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PAINLEVE' EQUATIONS AND THEIR CONNECTION WITH NONLINEAR EVOLUTION EQUATIONS.

Part II

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Уравнения Пенлеве и их связь с нелинейными эволюционными уравнениями. Часть II

Исследованы некоторые семейства частных решений третьего и пятого уравнений Пенлеве, обсуждаются результаты исследований общих и частных решений. Установлена связь между полученными решениями, найдено преобразование Беклунда.

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Painlevé Equations and Their Connection with Nonlinear Evolution Equations. Part II

INTRODUCTION

In the paper [1] we have discussed general and partial solutions of the first, second and fourth Painlevé equations. Here we deal with the third Painlevé equation

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w} \qquad (P_3)$$

and with the fifth Painlevé equation

$$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \delta\frac{w(w+1)}{w-1}. \qquad (P_5)$$

Some of the important overall features that these equations have are:

i) Both equations can have solutions with a fixed critical point.

ii) The equations P_3 and P_5 can have entire transcendental solutions (the equation P_3 for the parameters $\gamma = \alpha = 0$, and the equation P_5 for the parameters $\alpha = 0$, see [1]).

During the last decade the investigation of partial solutions of the equations P_3 and P_5 was especially intensive because solutions of some important physical problems are expressible in terms of the solutions of the equations P_3 and P_5. For instance, in the recent paper Newell and Flashka [2] have shown that the special case of the P_3 equation is derivable as similarity solutions of the Sine-Gordon equation. In the article [3] examples corresponding to equations for other Painlevé transcendents of the type P_3 and P_5 which result from the Regge-Lund model [4,5] and the Ernst equation [6] are presented. In addition, the investigation of the spin-spin correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for the two-dimensional Ising model in the scaling limits leads to the expressions for these scaling functions in terms of a Painlevé function of the third kind [7].

1. THE INTEGRABLE CASES OF THE THIRD PAINLEVÉ EQUATION

The first exceptional case of P_3 equation has been investigated in refs. [8,9]. Two conditions are indicated for the parameters of the equation for which the equation P_3 is completely integrable in terms of classical functions [9]. Indeed, if

a) $\alpha=\gamma=0$ and β, δ are any constants,

or (1)

b) $\beta=\delta=0$ and α, γ are arbitrary,

then we substitute

$$z=e^{t},$$

$$w=ve^{kt}$$

($k=1$ for the case a) and $k=-1$ for the case b)) and obtain that the third Painlevé equation corresponds to one of the equations

a) $vv''=v'^{2}+\beta v+\delta$ $(\alpha=\gamma=0)$

or

b) $vv''=v'^{2}+\alpha v^{3}+\gamma v^{4}$. $(\beta=\delta=0)$.

With the help of the substitution $v'=u$ we reduce the last equations to the equations of the first order

a) $vu\dfrac{du}{dv}=u^{2}+\beta v+\delta$

or (2)

b) $vu\dfrac{du}{dv}=u^{2}+\alpha v^{3}+\gamma v^{4}$.

After the integration of the equations (2) we obtain that the function $v(z)$ must be defined from the equality

a) $\displaystyle\int\frac{dv}{\sqrt{C_1 v^{2}-2\beta v-\delta}}=\pm t+C_2$

or (3)

b) $\displaystyle\int\frac{dv}{v\sqrt{\gamma v^{2}+2\alpha v+C_1}}=\pm t+C_2$,

where C_1 and C_2 are any constants.

Consequently, functions $v(t)$ and $w(z)$ respectively are expressible in terms of classical functions. For instance, if in the formula (3b) $C_1=\alpha=0$ then the solution of the eq. P_3 is

$$w(z)=\frac{1}{\sqrt{\gamma}\,z(\pm\ln z+C_2)}$$

and the point $z=0$ is a branch point of the solution.

It is a close connection between the cases a) and b) since the equation P_3 has certain symmetric properties. It is known that, if $w(z)=\phi(z,\alpha,\beta,\gamma,\delta)$ is a solution of the eq. P_3, then the functions

$$\begin{aligned}
&a)\ w(z)=\phi^{-1}(z,-\beta,-\alpha,-\delta,-\gamma),\\
&b)\ w(z)=\phi^{-1}(-z,\beta,\alpha,-\delta,-\gamma),\\
&c)\ w(z)=-\phi^{-1}(z,\beta,\alpha,-\delta,-\gamma),\\
&d)\ w(z)=-\phi^{-1}(-z,-\beta,-\alpha,-\delta,-\gamma)
\end{aligned}\tag{4}$$

are also solutions of the equation P_3.

Later the similar results were found by H.Airault [10]. The author proved the following theorem.

Theorem 1. Assume that $\beta=\delta=0$. Then the equation P_3 has two families of solutions

$$w(z)=\frac{z^{\lambda-1}}{Az^{2\lambda}+Bz^{\lambda}+D},\tag{5}$$

where

$$B=-\frac{\alpha}{\lambda^{2}},\quad 4AD=\frac{\alpha^{2}}{\lambda^{4}}-\frac{\gamma}{\lambda}$$

and

$$w(z)=\frac{1}{z(a\log^{2}z+b\log z+d)},\tag{6}$$

where

$$2a=\alpha,\quad b^{2}-4ad=\gamma.$$

Of course, the formulae (5-6) can be also obtained from (3).

In addition to the cases (1) N.Lukashevich has found that all solutions of the Riccati equation

$$\frac{dw}{dz}=aw^{2}+\frac{\alpha-a}{az}w+b\tag{7}$$

are simultaneously solutions of the equation P_3 is the parameters of the equation P_3 fulfil the conditions

$$\beta+\frac{\alpha-2a}{a}b=0,\quad \gamma=a^{2}\neq 0,\quad \delta+b^{2}=0.\tag{8}$$

If we take $w=-\frac{1}{a}\frac{u'}{u}=-\frac{1}{a}(\ln u')$, then the equation (7) can be rewritten in the linear form

$$u'' + \frac{a-\alpha}{az} u' + ab\,u = 0. \quad (9)$$

It is the equation for the Bessel function. For

$$\alpha = (2n+1)\ a = \pm(2n+1)\sqrt{\gamma}\ , \quad (10)$$

where n is an integer, it has solutions

$$u(z) = \sqrt{\frac{2}{\pi}}\ z^{n+\frac{1}{2}} \{(-1)^n C_1 \frac{d^n}{(\tau\, d\tau)^n}(\frac{\sin\tau}{\tau}) +$$

$$+ C_2 \frac{d^n}{(\tau\, d\tau)^n}(\frac{\cos\tau}{\tau}) \}, \quad (11)$$

where $\tau = z\sqrt{ab}$.

Therefore, the equation P_3 has solutions expressible in terms of classical functions and also under conditions (8)*.

It should be remarked that J.Myers, B.McCoy, G.A.Tracy and T.Wu have investigated almost analogical case /7,11/. Namely, they assume that the parameter α , β , γ and δ satisfy

$$\alpha\sqrt{-\delta} + \beta\sqrt{\gamma} = 0. \quad (12)$$

Under the assumption (12) there is no loss in generality if we consider in place of the equation P_3 the equation

$$w'' = \frac{1}{w}(w')^2 - \frac{1}{\theta}w' + \frac{2\nu}{\theta}(w^2-1) + w^3 - \frac{1}{w}, \quad (13)$$

where ν is a constant.

If we seek the one-parameter family of solutions of the eq. (13) that remain bounded as θ approaches infinity along the positive real axis, then we come to the following. The function $w(\theta,\nu,\lambda)$ for positive θ and $\mathrm{Re}\,\nu > -\frac{1}{2}$ has the representation

$$\frac{1-w(\theta,\nu,\lambda)}{1+w(\theta,\nu,\lambda)} = G(t;\nu,\lambda), \quad (14)$$

where $t = 2\theta$,

---

*Remark. It can be found also other conditions on the parameters α , β , γ , δ for which the equation P_3 has partial solutions expressible in terms of classical functions (see Sections 2,3).

$$G(t;\nu,\lambda) = \sum_{n=0}^{\infty} \lambda^{2n+1} g_{2n+1}(t;\nu), \quad (15)$$

$$g_1(t;\nu) = \int_1^{\infty} dy \frac{\exp(-ty)}{(y^2-1)^{1/2}} \cdot (\frac{y-1}{y+1})^{\nu}, \quad (16)$$

and for $n \geq 1$

$$g_{2n+1}(t;\nu) = (-1)^n \int_1^{\infty} dy_1 \ldots \int_1^{\infty} dy_{2n+1} \left[\prod_{j=1}^{2n+1} \frac{\exp(-ty)}{(y_j^2-1)^{1/2}} \cdot (\frac{y_j-1}{y_j+1})^{\nu} \right] \times$$

$$\times \left[\prod_{j=1}^{2n} (y_j + y_{j+1})^{-1} \right] \cdot \left[\prod_{j=1}^{n} (y_{2j}^2 - 1) \right],$$

$|\lambda| < R(t)$, $R(t)$ is the radius of convergence of (15). This assumption was proved via a straightforward substitution /11/.

N.Lukashevich /12/ has found the necessary and sufficient conditions by which the solutions of the eq. P_3 have a pole in the point $z = 0$: a pole, a critical pole, a critical algebraic point (i.e., the point $z = 0$ is a branch point of a solution and $w \to 0$ by $z \to 0$) or by which solutions of the equation P_3 are holomorphic functions in the point $z = 0$.

If the point $z = 0$ is a pole of solutions of the eq. P_3 or solutions are holomorphic in this point, then such solutions of the equation P_3 have the form

$$w = \frac{v(z)}{u(z)}, \quad (17)$$

where $v(z)$ and $w(z)$ are entire functions. But the general representation of the solution of P_3 and its complete classification have not been so far obtained.

In addition, systems equivalent to the equation P_3 were found. For example, for parameter $\gamma \neq 0$ the systems are

$$z\frac{dw}{dz} = a w_1 + \sqrt{\gamma}\, z w^2 ,$$

$$z w \frac{dw_1}{dz} = \delta z + \beta w + (a-1) w w_1 + z w_1^2 , \quad (18)$$

$$a = \frac{\alpha}{\sqrt{\gamma}} - 1, \quad \gamma \neq 0$$

and

$$z\frac{dw}{dz} = -(\frac{\alpha}{\sqrt{\gamma}} + 1) w + z w_2 - z\sqrt{\gamma}\, w^2 ,$$

$$z w \frac{dw_2}{dz} = -(\frac{\alpha}{\sqrt{\gamma}} + 2) w w_2 + \beta w + \delta z + z w_2^2 \quad . \tag{19}$$

The investigation of these systems allows us to prove the following theorem.

Theorem 2. When $\gamma \neq 0$, $z w_1' - (\frac{\alpha}{\sqrt{\gamma}} - 2) w_1 - \beta \neq 0$ (the function w_1 is defined through the system (18)), $z w_2' + (\frac{\alpha}{\sqrt{\gamma}} + 2) w_2 - \beta \neq 0$ (the function w_2 is defined by the system (19)), than all solutions of the P_3 equation (except for the rational solutions), which have in the point $z=0$ a pole or which are a holomorphic function in the point $z=0$, have an infinite number of poles with residues $\pm \frac{1}{\sqrt{\gamma}}$. When $\gamma = 0$, $\alpha \neq 0$ then all solutions, except for rational solutions, have an infinite number of poles with residues equal to zero /12/.

2. RATIONAL SOLUTION OF THE EQUATION P_3

If we take in (17)

$$v(z) = \sum_{j=0}^{n} a_j z^j , \quad a_n \neq 0,$$
$$u(z) = \sum_{j=0}^{m} b_j z^j , \quad b_m \neq 0, \tag{20}$$

then the function $w(z)$ is a rational function. It should be emphasized that three cases are possible, namely

$$\begin{aligned} &a)\ n = m, && \text{if} \quad \gamma\delta \neq 0, \\ &b)\ n > m, && \text{if} \quad \gamma = \alpha = 0, \\ &c)\ n < m, && \text{if} \quad \beta = \delta = 0. \end{aligned} \tag{21}$$

In the last two cases (21b,c) the equation P_3 is completely integrable, therefore the rational solutions exist and can be obtained from the general formulae (3) for the solutions of the equation P_3. For instance, if $\beta = \delta = 0$ (21c), and $\alpha = \pm k\sqrt{\gamma}$, $k > 0$, k is an integer and the equation P_3 has a rational solution

$$w(z) = \frac{\alpha}{-\gamma z + c z^{1 \pm k}} ,$$

where c is an arbitrary constant and so on (see Section 3, formulae (3), (5), (6)).

Besides these rational solutions the equation P_3 has for the parameters $\gamma\delta \neq 0$ rational solutions of the type

$$w(z) = \frac{\sum_{j=0}^{n} a_j z^j}{\sum_{j=0}^{n} b_j z^j} , \ a_n, \ b_n \neq 0. \tag{22}$$

If $n = m = 0$, then

$$w(z) = \pm \sqrt{-\frac{\beta}{\alpha}} \quad \text{is a solution} \tag{23}$$

of the equation P_3 for the parameters $\alpha\beta \neq 0$, $\gamma\beta^2 + \delta\alpha^2 \neq 0$. *

If $n = m = 1$, a solution of the eq. P_3 has the form

$$w(z) = \frac{z + a}{bz + c} . \tag{24}$$

The straight substitution gives us three possibilities for values of the coefficients a, b and c

a) if $\frac{\alpha}{\sqrt{-\delta\gamma}} + \frac{\beta}{\delta} = -\frac{4}{\gamma}$, then $bac \neq 0$,

$$\alpha = \frac{b^2(3c - ab)}{\Delta^2}, \quad \beta = \frac{c - 3ab}{\Delta^2}, \quad \gamma = \frac{b^4}{\Delta^2}, \quad \delta = -\frac{1}{\Delta^2},$$

$\Delta \equiv ab - c \neq 0$;

b) if $\alpha\beta \neq 0$, $9\gamma - \alpha^2 = 0$ and $\delta + \beta^2 = 0$, then

$$a = 0, \quad b^2 = \frac{\alpha}{3\beta}, \quad c = \frac{1}{\beta};$$

c) if $\gamma - \alpha^4 = 0$, $9\delta + \beta = 0$, then

$$b^2 = \frac{3\alpha}{\beta}, \quad a^2 = \frac{3\alpha}{\gamma\beta}, \quad c = 0.$$

These rational solutions (with $n = m$, (22)) can be found also from the system

* Remark. If $\alpha, \beta, \gamma, \delta$ and $w(z)$ are real and the conditions $\alpha\beta < 0$, $\gamma\beta^2 + \delta\alpha^2 = 0$ are fulfilled, then all solutions of the equation P_3 which fulfil the initial conditions $|w(z_0)| < \sqrt{-\frac{\beta}{\alpha}}$, $|z_0| < \infty$ and $|w'(z_0)| < \infty$ can be continued for all z possibly except for the point $z=0$.

$$zuu'' = zu'^2 - uu' - \gamma z v^2 - \alpha uv,$$
$$zvv'' = zv'^2 - vv' + \delta z u^2 + \beta uv, \tag{25}$$

which was obtained by Painlevé [13] for the functions $u(z)$ and $v(z)$ from (17). But now one must look for a solution of the system (25) in the form

$$u(z) = \zeta(z) \exp g(z),$$
$$v(z) = \eta(z) \exp g(z), \tag{26}$$

where $\zeta(z)$ and $\eta(z)$ are polynomials and $g(z)$ is an entire function. In order words, for the existence of a rational solution of the equation P_3 for $\gamma\delta \neq 0$ it is necessary and sufficient that the system

$$z\zeta\zeta'' = z\zeta'^2 - \zeta\zeta' - (2\lambda z + \mu)\zeta^2 - \gamma z \eta^2 - \alpha\zeta\eta,$$
$$2\eta\eta'' = z\eta'^2 - \eta\eta' - (2\lambda z + \mu)\eta^2 + \delta z \zeta^2 + \beta\zeta\eta, \tag{27}$$

where $g'(z) = \lambda z + \mu$ and λ, μ are any constants, has a polynomial solution for some μ and $\lambda = \pm \frac{1}{2}\sqrt{-\gamma\delta}$ [12].

Remark. It should be pointed out that by the restriction $g(z) = \text{const}$ one cannot find the rational solution of the equation P_3 of type (22) from the system (25). The system (25) has polynomial solutions of the type (22) with $m < n$ or $n > m$ only. But if we look for the rational solutions of the equation P_3 system (27), then we can find all rational solutions of the equation P_3. It is the ground that the statement in [10] that the equation P_3 can have rational solutions for $\gamma = \alpha = 0$ $(n > m)$ or for $\beta = \delta = 0$ $(n < m)$ only is wrong. This statement is right for the system (17) only. For instance, if we take

$$\alpha = 5, \ \beta = -1, \ \gamma = 1, \ \delta = -1,$$

then the equation P_3 has the rational solution

$$w(z) = \frac{z+1}{z+2}. \tag{28}$$

$w(z)$ generates also a solution of the system (25) if we take

$$u(z) = (z+2)\exp\left(-\frac{z(z+12)}{4}\right)$$

and

$$v(z) = (z+1)\exp\left(-\frac{z(z+12)}{4}\right),$$

i.e., the function $g(z) = -\frac{1}{4}z(z+12)$ is nontrivial in this case.

3. BÄCKLUND TRANSFORMATION FOR THE EQUATION P_3 AND ITS APPLICATION

At first one considers the equation P_3 with parameters

$$\gamma\delta \neq 0. \tag{29}$$

Under the assumption (29) there is no loss in generality if we take $\gamma = 1$ and $\delta = -1$ [9]. In 1975 V.Gromak [16] has found that the system

$$z\frac{dw}{dz} = (\alpha\epsilon - 1)w + zv + \epsilon z w^2,$$
$$zw\frac{dv}{dz} = \beta w - z + (\alpha\epsilon - 2)wv + zv^2, \tag{30}$$

where $\epsilon^2 = \frac{1}{\gamma} = 1$, is equivalent to the equation P_3 for parameters $\gamma = 1$ and $\delta = -1$. Using this system he proved the theorem.

Theorem 3. Let $w(z)$ be a solution of the equation P_3 for any α, β and $\gamma = 1$, $\delta = -1$ and the function $R \neq 0$, where

$$R(z, w, w') = \frac{dw}{dz} - \epsilon w^2 - \frac{1}{z}(\alpha\epsilon - 1)w + 1, \tag{31}$$

then the function

$$w_1(z) = \frac{2zR(R-z)}{2z\frac{dR}{dz} + R(\sigma(\beta - \alpha\epsilon + 2) - \eta(\beta + \alpha\epsilon - 2)) - 2\sigma(\beta - \alpha\epsilon + 2)} \tag{32}$$

is a solution of the equation P_3 for the parameters

$$\alpha_1 = \frac{\epsilon}{2}[\eta(\beta + \alpha\epsilon - 2) - \sigma(\beta - \alpha\epsilon + 2) + 4],$$
$$\beta_1 = \frac{\eta}{2}[\beta + \alpha\epsilon - 2] + \frac{\sigma}{2}[\beta - \alpha\epsilon + 2],$$
$$\gamma_1 = 1, \quad \delta_1 = -1,$$

where $\epsilon^2 = \eta^2 = \sigma^2 = 1$.

Essentially it is a Bäcklund transformation (B.T.) for the equation P_3 under the condition $\gamma\delta\neq 0$.

The B.T. for the equation P_3 in the cases

$$\text{if } \gamma=0 \text{ and } \delta\alpha\neq 0 \text{ or} \qquad \text{if } \delta=0 \text{ and } \beta\gamma\neq 0, \tag{33}$$

was found also by V.Gromak[14].

If $\gamma=0$, $\alpha\delta\neq 0$ we take $z=\lambda x$, $w=\mu y$, where

$$\lambda=\sqrt[4]{-\frac{1}{\alpha^2\delta}},\quad \mu=\sqrt[4]{-\frac{\delta}{\alpha^2}},$$

and obtain the following equation for the function

$$xyy''=xy'^2-yy'+y^3+\tilde\beta y-x, \tag{34}$$

where $\tilde\beta=\beta\sqrt{\frac{1}{\delta}}$.

On the other hand, we also get the equation for the second case

$\delta=0$, $\beta\gamma\neq 0$,

if we take $z=\lambda x$, $w=\frac{\mu}{y}$, where now $\lambda=\sqrt[4]{\frac{1}{\beta^2\gamma}}$, $\mu=\sqrt[4]{\frac{\gamma}{\beta^2}}$ and $\tilde\beta=-\alpha\sqrt{-\frac{1}{\gamma}}$.

Therefore in both cases (33) the equation P_3 can be reduced to the equation (34).

It can be proved that the system (30) is equivalent to the equation (34). The investigation of this system leads to the following theorem.

Theorem 4. Let $y_0=y(x,\tilde\beta)$ be a solution of the equation (34) for any parameter $\beta_0=\tilde\beta$ then the function

$$y_1(x)=\frac{(\epsilon-\beta_0)\,y_0+x-\epsilon x y_0'}{y_0^2},\quad \epsilon^2=1 \tag{35}$$

is a solution of the equation (34) for a parameter

$$\beta_1=\beta_0-2\epsilon. \tag{36}$$

Theorem 4 gives us the B.T. for the equation P_3 in cases (33)*.

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* Remark. Besides the theorem 4 the following theorem is true.

Theorem. If we look for the general solution of the equation (34) for any parameter $\tilde\beta$ it is enough to build the general solution of the equation (34) in the region $[\operatorname{Re}\beta_0, \operatorname{Re}\beta_1]$, where β_0 is any constant and $\beta_1=\beta_0-2\epsilon$, $\epsilon^2=1$.

The formulae (35), (36), i.e., B.T., can be used for the construction of partial classes of the solutions of the equation (34) and, respectively, of the equation P_3.

If we take

$$y_0(x)=\sqrt[3]{x} \quad \text{for} \quad \beta_0=0, \tag{37}$$

we obtain after the first step

$$y_1(x)=\frac{\mp 2+3\sqrt[3]{x^2}}{3\sqrt[3]{x}} \quad \text{for} \quad \beta_1=\pm 2, \tag{38}$$

after the second step

$$y_2(x)=\frac{\mp 24x+20\sqrt[3]{x}+9z\sqrt[3]{x^2}}{(2\mp 3\sqrt[3]{x^2})^2} \quad \text{for} \quad \beta_2=\pm 4 \tag{39}$$

and so on.

The solutions of the equation P_3 corresponding to the solution $y_0(x)$ (37) of the equation (34) are

$$w(z)=hz^{1/3}, \tag{40}$$

where h is defined from the equation

$\alpha h^3+\delta=0$

by the parameters $\beta=\gamma=0$, $\alpha\delta\neq 0$; and

$$\tilde w(z)=hz^{-1/3}, \tag{41}$$

where h is a solution of the equation

$\gamma h^3+\beta=0$

if in the equation P_3 the parameters α and δ are zero.

If one takes the solution $y_1(x)$ (38) of the equation (34) then the corresponding solution of the equation P_3 for $\gamma=0$, $\alpha\delta\beta\neq 0$, $16\delta^2-9\beta^2=0$ is, for instance,

$$w(z)=a\,\frac{9a\alpha\sqrt[3]{z^2}+4}{4\sqrt[3]{z}}, \tag{42}$$

where a is an arbitrary constant*.

The list of the solutions of the equation P_3 which are rational functions of the $\sqrt[3]{z}$ may be continued easily in the same way.

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* Remark. The solutions (40-42) are obtained by straight substitution in [12].

The B.T. can be applied also by the extraction of the values of the parameters for which the equation P_3 has always solutions which are expressible in terms of classical transcendents. The first results in this way were obtained in [15].

Theorem 5. Let be

$$\beta+\alpha\epsilon_1=2(2n+1)\epsilon_2, \quad \gamma=1,\ \delta=-1, \tag{43}$$

where n is an integer, $\epsilon_j^2=1$, fulfilled then the eq. P_3 for such parameters α, β, γ and δ has solutions which are rational functions of the Bessel function.

Indeed, if n=1 in (43) then all solutions of the eq.

$$w'-\epsilon_1 w^2-z^{-1}(\alpha\epsilon_1-1)w-\epsilon_2=0 \tag{44}$$

are solutions of the eq. P_3 for parameters

$$\beta+\alpha\epsilon_1=2\epsilon_2, \quad \gamma=1,\ \delta=-1, \quad \epsilon_1^2=\epsilon_2^2=1.$$

Using these solutions as start solutions by the B.T. one obtains new partial solutions of the eq. P_3 for the parameters

$$\beta+\alpha\epsilon_1\epsilon_2=-6\epsilon_2, \quad \gamma=1,\ \delta=-1.$$

They are solutions of the equation

$$(w')^3+\sum_{j=1}^{3} P_j(z,w)(w')^{3-j}=0, \tag{45}$$

where

$$P_1=\epsilon_1 w^2+z^{-1}(3+\alpha\epsilon)w+\epsilon_2,$$

$$P_2=-w^4+2z^{-1}(\epsilon_1-\alpha)w-(2\epsilon_1\epsilon_2 z^2+\alpha^2+6\alpha\epsilon_1+13)z^{-2}w^2 - 2(\alpha+7)\epsilon_2\epsilon_1 z^{-1}w-1,$$

$$P_3=-\epsilon_1 w^6-(1+3\alpha\epsilon_1)z^{-1}w^5-(3\epsilon_2 z^2+3\epsilon_1\alpha^2+10\alpha+15\epsilon_1)\times z^{-2}w^4-6(\alpha+3\epsilon_1)\epsilon_2 z^{-1}w^3-z^{-3}w^3(\alpha^3\epsilon_1+9\alpha^2+23\alpha\epsilon_1+15) -(9\epsilon_1 z^2+3\alpha^2\epsilon_2+26\alpha\epsilon_1\epsilon_2+63\epsilon_2)z^{-2}w^2-(3\alpha\epsilon_1+7)z^{-1}w-\epsilon_2.$$

All solutions of the equation (44) are expressible in terms of Bessel functions, therefore all solutions of the eq. (45) are also expressible in terms of Bessel functions. Using the induction we have now the proof of theorem 5*.

4. THE CONNECTION BETWEEN THE EQUATIONS P_3 AND P_5. THE BÄCKLUND TRANSFORMATION FOR THE EQUATION P_5

Solving the system (30) with respect to the function u(z), one proves the theorem concerning the connection between the solutions of the equations P_3 and P_5 [16].

Theorem 6. If function w(z) is a solution of the equation P_3 for any α, β and $\gamma=1$, $\delta=-1$ so that w(z) satisfies the condition

$$R(z,w,w')\equiv\frac{dw}{dz}-\epsilon w^2-\frac{1}{z}(\alpha\epsilon-1)w+1\neq 0, \quad \epsilon^2=1 \tag{46}$$

then the function

$$u(\tau)=1-\frac{2}{R(z,w,w')}, \quad 2\tau=z^2 \tag{46'}$$

is a solution of the equation P_5

$$u_{\tau\tau}=\frac{3u-1}{2u(u-1)}u'^2-\frac{u'}{\tau}+\frac{a}{\tau^2}u(u-1)^2+\frac{b}{\tau^2}\frac{(u-1)^2}{u}+\frac{c}{\tau}u+\frac{du(u+1)}{u-1} \tag{47}$$

for the parameters

$$a=\frac{1}{32}(\beta-\alpha\epsilon+2)^2, \quad b=-\frac{1}{32}(\beta+\alpha\epsilon-2)^2, \quad c=-\epsilon,\ d=0, ** \tag{48}$$

* Remark. This family of solutions of the eq. P_3 produce a family of solutions of the equation P_5 (see Section 4, th.6).

** Remark. In this part we use new notations for the equation P_5 and its solutions for more clear description of the results. Namely, under the equation P_5 we understand the equation (47), its solution is a function $u(\tau)$, and instead of parameters α, β, γ and δ we write in this equation parameters a, b, c and d, respectively.

Theorem 7. Let $u=u(\tau)$ be a solution of the equation P_5 for some parameters a, b and $c=\pm 1$ and $d=0$ for which the function M is

$$M(\tau,u,u')=\tau\frac{du}{d\tau}-\sqrt{2a}\,u+(\sqrt{2a}+\sqrt{-2b})u-\sqrt{-2b}\neq 0, \tag{49}$$

then a function

$$w(z)=\frac{\sqrt{2\tau}\,u(\tau)}{M(\tau,u,u')},\qquad 2\tau=z^2 \tag{49'}$$

is a solution of the equation P_3 for the patameters

$\alpha=2c(\sqrt{2a}-\sqrt{-2b}-1),$

$\beta=2(\sqrt{2a}+\sqrt{-2b}),\quad \gamma=1,\ \delta=-1$.*

The theorems 6,7 give us the connection between the equation P_3 for parameters $\gamma\delta\neq 0$ and the solutions of the eq. P_5 for parameter $d=0$ $(\delta=0)$ ** .

Using both these theorems we obtain a B.T. for the equation P_5 in the case $d=0$, $c=\pm 1$ (or $\delta=0$, $\gamma=\pm 1$) /16/.

Theorem 8. Let $u(\tau)$ be a solution of the equation P_5 (47), for any parameters a, b and $c=\pm 1$, $d=0$ then the function

$$u_1(\tau)=1+2M^2[2\tau u\frac{dM}{d\tau}-M^2-[2u+2\tau u'-2\epsilon cu(\sqrt{2a}-\sqrt{-2b}-1)]M+2\epsilon\tau u^2]^{-1}, \tag{50}$$

where $\epsilon^2=1$, $M=M(\tau,u,u')$ is defined by (49) and $M\neq 0$,*** is a solution of the equation P_5 for the parameters

$$a_1=\frac{1}{8}[\sqrt{2a}+\sqrt{-2b}-\epsilon c(\sqrt{2a}-\sqrt{-2b}-1)+1]^2,$$
$$b_1=-\frac{1}{8}[\sqrt{2a}+\sqrt{-2b}+\epsilon c(\sqrt{2a}-\sqrt{-2b}-1)-1]^2, \tag{51}$$
$$c_1=-\epsilon,\quad d_1=0.$$

The connection between solutions of the equations P_3 and P_5 has been investigated independently by H.Airault /10/. In this paper the following lemma was proved.

Lemma 1. Let $w(z)$ be a solution of the equation P_3 ($\beta=\delta=0$, $\alpha=\tilde{b}-\tilde{a}$, $\gamma=1$)

$$w''=\frac{(w')^2}{w}-\frac{w'}{z}+\frac{(\tilde{b}-\tilde{a})}{z}w^2+w^3$$

and let $y(z)=\dfrac{(w'-c\cdot w/z)}{w^2}$, where $c=a+b-1$. Then the function $u(z)=(y+1)(y-1)^{-1}$ is a solution of the equation P_5 when $(\delta=\gamma=0)$ $c=d=0$ and

$$a=\frac{\tilde{a}^2}{2},\quad b=-\frac{\tilde{b}^2}{2}\quad (\beta=-\frac{\tilde{b}^2}{2},\ \alpha=\frac{\tilde{a}^2}{2}). \tag{52}$$

Moreover, $u(z)$ is a particular solution of

$$u'=tu+\frac{(u-1)(\tilde{a}u+\tilde{b})}{z},\qquad \text{where } t=2w.$$

This result is not surprising since both the equation P_3 for parameters $\beta=\delta=0$ (see Section 1) and the equation P_5 for parameters $c=d=0$ $(\delta=\gamma=0)$ are completely integrable. Indeed, N.Lukashevich /8/ has obtained that the solutions $u(\tau)$ of the equation P_5 for $c=d=0$ $(\delta=\gamma=0)$ and any a and b can be determined through

$$\int\frac{du}{(u-1)\sqrt{au^2+c_1u-b}}=\pm\sqrt{2}(c_2+\ln\tau). \tag{53}$$

For instance, a partial solution of the equation P_5 for the parameters $a=b=0$, $c^2=-2d$ is

$u(\tau)=\tilde{c}\exp(\pm\sqrt{-2d}\,\tau)$, where $\tilde{c}$ is a constant /8/.

The rational solutions of the equation P_5 for $c=d=0$ $(\delta=\gamma=0)$ which was obtained in /10/ from the rational solutions of the equation P_3 for parameters $\beta=\delta=0$ can be found directly from (53). The other rational solutions of the equation P_5 are constructed with the help of B.T. and described in Section 5.

The further investigation of the systems equivalent to the eq. P_5 allows us to prove the following theorem.

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* Remark. The eqs. (46) and (46) are investigated in /17/.

** Remark. The theorems 6,7 take place also for the general solutions of the eqs. P_3 and P_5, i.e., if $w=w(z,c_1,c_2)$ is a general solution of the eq. P_3 for any α, β, $\gamma=1$ and $\delta=-1$, then $u(\tau)$ defined by (46') is a general solution of the eq. P_5 for the parameters which satisfy the conditions (48) and vice versa.

*** Remark. It is easy to prove that under the condition $M\neq 0$ $(R\neq 0)$ the denominator in the (50) (and, respectively, (32)) is not equal to zero /16/.

Theorem 9. Let $u=u(z)$ be a solution of the equation P_5 for a parameter $d\neq 0$ so, that

$$\Phi \equiv \tau u' - q u^2 + (q-p+k\tau)u + p \neq 0,$$

where $p^2+2b=0$, $q^2=2a$, $k^2=-2d$, then the function

$$u_1(\tau) = 1-2k\tau u\Phi^{-1}(\tau) \qquad (54)$$

is a solution of the equation P_5 for the parameters

$$a_1 = -\frac{1}{16d}[c+k(1-p-q)]^2,$$

$$b_1 = \frac{1}{16d}[c-k(1-p-q)]^2, \qquad (55)$$

$$c_1 = k(p-q)$$

$$d_1 = d.$$

The formulae (54), (55) give us the B.T. for the equation P_5 in the case if the parameter $d\neq 0$ $(\delta\neq 0)$.

5. PARTIAL SOLUTIONS OF THE EQUATION P_5

Families of partial solutions of the equation P_5 can be constructed with the help of B.T. for the equation P_5 and with the help of the relation connecting the equation P_3 with P_5.

We give now some examples of the calculation of new families of the solutions of the equation P_5.

If we take as start solutions for the B.T. the solutions of the equation

$$zw' - 2w^2 + (3+z)w - 1 = 0 \qquad (56)$$

(these solutions are simultaneously solutions of the equation P_5 for the parameters $\alpha=2$, $\beta=-1/2$, $\delta=-1/2$, $\gamma=0$), then after the first step of the B.T. (54) we have that the equation P_5 has solutions which are rational functions of the Whittaker function[15]. In other words, the solutions to the equation

$$y'^2 - \frac{2}{z}yy' + \frac{2}{z}y' - \frac{1}{z^2}y^4 + \frac{z^2+4}{2z^2}y^3 - y^2 - \left(\frac{2}{z^2}+\frac{1}{2}\right)y + \frac{1}{z^2} = 0 \qquad (57)$$

are simultaneously solutions of the equation P_5 for parameters $\alpha_1=\frac{1}{2}$, $\beta_1=-\frac{1}{2}$ $\gamma_1=3$, $\delta_1=-\frac{1}{2}$. Repeating of the B.T. gives us other solutions of this family. All these solutions are rational functions of the Whittaker function (this follows from the form of the B.T. (54)).

The other family of solutions of the equation P_5 can be obtained if we take as start solutions of the B.T. the solutions of the equation

$$zw' - cw^2 + (c-a+kz)w + a = 0. \qquad (58)$$

All solutions of the equation (58) are simultaneously solutions of the equation P_5 under the condition

$$k(1-a-c) = \gamma,$$

where $a^2=-2\beta$, $c^2=2\alpha$, $k^2=-2\delta$.

All solutions of this family are rational functions of the Bessel function.

The family of the rational solutions of the equation P_5 for the parameter $\delta\neq 0$ can be also obtained with the help of B.T. (54).

In the paper[8] N.Lukashevich has proved that any rational solution of the equation P_5 for the parameter $\delta\neq 0$ has the form

$$w(z) = \lambda z + \mu + \frac{P_{n-1}(z)}{Q_n(z)}, \qquad (59)$$

where λ and μ are some constants, $P_{n-1}(z)$ and $Q_n(z)$ are polynomials of the n-1 and n degrees, respectively.

The first three solutions (for $P_{n-1}(z)\equiv z$) have the form

1. $w\equiv -1$ for parameters $\gamma=0$, $\alpha+\beta=0$, δ is any constant,
2. $w=z+1$ for parameters $\alpha=-\delta$, $\beta=-1/2$, $\gamma=-2\delta$, δ is any constant,
3. $w=-kz+a$ for parameters $\alpha=1/2$, $\gamma=k(a-2)$ where $a\neq 1$ and k is any constant.

We can take now these solutions and the solution $w=1/z$ (for parameters $\alpha=0$, $\beta=1/2$, $\gamma=-2$, $\delta=-1/2$) as start solutions for the B.T. (54).

If we take $w_0=\frac{1}{z}$ then after the B.T. we have

$$w_1 = \frac{1+a-z(a-k)}{1+a-z(a+k)},$$

where $a^2=1$, $k^2=1$. It is the solution of the equation P_5 for the parameters

$$\alpha_1 = \frac{1}{8}[-2+k(1-a)]^2,$$

$$\beta_1 = -\frac{1}{8}[-2-k(1-a)]^2,$$

$$\gamma_1 = ka, \quad \delta_1 = \delta.$$

Continuing this procedure we obtain the family of the rational solutions of the equation P_5 for a parameter $\delta \neq 0$.

The equation P_5 has a symmetrical property, namely, if the function $w = \phi(z, \alpha, \beta, \gamma, \delta)$ is a solution of the equation P_5 then the function $\tilde{w} = \phi^{-1}(z, -\beta, -\alpha, -\gamma, \delta)$ is also a solution of the equation P_5. This can be used for the construction of the new solutions.

For the computation of the partial solutions of the equation P_5 the connection between the equations P_3 and P_5 (see Section 4, Th.6 and 7) can also be used. Indeed, each solution of the equation P_3 with parameters $\gamma\delta \neq 0$ gives us two solutions of the equation P_5 with parameters $\delta = 0$, $\gamma = \pm 1$ (see (46′)) and each solution of the equation P_5 with parameters $\gamma = \pm 1$ and $\delta = 0$ gives us 4 solutions of the equation P_3 for the parameters $\gamma = 1$ and $\delta = -1$ (see (49′)).

In this way one can obtain the rational (in $\sqrt{z}$) solutions of the equation P_5 from rational solutions of the equation P_3 (Section 2).

Since both parameters γ and δ are not equal to zero the rational solutions of the equation P_3 have the form

$$w(z) = \frac{P_n(z)}{Q_n(z)},$$

where $P_n(z)$ and $Q_n(z)$ are polynomials of the n degree. For $n = 0$ the equation P_3 has a solution

$$w(z) \equiv \lambda,$$

where $\lambda = \pm\sqrt{-\beta/\alpha}$ (for parameters $\gamma = 1$, $\delta = -1$). Let be $\alpha = 3$, $\beta = -3$, $\gamma = 1$, $\delta = -1$ and $w(z) = 1$ then we obtain two solutions of the equation P_5 (see Th.6)

$$u_0(\tau) = 1 + \sqrt{2\tau}, \quad a = \frac{1}{2}, \quad b = -\frac{1}{8}, \quad c = -1, \quad d = 0; \quad \epsilon = 1,$$

$$\tilde{u}_0(\tau) = \frac{2}{2 + \sqrt{2\tau}} \quad a = \frac{1}{8}, \quad b = -2, \quad c = 1, \quad d = 0, \quad \epsilon = -1.$$

On the other hand, each of these solutions determines four solutions of the equation P_3 through the Th.7. If we take one of them,

$$w_1(z) = \frac{z+1}{z+2} \quad (\alpha = 5, \ \beta = -1, \ \gamma = 1, \ \delta = -1)$$

and apply once more the Th.6 then we obtain two solutions of the equation P_5

$$u_1(\tau) = 1 + \sqrt{2\tau}; \ (a = \frac{1}{2}, \ b = -\frac{1}{8}, \ c = -1, \ d = 0; \ \epsilon = 1) \tag{60}$$

$$\tilde{u}_1(\tau) = \frac{2\tau + 4\sqrt{2\tau} + 3}{\tau\sqrt{2\tau} + 6\tau + 6\tau\sqrt{2\tau} + 3} \quad (a = \frac{9}{8}, \ b = -2, \ c = 1, \ d = 0; \ \epsilon = -1).$$

The first solution is equal to $u_0(\tau)$ and the second solution is a new solution.

The application of the Theorem 7 to the solution (60) gives us the following solution of the equation P_3

$$w_2(z) = -\frac{z^5 + 10z^4 + 39z^3 + 72z^2 + 60z + 18}{z^5 + 9z^4 + 32z^3 + 54z^2 + 42z + 12}$$

for parameters $\alpha = -3$, $\beta = 7$, $\gamma = 1$ and $\delta = -1$. Continuing this procedure we obtain the family of the solutions of the equation P_5 which are rational functions of the $\sqrt{\tau}$ ($\sqrt{z}$).

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