(x) = \int e^{-i(x\cdot\eta - g(\eta))}\sigma(g'(\eta),\eta)\,d\eta. \tag{3.4}$$

These considerations suggest that we may expect the distribution (3.4) (the integral being regularized, see refs. /1,5/) to have symbol σ. Before proving it we shall write (3.4) in another form - as an integral on the manifold Λ. The manifold Λ is given by $y = g'(\eta)$ and, since g is homogeneous of degree 1, $g(\eta) = \eta\cdot g'(\eta) = \eta\cdot y$, when $(y,\eta)\in\Lambda$. We write in (3.4) y instead of

$g'(\eta)$, and $\eta\cdot y$ instead of $g(\eta)$ and get

$$u(x) = \int_{\Lambda\subset R^{2n}_{y,\eta}} e^{i(x-y)\eta}\cdot\sigma(y,\eta)d\eta. \qquad (3.5)$$

Theorem 3.1. *Let Λ be a Lagrangian manifold and $\mathcal{G}(x,\xi) = \operatorname{asym}\sigma(x,\lambda\xi)$ an associated symbol satisfying the conditions of the definition 2.2. Then the distribution determined by the formulas (3.4) or (3.5) (the integral being regularized) belongs to the class $\mathcal{L}_0$ and has for symbol $\mathcal{G}(x,\xi)$.*

We omit the proof which is a straightforward, but tiresome application of the stationary phase method.

4. THE SYMBOL OF A LAGRANGIAN DISTRIBUTION IN THE GENERAL CASE

In this section we roughly sketch some ideas of how to define the symbol of a Lagrangian distribution when the condition of Λ to project one-to-one on the ξ-space is not fulfilled.

Let Λ be an arbitrary homogeneous Lagrangian manifold in R^{2n} and u - a distribution for which $\operatorname{sing\,spec} u\subset\Lambda$. According to a well known property of Lagrangian manifold (see Appendix 2) we may find a covering $\{U_\alpha\}$ of Λ (U_α are open in $R^{2n}_{x,\xi}$) and local coordinates κ_α in R^n_x so that the image Λ^α_α of $\Lambda_\alpha = U_\alpha\cap\Lambda$ by the mapping

$$x^{(\alpha)} = \kappa_\alpha(x), \qquad \xi^{(\alpha)} = (D\kappa_\alpha^*)^{-1}\xi.$$

is represented by $x^{(\alpha)} = g'_\alpha(\xi^{(\alpha)})$ with g_α homogeneous of degree 1. Denote $U^\alpha_\alpha = (\kappa_\alpha,(D\kappa^*_\alpha)^{-1})U_\alpha$, $\Lambda_{\alpha\beta} = U_\alpha\cap U_\beta\cap\Lambda$, $\Lambda^\alpha_{\alpha\beta} = (\kappa_\alpha,(D\kappa^*_\alpha)^{-1})\Lambda_{\alpha\beta}$, $\Lambda^\beta_{\alpha\beta} = (\kappa_\beta,(D\kappa^*_\beta)^{-1})\Lambda_{\alpha\beta}$. Then $\Lambda^\alpha_{\alpha\beta}$ is represented by $x^{(\alpha)} = g'_\alpha(\xi^{(\alpha)})$ and $\Lambda^\beta_{\alpha\beta}$ by $x^{(\beta)} = g'_\beta(\xi^{(\beta)})$.

Denote $u_\alpha = u\circ\kappa_\alpha^{-1}$. Then u_α is a primitive Lagrangian distribution, so $\mathcal{G}[u_\alpha]$ is defined. Consider the symbols $\mathcal{G}[u_\alpha]$ and $\mathcal{G}[u_\beta]$ which map Λ^α_α resp. Λ^β_β into $\mathfrak{A}(R^1)$. The question we ask is: how does the symbol change when local coordinates are changed, in other words how $\mathcal{G}[u_\alpha]|_{\Lambda^\alpha_{\alpha\beta}}$ and $\mathcal{G}[u_\beta]|_{\Lambda^\beta_{\alpha\beta}}$ are connected.

Suppose that the distributions u_α are not only primitive but belong to the class $\mathcal{L}_0$. Take functions $\sigma_\alpha(x^{(\alpha)},\xi^{(\alpha)})$ such that $\sigma_\alpha(x^{(\alpha)},\lambda\xi^{(\alpha)})\in\mathcal{G}[u_\alpha]$. According to the previous section

$$u_\alpha(x^{(\alpha)}) = \int e^{i(x^{(\alpha)}\eta - g_\alpha(\eta))}\cdot\sigma_\alpha(g'_\alpha(\eta),\eta)d\eta.$$

Then for $u_\beta = u_\alpha\circ\kappa_\alpha\circ\kappa_\beta^{-1}$ we get the expression

$$u_\beta(x^{(\beta)}) = \int e^{i(\kappa(x^{(\beta)})\cdot\eta - g_\alpha(\eta))}\cdot\sigma_\alpha(g'_\alpha(\eta),\eta)d\eta$$

where we denote $\kappa = \kappa_\alpha\circ\kappa_\beta^{-1}$. For simplification of the notation we shall write (z,ζ) instead of $(x^{(\beta)},\xi^{(\beta)})$. Take a point $(z^\circ,\zeta^\circ)\in\Lambda^\beta_{\alpha\beta}$. Then $z^\circ = g'_\beta(\zeta^\circ)$. Let $\psi\in 1^\infty_0(z^\circ)$. We have to calculate the asymptotic

$$\begin{aligned} F[u_\beta\psi](\lambda\zeta^\circ) &= \iint e^{-i\lambda\zeta^\circ\cdot z}\, e^{i(\kappa(z)\cdot\eta - g_\alpha(\eta))}\cdot\sigma_\alpha(g'_\alpha(\eta),\eta)\psi(z)dzd\eta \\ &= \lambda^n\iint e^{i\lambda(\kappa(z)\eta - \zeta^\circ\cdot z - g_\alpha(\eta))}\sigma_\alpha(g'_\alpha(\eta),\lambda\eta)\psi(z)dzd\eta. \end{aligned} \qquad (4.1)$$

Let us apply the stationary phase method. The stationary points (z,η) satisfy the equations

$$(\kappa'(z))^*\eta - \zeta^\circ = 0$$
$$\kappa(z) - g'_\alpha(\eta) = 0$$

hence

$$\begin{aligned} \eta &= (\kappa'(z)^*)^{-1}\zeta^\circ \\ \kappa(z) &= g'_\alpha((\kappa'(z)^*)^{-1}\zeta_0). \end{aligned} \qquad (4.2)$$

As was already mentioned $\Lambda^\alpha_{\alpha\beta}$ is represented by $x^{(\alpha)} = g'_\alpha(\xi^\alpha)$. On the other hand the mapping

$$x^{(\alpha)} = \kappa(z), \quad \xi^{(\alpha)} = (D\kappa^*)^{-1}\zeta$$

transforms $\Lambda^\beta_{\alpha\beta}$ into $\Lambda^\alpha_{\alpha\beta}$, hence $\Lambda^\beta_{\alpha\beta}$ is represented by

$$\kappa(z) = g'_\alpha((D\kappa^*)^{-1}\zeta). \qquad (4.3)$$

We see from (4.2) and (4.3) that if (z,η) is a stationary point of (4.1), than $(z,\zeta^\circ)\in\Lambda^\beta_{\alpha\beta}$ and consequently $z = g'_\beta(\zeta^\circ) = z^\circ$. Thus there is a single stationary point (z°,η°) where $\eta^\circ = (\kappa'(z^\circ)^*)^{-1}\zeta_0$. Denote by H_0 the Hessian at (z°,η°) of the phase function

$$\Phi(z,\eta) = \kappa(z)\cdot\eta - \zeta^\circ\cdot z - g_\alpha(\eta).$$

(Prove that H_0 is nonsingular!) The stationary phase method gives us the following formula for the asymptotic of (4.1)

$$e^{-i\lambda z^{\circ}\cdot\zeta^{\circ}}\sigma_{\beta}(z^{\circ},\zeta^{\circ}) \approx F[u_{\beta}\psi](\lambda\zeta^{\circ}) \approx$$

$$\approx (2\pi\lambda)^{n/2} |\det H_0|^{-1/2} \exp\{i[\lambda\Phi(z^{\circ},\eta^{\circ}) + \frac{\pi}{4}\operatorname{sgn} H_0]\} \times \quad (4.4)$$

$$\times \sum_{j=0}^{\infty} \frac{(i/2)^j}{j!} (D_{z,\eta} H_0^{-1} D_{z,\eta})^j [\sigma_{\alpha}(g_{\alpha}'(\eta),\lambda\eta)\psi(z)]|_{\substack{z=z_0^{\circ}\\ \eta=\eta^{\circ}}} \lambda^{-j}.$$

This relation may be differentiated with respect to z°,η°. Denote by S_{α} the suit of all the derivatives $D^{\mu}\sigma_{\alpha}(x_0^{(\alpha)},\lambda\xi_0^{(\alpha)})$ of $\sigma_{\alpha}(x^{(\alpha)},\xi^{(\alpha)})$ with respect to $(x^{(\alpha)},\xi^{(\alpha)})$ taken in the point $(x_0^{(\alpha)}, \lambda\xi_0^{(\alpha)})$ and by S_{β} similarly for σ_{β}. Here $(x_0^{(\alpha)},\xi_0^{(\alpha)})$ is the image of $(z^{\circ},\zeta^{\circ})$ by $(\kappa,(D\kappa^{*})^{-1})$. Then (4.4) and its derivatives give us a relation of the form

$$S_{\beta} = T_{\alpha\beta}S_{\alpha}, \quad (4.5)$$

where $T_{\alpha\beta}$ is some linear mapping of an infinite product $\mathfrak{A}^{\infty} = \mathfrak{A}(\mathbf{R}^1)\times\mathfrak{A}(\mathbf{R}^1)\times\ldots$ into itself. Now consider a fibre bundle $\mathcal{F}(\Lambda)$ with base Λ, fibre $\mathfrak{A}^{\infty}$ and transition formulas (4.5). The symbol of u is defined as a section of $\mathcal{F}(\Lambda)$ which in local coordinates is given by σ_{α} and its derivatives.

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