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ON A DECOMPOSITION OF DISTRIBUTIONS GENERATED BY THE ROTATION GROUP

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I. INTRODUCTION

Expansions of distributions in terms of special functions were first studied by Zemanian[1] in the one-dimensional case. In connection with light-cone expansions of current products and the corresponding expansion for their matrix elements higher dimensional expansions of distributions are needed[2,3,4]. An important case as the harmonic analysis on the space $\mathcal{S}'(\boldsymbol{R}^3)$ has been performed in ref.[5].

The purpose of this paper is to investigate an expansion of distributions on the space $\mathcal{S}'(\boldsymbol{R}^d)$ on the basis of the rotation group $SO(d)$. In the second section the expansion of test functions $\phi(x)\in \mathcal{S}(\boldsymbol{R}^d)$ in terms of harmonic polynomials is given. Taking into account this analysis in section III we associate to the expansion of test functions in terms of harmonic polynomials a decomposition of the distribution $f(x)\in\mathcal{S}'(\boldsymbol{R}^d)$ in a series of distributions $f^m(x)\in\mathcal{S}'(\boldsymbol{R}^d)$. We proof the theorem that this decomposition of $f(x)$ is available in the strong topology of $\mathcal{S}'(\boldsymbol{R}^d)$. Furthermore we describe the general structure of this decomposition if the distribution $f(x)$ is invariant with respect to a rotation $g\in SO(k)$. The fourth and last section is devoted to the validity of this decomposition after Fourier Transform and to some applications to analytical function expansion. According to Müller[6] we use the following notations. By spherical harmonics $Y_m(\alpha)$ we call the eigenvectors of the spherical part Δ_α of the d-dimensional Laplace operator Δ

$$\Delta_\alpha Y_m(\alpha) = -m_0(m_0+d-2)\,Y_m(\alpha) \tag{1}$$

$$m=(m_0, m(d))=(m_0, m_1, \ldots, m_{d-2}), \quad m_0\ge m_1\ge \ldots \ge m_{d-3}\ge |m_{d-2}|\ge 0,$$

where r, α, $\alpha=(\alpha_1, \ldots, \alpha_d)$, $x_i=\alpha_i r$ are the spherical coordinates of $x\in\boldsymbol{R}^d$.

Remarks. 1. They form a complete and closed set of orthogonal functions on the sphere S^{d-1}.

2. For each m_0 there exist

$$N(m_0)=(2m_0+d-2)(m_0+d-3)!/(m_0!(d-2)!)$$

linear independent spherical harmonics.

3. For the spherical harmonics the integral representation

$$Y_m(a)=\frac{1}{r_{m_0}}\int_{S^{d-1}} d\Omega(a')Y_m(a')(a,a')^{m_0}, \quad r_{m_0}=\frac{\pi^{\frac{d}{2}}\,2^{1-m_0}\,\Gamma(m_0+1)}{\Gamma(m_0+\frac{d}{2})} \tag{2}$$

is valid, where $d\Omega(a)$ is the invariant measure on the sphere S^{d-1}. $|\Omega|$ is the total surface.

We define the harmonic polynomials $H_m(x)$ by

$H_m(x)=r^{m_0}Y_m(a)$.

II. EXPANSION OF TEST FUNCTIONS

Besides the usual space $\mathcal{S}(\mathbf{R}^d)$ with the topology given by

$$||\phi(x)||_{p,\beta}=\sup_{x\in\mathbf{R}^d}(1+x^2)^p|\mathfrak{D}^\beta\phi(x)|, \qquad \beta=(\beta_1,\dots,\beta_d)$$

$$|\beta|=\sum_{i=1}^{d}\beta_i$$

the space $\mathcal{S}_{R_+}$ is used, described by the seminorm system

$$||\phi(t)||^{(n)}_{k,\ell,s}=\sup_{t\in\mathbf{R}_+} t^{\frac{(n-s)_+}{2}}(1+t)^k|\mathfrak{D}^\ell\phi(t)|$$

$$(n-s)_+=\max(0,n-s).$$

Proposition 1

For every function $\phi(x)\in\mathcal{S}(\mathbf{R}^d)$ there exists the expansion

$$\phi(x)=\sum_{m\geq 0}\phi_m(x^2)H_m(x) \tag{3}$$

with

$$\phi_m(x^2)=(x^2)^{-\frac{m_0}{2}}\int_{S^{d-1}} d\Omega(a)\bar{Y}_m(a)\phi(r,a), \qquad r=\sqrt{x^2} \tag{4}$$

converging in the space $\mathcal{S}(\mathbf{R}^d)$.

Proof. At first it will be shown that the expansion coefficients $\phi_m(x^2)$ of the function $\phi(x)\in\mathcal{S}(\mathbf{R}^d)$ are elements of the space $\mathcal{S}_{R_+}$. By construction it is clear that

$$\tilde{\phi}_m(r)=\int d\Omega(a)\bar{Y}_m(a)\phi(r,a) \tag{5}$$

belongs to the space $\mathcal{S}(\mathbf{R}^1)$. To show that the function ϕ_m is an element of $\mathcal{S}_{R_+}$ the properties of $\tilde{\phi}_m$ at the point $r=0$ must be investigated. For computing the derivatives $(\frac{\partial}{\partial r})^n\tilde{\phi}_m(r)$ in eq. (5) we apply the chain rule

$$\left(\frac{\partial}{\partial r}\right)^n=\left(\frac{x_i}{r}\frac{\partial}{\partial x_i}\right)^n=\sum_{i_1,\dots,i_n=1}^{n}a_{i_1}\dots a_{i_n}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_n}}$$

so that

$$\left(\frac{\partial}{\partial r}\right)^n\tilde{\phi}_m(r)\Big|_{r=0}=\int d\Omega(a)\bar{Y}_m(a)\sum a_{i_1}\dots a_{i_n}\frac{\partial^n\phi(x)}{\partial x_{i_1}\dots\partial x_{i_n}}\Big|_{r=0}.$$

Because of the orthogonality relation $\int d\Omega(a)\bar{Y}_m(a)a_{i_1}\dots a_{i_n}=0$ for $m_0>n$ we conclude

$$\left(\frac{\partial}{\partial r}\right)^n\tilde{\phi}_m(0)=0, \qquad n<m_0. \tag{6}$$

The symmetry properties $Y_m(a)=(-1)^{m_0}Y_m(-a)$ yield $\tilde{\phi}_m(-r)=(-1)^{m_0}\tilde{\phi}_m(r)$ so that $\phi_m(r^2)=r^{-m_0}\tilde{\phi}_m(r)$ is an even function. Now it is obvious that the expansion coefficients $\phi_m(t)$ belong to the space $\mathcal{S}_{R_+}$.

As the next step the seminorms $||\phi_m(t)||^{(m_0)}_{k,\ell,s}$ have to be estimated. At first they can be related to $\mathcal{S}(\mathbf{R}^1)$ seminorms. Using the commutator relation $[\frac{d}{dt},t^n]=nt^{n-1}$ we have for $k\geq 2\ell+s$

$$||\phi_m(r^2)||^{(m_0)}_{k,\ell,s}=\sup|(1+r^2)^k r^{m_0-s}\left(\frac{1}{2r}\frac{d}{dr}\right)^\ell\phi_m(r^2)|$$

$$\leq C\left(m_0+\frac{d-2}{2}\right)^\ell\max_{s_1\leq\ell}||r^{m_0-s-2\ell}\phi_m(r^2)||_{k+\ell;s_1} \tag{7}$$

and for $k<2\ell+s$

$$\|\phi_m(r^2)\|^{(m_0)}_{k,\ell,s} \le C\left(m_0+\frac{d-2}{2}\right)^{\frac{(m_0-s)_+}{2}} \cdot \max_{s_1 \le \frac{(m_0-s)_+}{2}} \left\|\left(\frac{1}{r}\frac{d}{dr}\right)^{\ell-\frac{(m_0-s)_+}{2}} \phi_m(r^2)\right\|_{k+\ell,s_1}$$

$$\le C\left(m_0+\frac{d-2}{8}\right)^{\ell} \max_{s_1 \le \frac{(m_0-s)_+}{2}} \left\|\left(\frac{1}{r}\frac{d}{dr}\right)^{\ell-\frac{(m_0-s)_+}{2}} \phi_m(r^2)\right\|_{k+\ell,s_1} . \qquad (8)$$

The further estimate with respect to $\mathcal{S}(R^d)$ seminorms is obtained if the bounds of the spherical harmonics $|Y_m(\alpha)| < (N(m_0)/|\Omega|)^{1/2}$, the definition of the expansion coefficients (4), eq. (6), and the Taylor Theorem are used. From ineq. (7), respectively, in eq. (8) we obtain

$$\|\phi_m(t)\|^{(m_0)}_{k,\ell,s} \le C\left(m_0+\frac{d-2}{2}\right)^{\ell+\frac{d-2}{2}} \max_{|\beta|<3\ell+s} \|\phi(x)\|_{k+\ell,\beta} . \qquad (9)$$

As further auxiliary step we regard the derivatives of the harmonic polynomials $\mathfrak{D}^\beta H_m(x)$. It is clear that $\mathfrak{D}^\beta H_m(x)$ are also harmonic polynomials, of cource, of degree $m_0-|\beta|$ because the operators $\mathfrak{D}^\beta$ and Δ commute. Therefore

$$\mathfrak{D}^\beta H_m(x) = \begin{cases} \sum_{m'(\alpha)} a^m_{\beta,m'(\alpha)} H_{m_0-|\beta|,m'(\alpha)}(x) & |\beta| \le m_0 \\ 0 , & |\beta| > m_0 . \end{cases}$$

With the help of the orthogonality relation for the spherical harmonics and the representation (2) the coefficients $a^m_{\beta,m'(d)}$ can be calculated

$$a^m_{\beta,m'(d)} = \int d\Omega(\alpha)\bar{Y}_{m_0-|\beta|,m'(d)}(\alpha)[\mathfrak{D}^\beta H_m(x)|_{x=\alpha}]$$

$$= \frac{r_{m_0-|\beta|}}{r_{m_0}} \frac{m_0!}{(m_0-|\beta|)!} \int d\Omega(\alpha)\alpha_1^{\beta_1}\dots\alpha_d^{\beta_d}\bar{Y}_{m_0-|\beta|,m'(d)}(\alpha)Y_m(\alpha).$$

Using the bound of the spherical harmonics this implies

$$|a^m_{\beta,m'(d)}| \le 2^{|\beta|}\left(m_0+\frac{d-2}{2}\right)^{d-2+|\beta|}.$$

Now it is possible to estimate each term of the series (3)

$$\|H_m(x)\phi_m(x^2)\|_{p,q}$$

$$= \sup_{x\in R^d}(1+x^2)^p \left| \sum_{0\le|\beta'|\le|\beta|} C_\beta(\mathfrak{D}^{\beta-\beta'}H_m(x))\mathfrak{D}^{\beta'}\phi_m(x^2)\right|$$

$$\le C\left(m_0+\frac{d-2}{2}\right)^{|\beta|+\frac{5}{2}(d-2)} \max_{|\beta'|\le|\beta|} \|\phi_m(t)\|^{(m_0)}_{p+|\beta|,|\beta'|,|\beta|} \qquad (10)$$

$$\le C\left(m_0+\frac{d-2}{2}\right)^{2|\beta|+3(d-2)} \max_{|\beta'|\le 4|\beta|} \|\phi(x)\|_{p+2|\beta|,\beta'} .$$

To show that the series (3) converges in $\mathcal{S}(R^d)$ we need a better m_0 independent estimate. For this reason we discuss $\left(m_0+\frac{d-2}{2}\right)^{2n}\phi_m(x^2)$. From the definition (1) we get

$$\left(m_0+\frac{d-2}{2}\right)^{2n}\phi_m(x^2) = r^{-m_0}\int d(\alpha)\left[\left(\frac{d-2}{2}-\Delta_\alpha\right)^n \bar{Y}_m(\alpha)\right]\phi(r,\alpha)$$

$$= r^{-m_0}\int d\Omega(\alpha)\bar{Y}_m(\alpha)\left[\left(\frac{d-2}{2}-\Delta_\alpha\right)^n\phi(r,\alpha)\right]. \qquad (11)$$

Remark that the function $\left(\frac{d-2}{2}-\Delta_\alpha\right)^n\phi(r,\alpha)$ is nothing else than the function $\phi^{(n)}(x) = \{\left(\frac{d-2}{2}+\sum_{i=1}^d x_i\frac{\partial}{\partial x_i}\right)^2 - x^2\Delta\}^n\phi(x)\in\mathcal{S}(R^d)$ in spherical coordinates /5/. Using ineq. (9) this leads to

$$\|\phi_m(t)\|^{(m_0)}_{k,\ell,s} \le C\left(m_0+\frac{d-2}{2}\right)^{\frac{d-2}{2}+\ell-2n} \max_{|\beta|\le 3\ell+s} \|\phi^{(n)}(x)\|_{k+\ell,\beta}$$

and, consequently, ineq. (10) can be written as

$$\|H_m(x)\phi_m(x^2)\|_{p,\beta} \le$$

$$\le C\left(m_0+\frac{d-2}{2}\right)^{2|\beta|+3(d-2)-2n} \max_{|\beta'|\le 4|\beta|} \|\phi^{(n)}(x)\|_{k+2|\beta|,\beta'} , \qquad (12)$$

where the constant c depends on p and β only. The last inequality is sufficient to show the convergence of the series (3) in $\mathcal{S}(\mathbf{R}^d)$. This follows immediatly from

$$\| \sum_{m\geq 0} H_m(x)\phi_m(x^2)\|_{p,\beta} \leq \sum_{m\geq 0} \| H_m(x)\phi_m(x^2)\|_{p,\beta}$$

$$\leq C \max_{|\beta'|\leq 4|\beta|} \|\phi^{(n)}(x)\|_{p+2|\beta|,\beta'} \sum_{m_0=0}^{\infty} (m_0+\frac{d-2}{2})^{2|\beta|+4(d-2)-2n} \quad (13)$$

$$\leq C' \max_{|\beta'|\leq 4|\beta|} \|\phi^{(n)}(x)\|_{p+2|\beta|,\beta'} \quad \text{for } n \geq 2(d-2)+|\beta|+1.$$

Trivially this shows that the series (3) is uniformly converging and represents a continuous function. Note that for fixed r and all spherical harmonics $Y_{m'}(\alpha)$

$$\int d\Omega(\alpha) Y_{m'}(\alpha) \{ (\sum_{m\geq 0} H_m(x)\phi_m(x^2)) - \phi(x)\} = 0$$

is true. From the continuity it follows

$$\sum_{m\geq 0} H_m(x)\phi_m(x^2) - \phi(x) \equiv 0$$

so that the series $\Sigma H_m(x)\phi_m(x^2)$ converges to the function $\phi(x)$ in $\mathcal{S}(\mathbf{R}^d)$.

□

<u>Definition 1</u>: $\mathcal{S}(\mathbf{R}_+;SO(d))$ denotes the set of all sequences $\{\phi_m(t)\}$ of functions $\phi_m(t)$ infinitely differentiable on $\mathbf{R}_+$ such that

$$\max_m (m_0+\frac{d-2}{2})^n \|\phi_m(t)\|^{(m_0)}_{k,\ell,s} < \infty \quad (14)$$

for any n, k, ℓ, s /5/.

Remark. The space $\mathcal{S}(\mathbf{R}_+;SO(d))$ is a locally convex topological vector space with the topology introduced by ineq.(14). According to definition 1 the proposition 1 changes to

<u>Proposition 1'</u>

The map

$$j_1 : \{\phi_m(t)\} \longrightarrow \phi(x) = \Sigma H_m(x)\phi_m(x^2)$$

is a topological isomorphism between the spaces $\mathcal{S}(\mathbf{R}_+;SO(d))$ and $\mathcal{S}(\mathbf{R}^d)$.

III. DECOMPOSITION OF DISTRIBUTIONS

<u>Definition 2</u>: By spherical harmonics of the distribution $f(x)\in\mathcal{S}'(\mathbf{R}^d)$ we call the linear functional $f_m(t)\in\mathcal{S}'_{\mathbf{R}_+}$ defined by

$$<f_m(t),\phi(t)> := <f(x), H_m(x)\phi(x^2)>, \quad \phi(t)\in\mathcal{S}_{\mathbf{R}_+}. \quad (15)$$

Remark. Because $H_m(x)\phi(x^2)\in\mathcal{S}(\mathbf{R}^d)$ for $\phi(t)\in\mathcal{S}_{\mathbf{R}_+}$ the definition 2 is correct. Due to ineq.(10) we have

$$|<f_m(t),\phi(t)>| \leq$$

$$\leq C\|f\|_{p,\beta} (m_0+\frac{d-2}{2})^{|\beta|+\frac{5}{2}(d-2)} \max_{\beta'\leq|\beta|} \|\phi(t)\|^{(m_0)}_{p+|\beta|,\beta',|\beta|} \quad (16)$$

so that the relation (15) defines a continuous linear functional on $\mathcal{S}'_{\mathbf{R}_+}$.

<u>Proposition 2</u>

The map

$$j_2: \ f(x) \longrightarrow \{f_m(t)\} \quad (17)$$

gives a topological isomorphism between the spaces $\mathcal{S}'(\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}_+;SO(d))$.

<u>Proof</u>. For $f(x)\in\mathcal{S}'(\mathbf{R}^d)$ it follows from ineq.(14) that $\{f_m(t)\}\in\mathcal{S}'(\mathbf{R}_+;SO(d))$. Since the set of all finite sequences $\{\phi_0(t),\ldots,\phi_m(t),0,0,\ldots\}$ is dence in $\mathcal{S}(\mathbf{R}_+;SO(d))$ the map j_2 is a map from $\mathcal{S}'(\mathbf{R}^d)$ on $\mathcal{S}'(\mathbf{R}_+;SO(d))$. Now we consider a sequence $\{f_m(t)\}\in\mathcal{S}'(\mathbf{R}_+;SO(d))$. In view of ineq.(12) and (16) the series $\Sigma<f_m(t),\phi_m(t)>$ converges. Taking into account proposition 1 and definition 2 the reconstruction of the distribution f(x) is arrived at

$$<f(x),\phi(x)> = \Sigma<f_m(t),\phi_m(t)>.$$

□

Corollary 3

$j_1^* = j_2$ in the weak topology of dual pairs.

From the physical point of view it is interesting to know the common structure of the decomposition (17) if the distribution is invariant with respect to additional transformations (for instance, gauge transformation). Here we describe the case that the distribution $f(x)$ from $\mathcal{S}'(\boldsymbol{R}^d)$ is invariant under rotations $g \in SO(k)$.

Invariance of a distribution $f(x)$ under rotations $g \in SO(k)$ means

$$\langle f(x), \phi(x)\rangle = \int_{SO(k)} dg \langle f(x), \phi_g(x)\rangle , \tag{18}$$

where dg is the normalized invariant Haar measure on the group $SO(k)$ and $\phi_g(x)$ is defined by $\phi_g(x) = \phi(g \cdot x)$. In order to simplify formulae without loss of generality we propose that the rotation g acts on the subspace $\boldsymbol{R}^k$ of the first k variables.

Proposition 4

The map

$$j: f(x) \longrightarrow \{ f_m(t)\}_{m \geq 0,\ m_{d-k} = \dots = m_{d-2} = 0} \tag{19}$$

gives a topological isomorphism between the space of $SO(k)$ invariant distributions of $\mathcal{S}'(\boldsymbol{R}^d)$ and the space $\mathcal{S}'(\boldsymbol{R}_+; SO(d); SO(k))$.

Proof. Since proposition 1 and the invariance of the measure $d\Omega(\alpha)$ on the unit sphere S^{d-1} we have

$$\phi_g(x) = \sum_{m \geq 0} H_m(x)(x^2)^{-m_0/2} \int d\Omega(\alpha)\bar{Y}_m(\alpha)\phi(r, g\circ\alpha)$$

$$= \sum_{m \geq 0} H_m(x)(x^2)^{-m_0/2} \int d\Omega(\alpha)\bar{Y}_m(g^{-1}\circ\alpha)\phi(r,\alpha). \tag{20}$$

On the other hand, from the explicit form of $H_m(x)$ in terms of Gegenbauer polynomials [7] we can write

$$H_m(x) = W_{m_0 \dots m_{d-k}}(x^2, \alpha_{k+1}, \dots, \alpha_d) H_{m_{d-k} \dots m_{d-2}}(x_1, \dots, x_k), \tag{21}$$

$$W_{m_0 \dots m_{d-k-1}, 0}(x^2, \alpha_{k+1}, \dots, \alpha_d) = H_{m_0 \dots m_{d-k-1}, 0, \dots 0}(x).$$

Using the representation theory of compact groups we have

$$H_{m_{d-k}, \dots, m_{d-2}}(g^{-1}\circ x) = T^k_{m_{d-k}}(g) H_{m_{d-k}, \dots, m_{d-2}}(x)$$

$$= \sum_{m'} t^{k, m_{d-k}}_{m_{d-k+1}, \dots m_{d-2}, m'}(g^{-1}) H_{m_{d-k}, m'}(x). \tag{22}$$

$T^k_\ell(g)$ is the operator generating the unitary irreducible representations of $SO(k)$ on the space of homogeneous polynomials of degree ℓ in k dimensions, and $t^{k,\ell}_{m,m'}(g)$ are the matrix elements of the corresponding canonical matrix.

Remark. The orthogonality relation

$$\int_{SO(k)} dg\, t^{k,\ell}_{(m)}(g) \overline{t^{k,\ell'}_{(m')}(g)} = \begin{cases} 0 & (m) \neq (m') \\ \dfrac{1}{\dim T^k_\ell(g)} & (m) = (m') \end{cases} \tag{23}$$

is true.

Taking into account eq. (20)-(23) by standard considerations we conclude

$$\langle f(x), \phi(x)\rangle = \int_{SO(k)} dg \langle f(x), \phi_g(x)\rangle$$

$$= \sum_{m \geq 0} \int_{SO(k)} dg \langle f(x), H_m(x)(x^2)^{-m_0/2} \int d\Omega(\alpha)\bar{Y}_m(g^{-1}\circ\alpha)\phi(x)\rangle$$

$$= \sum_{m \geq 0} \int_{SO(k)} dg \langle f(x), H_m(x)(x^2)^{-m_0/2} \int d\Omega(\alpha)\bar{W}_{m_0 \dots m_{d-k}}(1, \alpha_{k+1} \dots \alpha_d) \times$$

$$\times \bar{Y}_{m_{d-k}, \dots m_{d-2}}(g^{-1}(\alpha_1, \dots \alpha_k))\phi(x)\rangle$$

$$= \sum_{m\geq 0} \int_{SO(k)} dg \sum_{m'} t^{k,m_{d-k}}_{m_{d-k+1},\ldots,m_{d-2},m'}(g^{-1}) \times$$

$$\times < f(x), H_m(x)(x^2)^{-m_0/2} \int d\Omega(\alpha)\bar{W}_{m_0 \ldots m_{d-2}}(1,\alpha_{k+1},\ldots,\alpha_d)\bar{Y}_{m'}(\alpha)\phi(x)>$$

$$= \sum_{\substack{m\geq 0 \\ m_{d-2}=\ldots=m_{d-k}=0}} < f(x), H_m(x)(x^2)^{-m_0/2} \int d\Omega(\alpha)\bar{W}_{m_0 \ldots m_{d-2}}(1,\alpha_{k+1},\ldots,\alpha_d)\phi(x)>$$

$$= \sum_{\substack{m\geq 0 \\ m_{d-k}=\ldots=m_{d-2}=0}} < f_m(t), \phi_m(t)> .$$

□

Our final aim is to have a decomposition of the distribution $f(x) \in S'(R^d)$ in terms of distributions on the same space.

Definition 3: Let $\{g_{m'}(t)\}$ be a sequence from $S'(R_+; SO(d))$ such that $g_{m'}(t) = f_m(t)$ for $m' = m$ and $g_{m'}(t) = 0$ otherwise. By the linear functional $f^m(x) \in S'(R^d)$ defined by

$$< f^m(x), \phi(x)> := \sum_{m'} < g_{m'}(t), \phi_{m'}(t)> = < f_m(t), \phi_m(t)>$$

for all $\phi(x) \in S(R^d)$ we call harmonic components of the distribution $f(x)$.

Notes. 1. Due to proposition 2 the definition is correct. By ineq.(9) and (16) follows

$$|< f^m(x), \phi(x)>| = |< f_m(t), \phi_m(t)>| \leq$$

$$\leq ||f_m||^{(m_0)}_{k,\beta,s} ||\phi_m||^{(m_0)}_{k,\beta,s} \leq C(m_0 + \frac{d-2}{2})^{\beta+\frac{d-2}{2}} ||f_m|| \cdot \max_{|\ell|\leq 3\beta+s} ||\phi(x)||_{k+\beta,\ell},$$

since $f^m(x)$ is a continuous linear functional.

2. Considering that $f_m(t)$ is generated by $f(x)$ we get from ineq.(11) and (12)

$$|< f^m(x), \phi(x)>| = |< f(x), H_m(x)\phi_m(x^2)>| \leq$$

$$\leq ||f||_{k,\beta} \cdot ||\phi_m(x^2)H_m(x)||_{k,\beta}$$

$$\leq C(m_0 + \frac{d-2}{2})^{2|\beta|+3(d-2)-2n} \max_{|\ell|\leq 4|\beta|+2n} ||\phi(x)||_{k+2|\beta|+n,\ell}. \qquad (24)$$

The constant c is m - independent and depends only on n, k, β, and the space dimension d.

3. Obviously $f^m(x)$ has the property

$$< f^m(x), \phi(x)> = < f^m(x), H_m(x)\phi_m(x^2)>$$

$$= < f(x), H_m(x)\phi_m(x^2)>. \qquad (25)$$

Now we can state our theorem.

Theorem.

The series

$$\sum_{m\geq 0} f^m(x) \qquad (26)$$

converges in the strong topology of $S'(R^d)$ to $f(x)$.

Proof. Considering proposition 1 and ineq.(24), we are allowed to conclude for $2n \geq 2|\beta| + 4(d-2)+2$

$$|< f - \sum_{m\leq \bar{m}} f^m(x), \phi(x)>| = |< f(x), \sum_{m>\bar{m}} H_m(x)\phi_m(x^2)>|$$

$$\leq C ||f||_{k,\beta} \max_{|\ell|\leq 4|\beta|+2n} ||\phi(x)||_{k+2|\beta|+n,\ell} \sum_{m_0=\bar{m}_0}^{\infty} (m_0 + \frac{d-2}{2})^{2|\beta|+4(d-2)-2n}$$

$$\leq \epsilon \cdot C ||f||_{k,\beta} \max_{|\ell|\leq 4|\beta|+2n} ||\phi(x)||_{k+2|\beta|+n,\ell}.$$

The inequality implies the theorem.

The series (26) is the desired decomposition of the distribution.

Remark. With the help of the proposition 4 it is easy to write the general structure of the decomposition (26) if the distribution $f(x)$ is invariant under rotations $g \in SO(k)$.

IV. APPLICATIONS OF THE DECOMPOSITION TO FOURIER TRANSFORM

The purpose of this section is to study the Fourier Transform of the series (26). First we proof the formula

$$\mathcal{F}[f^m(x)](g) = \mathcal{F}[f(x)]^m(g)$$

and second we give some applications to analytical function expansion.

Lemma: Suppose $f(t) \in C^\infty$. Then for all m the relation holds

$$H_m(\frac{\partial}{\partial x}) f(x^2) = 2^{m_0} H_m(x) (\frac{\partial}{\partial x^2})^{m_0} f(x^2), \quad x^2 = \sum_{i=1}^{d} x_i^2. \tag{27}$$

Proof. From the integral representation for the spherical harmonics (2) we have

$$H_m(\frac{\partial}{\partial x}) f(x^2) = \frac{1}{r_{m_0}} \int d\Omega(\alpha) Y_m(\alpha) (\alpha, \frac{\partial}{\partial x})^{m_0} f(x^2). \tag{28}$$

Simple computations show that

$$(\alpha, \frac{\partial}{\partial x})^{m_0} f(x^2) = [2(\alpha, x)]^{m_0} (\frac{\partial}{\partial x^2})^{m_0} f(x^2) + \Phi(f, x, \alpha), \tag{29}$$

where Φ is a polynom in α, however, of degree smaller than m_0. Substituting eq.(29) into eq.(28) and using the orthogonality relation for the spherical harmonics we get formula (27).

Notes. 1. The Fourier Transform of $H_m(x)\phi_m(x^2)$ can be expressed by

$$\mathcal{F}[H_m(x)\phi_m(x^2)](q) = H_m(i\frac{\partial}{\partial q}) \mathcal{F}[\phi_m(x^2)](q)$$

$$= (2i)^{m_0} H_m(q) (\frac{\partial}{\partial q^2})^{m_0} \mathcal{F}[\phi_m](q^2).$$

2. As a consequence of the proposition 1 we are allowed to write

$$\mathcal{F}[\phi](q) = \sum_{m \geq 0} H_m(q) \mathcal{F}[\phi]_m(q^2) = \sum_{m \geq 0} H_m(q)(2i)^{m_0} (\frac{\partial}{\partial q^2})^{m_0} \mathcal{F}[\phi_m](q^2). \tag{30}$$

Hence $\mathcal{F}[\phi]_m(q^2) = (2i)^{m_0} (\frac{\partial}{\partial q^2})^{m_0} \mathcal{F}[\phi_m(x^2)](q^2)$. Taking into account the property (25) we get

$$\mathcal{F}[f^m(x)](q) = \mathcal{F}[f(x)]^m(q). \tag{31}$$

Let us now propose that the distribution $f(x)$ has a compact support. It is well known that the Fourier Transform $\mathcal{F}[f](q)$ is an entire function of first order, polynomial bounded on the real axis. Remark that the support of $f^m(x)$ is not bigger than the support of $f(x)$ so that $\mathcal{F}[f^m](q)$ is also an entire function of an order and type as $f(x)$. Using eq.(25),(31) and the analytical properties of $\mathcal{F}[f^m(x)](q)$ simple computations show

$$< \mathcal{F}[f^m(x)](q), \phi(q) > =$$

$$< \{ (q^2)^{-m_0/2} \int d\Omega(\alpha) \bar{Y}_m(\alpha) \mathcal{F}[f^m(x)](q) \} H_m(q), \phi(q) >.$$

Finally we have

$$< \mathcal{F}[f(x)](q), \phi(q) > = \sum_{m \geq 0} < \mathcal{F}[f^m(x), \phi(q) >$$

$$= \sum_{m \geq 0} < f_m(q^2) H_m(q), \phi(q) > = < \sum_{m \geq 0} f_m(q^2) H_m(q), \phi(q) >.$$

This means, every first order analytical function which is polynomial bounded on the real axis can be expanded in a series in terms of harmonic polynomials uniformly converging in every compact subset of the complex plane. A similar result for the analytical function expansion is described in ref./4/.

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