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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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TOPOLOGICAL ALGEBRAS OF OPERATORS

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E5 - 4606

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Submitted to Journal of Functional Analysis

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Introduction

For a systematic study of topological non-normed algebras and their representations one has to regard also representations of algebras of unbounded operators. The impressive results of the theory of Banach algebras and their representations are obtained by a refined combination of algebraic and topological methods. Therefore one would like to have also on an algebra of unbounded operators a "natural" topology like the uniform topology (operator-norm topology) on an algebra of bounded operators.

In this paper topologizations of algebras of unbounded operators are compared. For an Op^*-algebra (Def. 2.1) the uniform topology $\mathfrak{T}_{\mathfrak{D}}$ is defined (Def. 4.1) which is expected to be a good generalization of the uniform topology on an algebra of bounded operators. With this topology in Sec. 5 the notions of an O^*-algebra and of an AO^*-algebra are defined, which are generalizations of the notions of a C^*-algebra and of a B^*-algebra. In Sec. 6 we prove for a large class of function algebras $C(\mathfrak{M})$ with the bicompact-open topology that they are AO^*-algebras. In Sec. 7 is investigated what topologies $\mathfrak{T}_{\mathfrak{D}}$ are related to representations of the algebra of polynoms.

The considerations in the last both sections are some modest steps to demonstrate that one can get nontrivial results about the structure of topological algebras and their representations with the help of the concepts developed in this paper.

In a forthcoming paper we hope to apply the ideas developed here to the investigation of representations of tensor algebras over a locally convex space, especially to the topological tensor algebra $\mathcal{S}_{\otimes}$ over the Schwartz space $\mathcal{S}$, which plays an important role in the quantum field theory/11,12/.

1. Definitions

An algebra $\mathcal{A}$ is a linear space over C (complex field) with a multiplication. We always assume $\mathcal{A}$ to contain a unit element e, $ea = ae = a$, $\forall a \in \mathcal{A}$. $\mathcal{A}$ is called a $*$-algebra if in $\mathcal{A}$ is defined an involution $a \to a^*$, $(\lambda a)^* = \bar{\lambda} a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$. A LK-algebra $\mathcal{A}$ is an algebra in which a locally convex topology is defined by a set of seminorms $\| \|_\gamma$, $\gamma \in \Gamma$ (index set), so that the multiplication is separately continuous, i.e. $a \to ba$ and $a \to ab$ are linear continuous mappings of $\mathcal{A}$ for any $b \in \mathcal{A}$.

The system of seminorms $\| \|_\gamma$, $\gamma \in \Gamma$, of a locally convex space E is said to be saturated if for $\gamma, \gamma' \in \Gamma$ there is a $\gamma'' \in \Gamma$ with $\|x\|_\gamma + \|x\|_{\gamma'} \leq M \|x\|_{\gamma''}$, $M > 0$. The multiplication $a, b \to ab$ is said to be jointly continuous if for every $\gamma \in \Gamma$ there are $\gamma', \gamma'' \in \Gamma$ so that $\|ab\|_\gamma \leq M \|a\|_{\gamma'} \|b\|_{\gamma''}$, $\forall a, b \in \mathcal{A}$.

A LK-algebra which is moreover a $*$-algebra is called LK^*-algebra if the involution $a \to a^*$ is a continuous mapping.

2. Algebras of operators

Let $\mathfrak{D}$ be a unitary space (uncomplete Hilbert space) with the scalar product $\langle\phi,\Psi\rangle$, $\langle\bar{\lambda}\phi,\Psi\rangle = \langle\phi,\lambda\Psi\rangle = \lambda\langle\phi,\Psi\rangle$ $\phi,\Psi\in\mathfrak{D}$, $\lambda\in C$ (the field of complex numbers) and $\|\phi\| = \langle\phi,\phi\rangle^{\frac{1}{2}}$. The Hilbert space $\mathcal{H}$ denotes the completion of $\mathfrak{D}$.

By $\mathcal{L}_a(\mathfrak{D})$ we denote the algebra of all linear operators of $\mathfrak{D}$. ($\mathcal{L}_a(\mathfrak{D})$ contains all linear operators A defined on $\mathfrak{D}$ with $A\mathfrak{D}\subset\mathfrak{D}$, without an assumption on the continuity).

Each operator $A\in\mathcal{L}_a(\mathfrak{D})$ can be regarded as an (unbounded) operator in $\mathcal{H}$ with the dense domain $\mathfrak{D}(A)=\mathfrak{D}$ By $\mathcal{L}_o(\mathfrak{D})$ we denote the set of all operators $A\in\mathcal{L}_a(\mathfrak{D})$ which are closable in $\mathcal{H}$, i.e. for which the closure $\bar{A}$ in $\mathcal{H}$ exists (/3/, XII, 4.7) and by $\mathcal{L}_+(\mathfrak{D})$ we denote the set of all operators $A\in\mathcal{L}_a(\mathfrak{D})$ for which an operator $A^+\in\mathcal{L}_a(\mathfrak{D})$ exists, that $\langle\phi,A\Psi\rangle = \langle A^+\phi,\Psi\rangle$, $\forall\,\phi,\Psi\in\mathfrak{D}$ holds. With other words, $\mathcal{L}_+(\mathfrak{D})$ contains all linear operators A with the domain $\mathfrak{D}(A)=\mathfrak{D}$ which satisfy the conditions:

1. $\mathfrak{D}$ is invariant for A, $A\mathfrak{D}\subset\mathfrak{D}$
2. For A exists the adjoint operator A^*
3. The domain $\mathfrak{D}(A^*)$ of the adjoint operator contains $\mathfrak{D}$, $\mathfrak{D}(A^*)\supset\mathfrak{D}$,
   and it holds $A^*\mathfrak{D}\subset\mathfrak{D}$.
   The restriction of A^* to the domain $\mathfrak{D}$ is denoted by A^+.

Let us remark that $\mathcal{L}_o(\mathfrak{D})$ in general is not an algebra, i.e.

if $A, B \in \mathcal{L}_0(\mathcal{D})$ then it is possible that $A \cdot B \notin \mathcal{L}_0(\mathcal{D})$, because the closure $\overline{A \cdot B}$ does not exist. But it holds the following

<u>Lemma 2.1</u>. $\mathcal{L}_+(\mathcal{D})$ is an algebra. Equipped with the involution $A \to A^+$ $\mathcal{L}_+(\mathcal{D})$ becomes a $*$ -algebra. It holds $\mathcal{L}_+(\mathcal{D}) \subset \mathcal{L}_0(\mathcal{D})$.

<u>Proof</u>. If $A, B \in \mathcal{L}_+(\mathcal{D})$ then exist A^+ and B^+ and it holds $\langle \phi, AB\psi \rangle = \langle B^+A^+\phi, \psi \rangle$, $\forall \phi, \psi \in \mathcal{D}$. From this we see that $(AB)^+$ exist and consequently $AB \in \mathcal{L}_+(\mathcal{D})$ and $(AB)^+ = B^+ A^+$. Furthermore it is trivial that $\lambda A + \mu B \in \mathcal{L}_+(\mathcal{D})$ and that $(\lambda A + \mu B)^+ = \bar{\lambda} A^+ + \bar{\mu} B^+$. Because for every $A \in \mathcal{L}_+(\mathcal{D})$ the adjoint A^* exists, A is closable and consequently, $A \in \mathcal{L}_0(\mathcal{D})$. Further it holds the

<u>Lemma 2.2</u>.

i) For $\mathcal{D} = \mathcal{H}$ it is $\mathcal{L}_+(\mathcal{D}) = \mathcal{L}_0(\mathcal{D}) = \mathcal{B}(\mathcal{H})$, the $*$ -algebra of all bounded operators of $\mathcal{H}$.

ii) If only one operator $A \in \mathcal{L}_+(\mathcal{D})$ is closed, then it is $\mathcal{D} = \mathcal{H}$ and consequently, $\mathcal{L}_+(\mathcal{D}) = \mathcal{B}(\mathcal{H})$, the $*$ -algebra of all bounded operators.

<u>Proof</u>;

i) Follows immediately from the closed graph theorem.

ii): We equipped $\mathcal{D}$ with the scalar product $(\phi, \psi) = \langle \phi, \psi \rangle + \langle A\phi, A\psi \rangle$. Because A is assumed to be closed, we obtain a complete Hilbert space $\mathcal{H}_1 = \mathcal{D}$ (/3/ chap. XII,4). A becomes a continuous mapping from $\mathcal{H}_1$ into $\mathcal{H}$ and consequently $\langle \phi, A\psi \rangle$, $\phi \in \mathcal{H}$ arbitrary, depends continuously on $\psi \in \mathcal{H}_1$

(continuous with respect to the scalar product (Ψ', Ψ)) and therefore by the Riesz theorem exists an element $\chi \in \mathcal{H}_1$ with $\langle \phi, A\Psi \rangle = (\chi, \Psi)$. Then it follows $\langle \phi, A\Psi \rangle =$ $= \langle \chi, \Psi \rangle + \langle A\chi, A\Psi \rangle = \langle \chi + A^+ A\chi, \Psi \rangle$. Consequently, it holds $\phi \in \mathfrak{D}(A^*)$ and because ϕ was arbitrary of $\mathcal{H}$ we have $\mathfrak{D}(A^*) = \mathcal{H} = \mathcal{H}_1 = \mathfrak{D}$. Important for the proof of Theorem 4.3 is the following theorem:

<u>Theorem 2.1</u>. Let $A = A^+$ be a symmetric operator of $\mathcal{L}_+(\mathfrak{D})$. If in $\mathfrak{D}$ exists a norm $\| \ \|_1$ stronger then the Hilbert norm $\| \ \|$, i.e. $\| \phi \| \le c_1 \| \phi \|_1$, $c_1 > 0$, such that A is continuous with respect to $\| \ \|_1$, $\| A\phi \|_1 \le K \| \phi \|_1$, then A is also a bounded operator with respect to the norm $\| \ \|$.

<u>Proof</u>: For any $\phi \in \mathfrak{D}$ it holds

$$\| A^\nu \phi \| \le c_1 \| A^\nu \phi \|_1 \le c_1 K^\nu \| \phi \|_1 , \quad \nu = 0, 1, \dots , \tag{2.1}$$

i.e. every $\phi \in \mathfrak{D}$ is an analytic vector for A and consequently A, with the domain $\mathfrak{D}$, is an essentially self-adjoint (cf. /4,5/) and thus it exists a spectral decomposition

$$A\phi = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda \phi , \quad \phi \in \mathfrak{D} . \tag{2.2}$$

We suppose now that A is an unbounded operator, say $F_\lambda = I - E_\lambda \neq 0$ for all $\lambda < +\infty$, then there is a vector $\phi_\lambda \in \mathfrak{D}$ with $F_\lambda \phi_\lambda \neq 0$ ($F_\lambda \phi_\lambda$ is in general not an element of $\mathfrak{D}$) and it holds for $\nu = 0, 1, 2, \dots$, $\lambda > 0$,

$$\lambda^{\nu}\,\|F_\lambda \phi_\lambda\| \leq \|A^{\nu} F_\lambda \phi_\lambda\| \leq \|A^{\nu}\phi_\lambda\| \leq c_1 K^{\nu}\,\|\phi_\lambda\|_1 .$$

For $\lambda > K$ that is a contradiction.

Hence A cannot be unbounded.

Remark: Without the assumption $A = A^+$ the last theorem does not hold (see also Example 5.2, Statement 2).

In this paper we use the following notations:

Definition 2.1.

A subalgebra $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ of $\mathcal{L}_a(\mathfrak{D})$ is called an Op-algebra if $\mathcal{A} \subset \mathcal{L}_o(\mathfrak{D})$ and an Op*-algebra if $\mathcal{A}$ is a $*$ - subalgebra of $\mathcal{L}_+(\mathfrak{D})$.

From Lemma 2.1 it can be seen that for $\mathfrak{D}$ exists a maximal Op*-algebra. We remark yet that any Op*-algebra is an Op - algebra.

3. Topologization of the domain

Let $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ be an Op -algebra then by $t_{\mathcal{A}}$ we denote the weakest locally convex topology in $\mathfrak{D}$ with respect to which every operator $A \in \mathcal{A}$ is a continuous linear mapping of the locally convex space $\mathfrak{D}[t_{\mathcal{A}}]$ into $\mathfrak{D} \subset \mathcal{H}$ (equipped with the Hilbert space topology).

Lemma 3.1.

1) The topology $t_{\mathcal{A}}$ is defined by the seminorms $\|\phi\|_A = \|A\phi\|$, $A \in \mathcal{A}$, $\phi \in \mathfrak{D}$, and is stronger than the Hilbert space topology defined by the norm $\|\ \|$.

ii) Every operator $A \in \mathcal{A}$ is not only a continuous operator of $\mathcal{D}[t_{\mathcal{A}}]$ into $\mathcal{H}$, but also a continuous linear transformation of the LK-space $\mathcal{D}[t_{\mathcal{A}}]$ into itself.
iii) If every operator $A \in \mathcal{A}$ is bounded (with respect to $\| \ \|$) then $t_{\mathcal{A}}$ coincides with the Hilbert space topology defined by $\| \ \|$.

Proof:

i) Follows immediately from the definition of $t_{\mathcal{A}}$. $t_{\mathcal{A}}$ is stronger than the Hilbert space topology, because $\mathcal{A}$ contains the identity I.

ii): For an arbitrary $B \in \mathcal{A}$ and $\| \ \|_A$ it holds $\|B\phi\|_A = \|AB\phi\| \leq \|\phi\|_C$, because $C = AB \in \mathcal{A}$. Therefore B is a continuous operator in $\mathcal{D}[t_{\mathcal{A}}]$.

iii) Follows from $\|\phi\|_A \leq \|A\| \, \|\phi\|$ for bounded A and the fact that $\| \ \|$ is also a seminorm of the system $\{\| \ \|_A : A \in \mathcal{A}\}$.

Lemma 3.2.
i) Let Σ be an (algebraic) linear basis in the Op-algebra $\mathcal{A}(\mathcal{D})$, then the topology $t_{\mathcal{A}}$ is already defined by the seminorms $\| \ \|_A$ with $A \in \Sigma$.
ii) If $\mathcal{A}(\mathcal{D})$ is an Op^*-algebra and the algebraic basis Σ contains with A,B also the operator $I + A^+A + B^+B$, then the system $\|\phi\|_A$, $A \in \Sigma$, of seminorms is saturated (see Sec. 1). Especially the system of all seminorms $\| \ \|_A$, $A \in \mathcal{A}$, is saturated.

Proof:

i) Each $A \in \mathcal{A}$ can be written as $A = \alpha_1 A_1 + \dots + \alpha_n A_n$ with $A_i \in \mathcal{A}$, α_i complex number, and it follows $\|\phi\|_A \leq |\alpha_1| \|\phi\|_{A_1} + \dots + \alpha_n \|\phi\|_{A_n}$. This proves i).

ii) For $A, B \in \Sigma$ it holds

$$(\|\phi\|_A + \|\phi\|_B)^2 \leq 2(\langle A^+A\phi,\phi\rangle + \langle B^+B\phi,\phi\rangle)$$

$$\leq 2 \langle C\phi,\phi\rangle \leq 2\|C\phi\|\|\phi\|$$

with $C = I + A^+A + B^+B$. Because $\|\phi\| \leq \|C\phi\|$ it follows $\|\phi\|_A + \|\phi\|_B \leq \sqrt{2}\|\phi\|_C$, q.e.d.

The locally convex space $\mathfrak{D}[t_{\mathfrak{A}}]$ is in general not complete. Its completion (/7/, 1.5) we denote by $\tilde{\mathfrak{D}}[\tilde{t}_{\mathfrak{A}}]$

<u>Lemma 3.3</u>.
Let $\mathfrak{A}(\mathfrak{D})$ be an Op-algebra, $\mathfrak{D}$ dense in the Hilbert space $\mathcal{H}$. The naturally given continuous imbedding of $\mathfrak{D}[t_{\mathfrak{A}}]$ into $\mathcal{H}$ can be uniquely extended to a continuous imbedding of $\tilde{\mathfrak{D}}[\tilde{t}_{\mathfrak{A}}]$ into $\mathcal{H}$ and it holds then

$$\tilde{\mathfrak{D}}[\tilde{t}_{\mathfrak{A}}] = \bigcap_{A\in\mathfrak{A}} \mathfrak{D}(\bar{A}),$$

where $\mathfrak{D}(\bar{A})$ is the domain of the closure $\bar{A}$ of the operator $A \in \mathfrak{A}$.

<u>Proof</u>.

We equipped $\mathfrak{D}(\bar{A})$ with the norm $\|\phi\|'_A = [\langle\phi,\phi\rangle + \langle\bar{A}\phi,\bar{A}\phi\rangle]^{\frac{1}{2}}$. $\mathfrak{D}(\bar{A})$ is complete with respect to the norm $\|\ \|'_A$ (/3/ chap. XII,4) and $\mathfrak{D}$ is dense in $\mathfrak{D}(\bar{A})$. It is $\mathfrak{D} \subset \bigcap_{A\in\mathfrak{A}} \mathfrak{D}(\bar{A})$ and $\bigcap_{A\in\mathfrak{A}} \mathfrak{D}(\bar{A})$ is complete with respect to the topology $\tilde{t}_{\mathfrak{A}}$ defined by all norms $\|\ \|'_A$, $A \in \mathfrak{A}$. Further, $\mathfrak{D}$ is dense in $\bigcap_{A\in\mathfrak{A}} \mathfrak{D}(\bar{A})$ with respect to the topology $\tilde{t}_{\mathfrak{A}}$ and it is easy to see that the topology $\tilde{t}_{\mathfrak{A}}$ induces the topology $t_{\mathfrak{A}}$ on $\mathfrak{D}$. Consequently, $\bigcap_{A\in\mathfrak{A}} \mathfrak{D}(\bar{A})$ equipped with the topology $\tilde{t}_{\mathfrak{A}}$ is isomorphic to the completion of $\mathfrak{D}[t_{\mathfrak{A}}]$.

Examples:

Let $\mathcal{H} = L_2(R^1)$ be the Hilbert space of all quadratic summable functions $\mathfrak{D} = C_0^\infty(R^1)$ the space of infinitely differentiable functions with compact support and $\mathcal{A}_1(\mathfrak{D})$ the $*$-algebra of all differential operators $A = \sum_{n,m\geq 0} a_{nm} x^n \left(\frac{d}{dx}\right)^m$, only finite many $a_{n,m}$ are different from zero, then the completion of $\mathfrak{D}[t_{\mathcal{A}_1}]$ is the Schwartz space $\mathcal{S} = \tilde{\mathfrak{D}}[\tilde{t}_{\mathcal{A}_1}]$. If $\mathcal{A}_2(\mathfrak{D})$ is the $*$-algebra of all differential operators $A = \sum_{n\geq 0} f_n(x)\left(\frac{d}{dx}\right)^n$, only finite many $f_n(x)$ different from zero and $f_n(x)$ an arbitrary infinitely differentiable function, then $\mathfrak{D}[t_{\mathcal{A}_2}]$ is the Schwartz space $\mathfrak{D}$.

4. Topologization of Operator Algebras

In this section we investigate the properties of different topologies in an Op-algebra or an Op^*-algebra. Let $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ be an Op-algebra. A system $\mathfrak{S}$ of bounded sets $\mathcal{M}$ of the locally convex space $\mathfrak{D}[t_{\mathcal{A}}]$ is said to be admissible if 1. for $\mathcal{M}\in\mathfrak{S}$ and $A\in\mathcal{A}$ it holds $A\mathcal{M}\in\mathfrak{S}$ 2. $\bigcup_{\mathcal{M}\in\mathfrak{S}}\mathcal{M}$ is dense in $\mathcal{H}$ and 3. for $\mathcal{M},\mathcal{M}'\in\mathfrak{S}$ there is a $\mathcal{N}\in\mathfrak{S}$ with $\mathcal{M}\cup\mathcal{M}'\subset\mathcal{N}$.

Definition 4.1.

Let $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ be an Op-algebra and $\mathfrak{S}$ an admissible system of bounded sets of $\mathfrak{D}[t_{\mathcal{A}}]$, then we define in a topology $\tau^{\mathfrak{S}}$ by the seminorms

$$\tau^{\mathfrak{S}}:\ p_{B,\mathcal{M}}(A) = \sup_{\phi\in\mathcal{M}} \|A\phi\|_B \ ,\quad \mathcal{M}\in\mathfrak{S},\ B\in\mathcal{A},$$

and the topology $\mathfrak{T}_{\mathfrak{S}}$ by the seminorms

$$\mathfrak{T}_{\mathfrak{S}} : \quad \|A\|_{\mathcal{M}} = \sup_{\phi,\psi\in\mathcal{M}} |\langle\phi, A\psi\rangle| \, , \quad \mathcal{M}\in\mathfrak{S} \, .$$

If $\mathfrak{S} = \mathfrak{S}_{max}$ is the system of all bounded sets of $\mathcal{D}[t_{\mathcal{A}}]$ we write $\mathfrak{T}^{\mathcal{D}} \equiv \mathfrak{T}^{\mathfrak{S}_{max}}$ and $\mathfrak{T}_{\mathcal{D}} = \mathfrak{T}_{\mathfrak{S}_{max}}$.

Because of the properties 1-3. of $\mathfrak{S}$, it easy to see that $\mathcal{A}[\mathfrak{T}^{\mathfrak{S}}]$ and $\mathcal{A}[\mathfrak{T}_{\mathfrak{S}}]$ are locally convex spaces. From 1. it follows $p_{B,\mathcal{M}}(A) < \infty$ and $\|A\|_{\mathcal{M}} < \infty$ for $A, B \in \mathcal{A}$, $\mathcal{M} \in \mathfrak{S}$, from 2. it follows that $A = 0$ if $p_{B,\mathcal{M}}(A) = 0$ for all $B \in \mathcal{A}$, $\mathcal{M}\in\mathfrak{S}$ or $\|A\|_{\mathcal{M}} = 0$ for all $\mathcal{M}\in\mathfrak{S}$.

<u>Theorem 4.1.</u>

i) For an Op-algebra $\mathcal{A}(\mathcal{D})$ the multiplication is separately continuous with respect to $\mathfrak{T}^{\mathfrak{S}}$, i.e. $\mathcal{A}[\mathfrak{T}^{\mathfrak{S}}]$ is a LK-algebra.

ii) If $\mathcal{A}(\mathcal{D})$ is an Op*-algebra, then $\mathcal{A}[\mathfrak{T}_{\mathfrak{S}}]$ is a LK^*-algebra, i.e. the multiplication is separately continuous and the involution $A \to A^+$ is continuous with respect to $\mathfrak{T}_{\mathfrak{S}}$.

<u>Proof:</u>

i) For an arbitrary $B \in \mathcal{A}$ we have to show that $A \to BA$ and $A \to AB$ are continuous transformations in $\mathcal{A}[\mathfrak{T}^{\mathfrak{S}}]$. Let $p_{C,\mathcal{M}}(\cdot)$, $C \in \mathcal{A}$, $\mathcal{M}\in\mathfrak{S}$, be one of the seminorms for $\mathfrak{T}^{\mathfrak{S}}$, then it holds

$$p_{C,\mathcal{M}}(BA) = \sup_{\phi\in\mathcal{M}} \|CBA\phi\| = p_{CB,\mathcal{M}}(A)$$

and consequently $A \to BA$ is continuous. Further,

$$p_{C,\mathcal{M}}(AB) = \sup_{\phi\in\mathcal{M}} \|CAB\phi\| = p_{C,B\mathcal{M}}(A)$$

and this means $A \to AB$ is continuous, since $B\mathcal{M} \in \mathfrak{S}$.

ii) Let now $\mathcal{A}(\mathcal{D})$ be an Op^*-algebra, $B \in \mathcal{A}$ and $\mathcal{M} \in \mathfrak{S}$. Let $\mathcal{N}$ be a set of $\mathfrak{S}$ with $\mathcal{N} \supset \mathcal{M} \cup B\mathcal{M}$ and let $\mathcal{N}'$ be a set of $\mathfrak{S}$ with $\mathcal{N}' \supset \mathcal{M} \cup B^+\mathcal{M}$, then it holds

$$\|BA\|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} |\langle B^+\phi, A\psi\rangle| \leq \|A\|_{\mathcal{N}'}$$

$$\|AB\|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} |\langle \phi, AB\psi\rangle| \leq \|A\|_{\mathcal{N}}$$

$$\|A\|_{\mathcal{M}} = \|A^+\|_{\mathcal{M}}.$$

From that immediately follows the continuity of $A \to BA$, $A \to AB$ and $A \to A^+$ with respect to the topology $\mathfrak{T}_{\mathfrak{S}}$.

<u>Remark</u>. In general for an Op-algebra $\mathcal{A}(\mathcal{D})$ $\mathcal{A}[\mathfrak{T}_{\mathfrak{S}}]$ is not a LK-algebra, even not for the strongest $\mathfrak{T}_{\mathfrak{S}}$-topology $\mathfrak{T}_{\mathcal{D}}$ and for an Op^*-algebra the involution $A \to A^+$ is in general not continuous with respect to $\mathfrak{T}^{\mathcal{D}}$, i.e. $\mathcal{A}[\mathfrak{T}^{\mathcal{D}}]$ is in general not a LK^*-algebra. (See Examples 5.1 and 5.2).

But it holds the following

<u>Theorem 4.2.</u>

i) If every operator A of an Op-algebra $\mathcal{A}(\mathcal{D})$ is bounded, so it holds $\mathfrak{T}^{\mathcal{D}} = \mathfrak{T}_{\mathcal{D}} = \mathfrak{T}_{\|\,\|}$,

where $\mathfrak{T}_{\|\,\|}$ is the usual operator-norm topology.

ii) $\mathfrak{T}^{\mathcal{D}}$ is stronger than $\mathfrak{T}_{\mathcal{D}}$. For an Op^*-algebra it holds $\mathfrak{T}^{\mathcal{D}} = \mathfrak{T}_{\mathcal{D}}$ if and only if the multiplication $A, B \to A \cdot B$ is jointly continuous with respect to the topology $\mathfrak{T}_{\mathcal{D}}$.

Proof:

i) By Lemma 3.1 iii) $t_{\mathcal{A}}$ coincides with the Hilbert-Space topology defined by $\| \ \|$. Let $\mathcal{K} = \{\phi \in \mathcal{D} : \|\phi\| \leq 1\}$ be the unit ball of $\mathcal{D}$, then $\mathcal{K}$ is bounded with respect to $t_{\mathcal{A}}$ and for every bounded $\mathcal{M} \subset \mathcal{D}[t_{\mathcal{A}}]$ there is a positive α with $\frac{1}{\alpha}\mathcal{M} \subset \mathcal{K}$. Then follows for $B \in \mathcal{A}$

$$p_{B,\mathcal{M}}(A) = \sup_{\phi \in \mathcal{M}} \|BA\phi\| \leq \alpha \|B\| \|A\| .$$

Furthermore one has $\|A\| = p_{I,\mathcal{K}}(A)$, I the unity, and consequently, the system of seminorms $p_{B,\mathcal{M}}(A)$ is equivalent to the one norm $\|A\|$.

In the same way one proves the equivalence of $\mathcal{T}_{\mathcal{D}}$ and $\mathcal{T}_{\| \ \|}$.

ii) Let $\mathcal{M}$ be an arbitrary bounded set of $\mathcal{D}[t_{\mathcal{A}}]$, then it holds

$$\|A\|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} |\langle \phi, A\psi \rangle| \leq M \cdot p_{I,\mathcal{M}}(A),$$

where $M = \sup_{\phi \in \mathcal{M}} \|\phi\|$. Consequently, $\mathcal{T}^{\mathcal{D}}$ is stronger then $\mathcal{T}_{\mathcal{D}}$.

Let $\mathcal{A}(\mathcal{D})$ now be an Op^*-algebra. If $\mathcal{T}^{\mathcal{D}} = \mathcal{T}_{\mathcal{D}}$, then for every bounded $\mathcal{M} \subset \mathcal{D}[t_{\mathcal{A}}]$ there is a bounded $\mathcal{N} \subset \mathcal{D}[t_{\mathcal{A}}]$ with $p_{I,\mathcal{M}}(A) \leq \|A\|_{\mathcal{N}}$.

Therefore we obtain

$$\|AB\|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} |\langle \phi, AB\psi \rangle| = \sup_{\phi,\psi \in \mathcal{M}} |\langle A^+\phi, B\psi \rangle|$$

$$\leq p_{I,\mathcal{M}}(A^+)\, p_{I,\mathcal{M}}(B) \leq \|A^+\|_{\mathcal{N}} \|B\|_{\mathcal{N}}$$

$$\leq \|A\|_{\mathcal{N}} \|B\|_{\mathcal{N}} .$$

and the multiplication $A,B \to AB$ is jointly continuous. Vice versa, if the multiplication $A,B \to AB$ is jointly continuous with respect to $\mathfrak{T}_{\mathfrak{D}}$, then for every bounded $\mathcal{M} \subset \mathfrak{D}[t_{\mathcal{A}}]$ there is a bounded $\mathcal{N} \subset \mathfrak{D}[t_{\mathcal{A}}]$ so that

$$\|A^+B^+BA\|_{\mathcal{M}} \leq \|A^+\|_{\mathcal{N}} \|B^+\|_{\mathcal{N}} \|B\|_{\mathcal{N}} \|A\|_{\mathcal{N}} = \|A\|_{\mathcal{N}}^2 \|B\|_{\mathcal{N}}^2$$

for $A,B \in \mathcal{A}$ and consequently,

$$p_{B,\mathcal{M}}(A)^2 = \sup_{\phi \in \mathcal{M}} |\langle BA\phi, BA\phi\rangle| \leq$$

$$\leq \sup_{\psi,\phi \in \mathcal{M}} |\langle \psi, A^+B^+BA\phi\rangle| = \|A^+B^+BA\|_{\mathcal{M}} \leq$$

$$\leq \|A\|_{\mathcal{N}}^2 \|B\|_{\mathcal{N}}^2,$$

and thus $p_{B;\mathcal{M}}(A) \leq \|B\|_{\mathcal{N}} \|A\|_{\mathcal{N}}$.
Therefore $\mathfrak{T}_{\mathfrak{D}}$ is stronger than $\mathfrak{T}^{\mathfrak{D}}$ and since $\mathfrak{T}^{\mathfrak{D}}$ is stronger than $\mathfrak{T}_{\mathfrak{D}}$, it follows $\mathfrak{T}^{\mathfrak{D}} = \mathfrak{T}_{\mathfrak{D}}$.

It is remarkable that for an Op^*-algebra and the topology $\mathfrak{T}_{\mathfrak{D}}$ a certain converse of Theorem 4.2 can be proved, namely the

<u>Theorem 4.3</u>.
If in an Op^*-algebra $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ exists a norm $\|\ \|_0$ defining a stronger topology than $\mathfrak{T}_{\mathfrak{D}}$, then every operator $A \in \mathcal{A}$ is bounded.

We remark that $\|\ \|_0$ is not assumed to be a norm, which makes $\mathcal{A}$ to a normed $*$-algebra, i.e. we have not assumed the multiplication $A,B \to A\cdot B$ or the involution $A \to A^+$ to be continuous with respect to $\|\ \|_0$.

Proof:

Let $\mathfrak{M}$ be the system of all bounded sets of $\mathcal{D}[t_{\mathcal{A}}]$ with the property

$$\|A\|_{\mathcal{M}} \leq \|A\|_0 \qquad \text{for all} \quad A \in \mathcal{A} .$$

We prove: 1. In $\mathfrak{M}$ exists a maximal element $\mathcal{M}_0$, i.e. from $\mathcal{M}_0 \subset \mathcal{N} \in \mathfrak{M}$ follows $\mathcal{M}_0 = \mathcal{N}$.

2. $\mathcal{M}_0$ is a closed absolutely convex set, i.e. from $\phi, \psi \in \mathcal{M}$ it follows $\alpha\phi + \beta\psi \in \mathcal{M}$ for $|\alpha| + |\beta| = 1$, and $\mathcal{M}_0$ absorbs every bounded set $\mathcal{K}$ of $\mathcal{D}[t_{\mathcal{A}}]$, i.e. there is a $\lambda > 0$ with $\lambda\mathcal{K} \subset \mathcal{M}_0$.

ad 1): Let $\{\mathcal{N}_\gamma\}$ be a subset of $\mathfrak{M}$, ordered by $\subset$ and $\mathcal{N} = \bigcup_\gamma \mathcal{N}_\gamma$. $\mathcal{N}$ is an upper bound for $\{\mathcal{N}_\gamma\}$, i.e. $\mathcal{N}_\gamma \subset \mathcal{N}$. If we yet show that $\mathcal{N}$ is an element of $\mathfrak{M}$, then the existence of $\mathcal{M}_0$ follows by the Zorn's Lemma.

Let ϕ be a vector of $\mathcal{N}$, then $\phi \in \mathcal{N}_{\gamma_0}$ for a certain $\mathcal{N}_{\gamma_0}$ and for $A \in \mathcal{A}$ it holds

$$\|\phi\|_A^2 = \langle \phi, A^+A\phi \rangle \leq \|A^+A\|_{\mathcal{N}_{\gamma_0}} \leq \|A^+A\|_0$$

The right-hand side does not depend on $\mathcal{N}_{\gamma_0}$ and consequently, $\sup_{\phi \in \mathcal{N}} \|\phi\|_A^2 \leq \|A^+A\|_0$ and thus $\mathcal{N}$ is bounded in $\mathcal{D}[t_{\mathcal{A}}]$. Furthermore, since $\{\mathcal{N}_\gamma\}$ is ordered, for $\phi, \psi \in \mathcal{N}$ there is a $\mathcal{N}_{\gamma_1}$ with $\phi, \psi \in \mathcal{N}_{\gamma_1}$ and thus

$$|\langle \phi, A\psi \rangle| \leq \|A\|_{\mathcal{N}_{\gamma_1}} \leq \|A\|_0 .$$

From that it follows $\|A\|_{\mathcal{N}} \leq \|A\|_0$. Consequently $\mathcal{N}$ is an element of $\mathfrak{M}$.

ad 2): Since $\mathcal{M}_o$ is maximal in $\mathfrak{M}$, it is easy to see that $\mathcal{M}$ is closed and absolutely convex.

Let $\mathcal{K}$ be an arbitrary bounded set of $\mathcal{D}[t_{\mathcal{A}}]$, then also $\mathcal{M}_o \cup \mathcal{K}$ is bounded and because the norm $\| \ \|_o$ is stronger than the topology $\mathcal{T}_{\mathcal{D}}$, there is a positive $\mu \leq 1$ with

$$\|A\|_{\mathcal{M}_o \cup \mathcal{K}} \leq \frac{1}{\mu} \|A\|_o \qquad \text{for all} \qquad A \in \mathcal{A} .$$

We put $\mathcal{M}' = \mathcal{M}_o \cup (\mu \mathcal{K})$ and take $\phi, \psi \in \mathcal{M}'$. We have to distinguish three cases a) $\phi, \psi \in \mathcal{M}_o$,
b) $\phi \in \mathcal{M}_o$, $\psi \in \mu\mathcal{K}$ or $\psi \in \mathcal{M}_o$, $\phi \in \mu\mathcal{K}$ and
c) $\phi, \psi \in \mu \mathcal{K}$.

For a) it holds $|\langle \psi, A\phi \rangle| \leq \|A\|_{\mathcal{M}_o} \leq \|A\|_o$
for b) (or c)) it follows $|\langle \psi, A\phi \rangle| = \mu \, |\langle \frac{1}{\mu}\psi, A\phi \rangle| \leq \mu \|A\|_{\mathcal{M}_o \cup \mathcal{K}} \leq \|A\|_o$, i.e. $\mathcal{M}' = \mathcal{M}_o \cup (\mu\mathcal{K}) \in \mathfrak{M}$
Because $\mathcal{M}_o$ is maximal, it follows $\mu\mathcal{K} \subset \mathcal{M}_o$, i.e. $\mathcal{M}_o$ absorbs the set $\mathcal{K}$, and thus the property 2. is completely proved. Now we define in $\mathcal{D}$ the Minkowski functional for $\mathcal{M}_o$,

$$\|\phi\|_1 = \inf \{ t > 0 : \tfrac{1}{t}\phi \in \mathcal{M}_o \} .$$

Because $\mathcal{M}_o$ is absorbing and bounded, $\| \ \|_1$ is a norm in $\mathcal{D}$, which is stronger than $\| \ \|$, since $\mathcal{M}_o$ is also bounded with respect to the norm $\| \ \|$.

Furthermore, for $A^+ = A \in \mathcal{A}$ $A\mathcal{M}_o$ is also a bounded set in $\mathcal{D}[t_{\mathcal{A}}]$ and in consequence of 2. there is a $K>0$ with $\frac{1}{K} A \mathcal{M}_0 \subset \mathcal{M}_0$. From this follows

$$\| A\phi \|_1 \leqslant K \|\phi\|_1 \quad , \quad \text{i.e. } A$$

is continuous with respect to the norm $\| \ \|_1$ and in consequence of Theorem 2.1 A is bounded. Since every symmetric operator of $\mathcal{A}$ is bounded, so it is also every $B \in \mathcal{A}$, since $B \in \mathcal{A}$ can be written in the form $B = \frac{1}{2}(B^+ + B) + \frac{i}{2}(iB^+ - iB)$ where $B^+ + B$ and $iB^+ - iB$ are bounded operators.

An analogous theorem for an Op -algebra and the topology $\mathfrak{T}^{\mathfrak{D}}$ does not hold (see Example 5.2).

5. $\hat{O}^*$ -Algebras

The investigation in the foregoing section, especially Theorem 4,2 and 4,3, shows that for an Op^* -algebra the topology $\mathfrak{T}_{\mathfrak{D}}$ is a good candidate for a topology which is expected to play the same role as the operator-norm topology of an algebra of bounded operators.

Definition 5.1.
An Op^* -algebra $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ equipped with the topology $\mathfrak{T}_{\mathfrak{D}}$ is called $\hat{O}^*$-algebra and denoted by $\mathcal{A}_{\mathfrak{D}} = \mathcal{A}[\mathfrak{T}_{\mathfrak{D}}]$ The topology $\mathfrak{T}_{\mathfrak{D}}$ is called the uniform topology of $\mathcal{A}$. If an $\hat{O}^*$ -algebra $\mathcal{A}_{\mathfrak{D}}$ is a complete locally convex space it is called O^* -algebra.

The concept of an O^* -algebra is a generalization of the concept of an C^* -algebra. It is remarkable that in the case of unbounded operators the difference between

$\hat{O}^*$-algebra and O^*-algebra is much more essential than the difference between an uncomplete normed algebra of bounded operators and a complete normed algebra of bounded operators, a C^*-algebra. Whereas every normed algebra of bounded operators is a dense subalgebra of a C^*-algebra that does not hold in general for an $\hat{O}^*$-algebra, as it will be shown in Example 5.1.

The concept of an " $\hat{O}$ -algebra" we shall not define, because for that one would have to take the topology $\mathfrak{T}^{\mathfrak{D}}$, then an O_p-algebra is in general not an LK-algebra with respect to the topology $\mathfrak{T}_{\mathfrak{D}}$, as one can see from Example 5.2. Consequently, an $\hat{O}^*$-algebra regarded as " $\hat{O}$ -algebra" would carry another topology, since $\mathfrak{T}^{\mathfrak{D}}$ and $\mathfrak{T}_{\mathfrak{D}}$ do not coincide in general.

Our main objects are O_p^*-algebras ($\hat{O}^*$-algebras), but the O_p-algebras play a role by the investigation of extensions of an O_p^*-algebra $\mathcal{A}(\mathfrak{D})$ to a wider domain $\mathfrak{D}_1 \supset \mathfrak{D}$. As it is done in /6/, one can for any O_p^*-algebra $\mathcal{A}(\mathfrak{D})$ construct a maximal O_p-algebra $\mathcal{A}_*(\mathfrak{D}_*)$ so that every extension of $\mathcal{A}(\mathfrak{D})$ to a wider domain $\mathfrak{D}_1 \supset \mathfrak{D}$ is an restriction of $\mathcal{A}_*(\mathfrak{D}_*)$ to $\mathfrak{D}_1 \subset \mathfrak{D}_*$, analogously to the case of one symmetric operator (/3/, chap. XII,

The O_p-algebra $\mathcal{A}_*(\mathfrak{D}_*)$ is the following one: We define $\mathfrak{D}_* = \bigcap_{A \in \mathcal{A}} \mathfrak{D}(A^*)$, where $\mathfrak{D}(A^*)$ is the domain of the usual adjoint operator A^* of $A \in \mathcal{A}$, and the operator A_*, which is the restriction of $(A^+)^*$ to

$\mathfrak{D}_*$. It can be shown that the set $\mathcal{A}_*(\mathfrak{D}_*)$ of all operators A_* is an $O\rho$-algebra and that the mapping $A \to A_*$ is an algebraic isomorphism between $\mathcal{A}(\mathfrak{D})$ and $\mathcal{A}_*(\mathfrak{D}_*)$.

We shall investigate the problem of extension of $O\rho^*$-algebras in a forthcoming paper.

Definition 5.2.

A $*$- representation $a \to A(a)$ of a $*$-algebra $\mathfrak{R}$ is a $*$-homomorphism (/2/,IV) of $\mathfrak{R}$ onto an $O\rho^*$-algebra $\mathcal{A}(\mathfrak{D})$.

A $*$-representation of a LK^*-algebra (locally convex $*$-algebra) is said to the weakly continuous if $\langle \phi, A(a) \psi \rangle$ depends continuously on a for all $\phi, \psi \in \mathfrak{D}$. The $*$-representation is said to be uniformly continuous if $a \to A(a)$ is a continuous mapping of $\mathfrak{R}$ onto the $\hat{O}^*$-algebra $\mathcal{A}[\mathfrak{T}_{\mathfrak{D}}]$.

It is well-known that any $*$-representation of a Banach $*$-algebra $\mathfrak{R}$ with an identity element is uniformly continuous. One can now prove certain generalization of this fact

Theorem 5.1.

Let $a \to A(a)$ be a $*$-representation of a LK^*-algebra $\mathfrak{R}$. If for $\phi \in \mathfrak{D}$ $F_\phi(a) = \langle \phi, A(a) \phi \rangle$ is continuous in a, then $a \to A(a)$ is weakly continuous and if furthermore $\mathfrak{R}$ is a barraled space, (tonnelé/7/) so $a \to A(a)$ is also uniformly continuous.

Proof:

Let $\mathfrak{R}_h$ be the real subspace of the symmetric elements $a = a^*$ of $\mathfrak{R}$. For $a \in \mathfrak{R}_h$ it holds

$$4\langle\phi, A(a)\psi\rangle = F_{\phi+\psi}(a) - F_{\phi-\psi}(a) - iF_{\phi+i\psi}(a) + iF_{\phi-i\psi}(a)$$

and consequently $\langle\phi, A(a)\psi\rangle$ depends oontinuously on $a \in \mathfrak{R}_h$ for any $\phi, \psi \in \mathfrak{D}$. Since $a = \frac{1}{2}(a^* + a) + \frac{i}{2}(ia^* - ia)$ and the involution $a \to a^*$ is continuous, so also $\langle\phi, A(a)\psi\rangle$ depends continuously on all $a \in \mathfrak{R}$.

Let $\mathcal{M}$ be a bounded set of $\mathfrak{D}[t_{\mathfrak{A}}]$ and $\varepsilon > 0$. We define

$$\mathcal{U}_{\mathcal{M},\varepsilon} = \{a \in \mathfrak{R} : \|A(a)\|_{\mathcal{M}} \leq \varepsilon\} = \bigcap_{\phi,\psi \in \mathcal{M}} \{a \in \mathfrak{R} : |\langle\phi, A(a)\psi\rangle| \leq \varepsilon\}$$

Every set $\{a \in \mathfrak{R} : |\langle\phi, A(a)\psi\rangle| \leq \varepsilon\}$ is absolutely convex and closed, since $|\langle\psi, A(a)\phi\rangle|$ is continuous in a and thus $\mathcal{U}_{\mathcal{M},\varepsilon}$ is closed and absolutoly convex. Furthermore, $\mathcal{U}_{\mathcal{M},\varepsilon}$ is also an absorbing set, then for any $a \in \mathfrak{R}$ it is $a\|A(a)\|_{\mathcal{M}}^{-1}\varepsilon \in \mathcal{U}_{\mathcal{M},\varepsilon}$ if not already $\|A(a)\|_{\mathcal{M}} = 0$. If $\mathfrak{R}$ is barreled $\mathcal{U}_{\mathcal{M},\varepsilon}$ is a neighbourhood and oonsequently, $a \to A(a)$ is uniformly continuous.

Theorem 5.2.

Every $*$-representation of a F^*-algebra $\mathfrak{R}$ is uniformly continuous.

A F^*-algebra is a complete LK^*-algebra which topology is defined by an enumerable system of seminorms $p_n(a)$ for which $p_n(a \cdot b) \leq p_n(a)\, p_n(b)$ holds, $n = 1, 2, \ldots$

Proof:

A F^*-algebra is a barreled space. For $\phi \in \mathfrak{D}$ $\langle\phi, A(a)\phi\rangle$ is a positive functional on $\mathfrak{R}$ and in consequence of [8], Theorem 1, it is continuous. Thus we can apply the foregoing theorem.

With the following definition we give a generalization of the notion of a B^*-algebra.

Definition 5.2.
A $A\hat{O}^*$-algebra resp. AO^*-algebra (abstract O^*-algebra) we call a LK^*-algebra which is algebraically and topologically $*$-isomorphic to an $\hat{O}^*$-algebra resp. to an O^*-algebra.

Doubtless, it would be an interesting result, if one can find an abstract characterization of an AO^*-algebra, like the property $\|x^*x\| = \|x\|^2$ of a B^*-algebra.

In this paper we must restrict ourselves to give some nontrivials examples of AO^*-algebras in section 6 and 7.

As a corollary of Theorem 4.3 we note here the following property of an $A\hat{O}^*$-algebra:

Theorem 5.3.
If the topology $\mathcal{T}$ of an $A\hat{O}^*$-algebra $\mathfrak{R}$ is not a norm-topology, then there does not exists a norm in $\mathfrak{R}$ defining a stronger topology than $\mathcal{T}$.

This is a strange result, because in a normed $A\hat{O}^*$-algebra it can exist a stronger norm. For example the algebra $C^1[0,1]$ equipped with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$ is an $A\hat{O}^*$-algebra (namely a dense subalgebra of the B^*-algebra $C[0,1]$) and in $C^1[0,1]$ exists a stronger norm, namely $\|f\|_1 = \|f\| + \|\frac{d}{dx}f\|$. As conclusion of this section we give two examples.

E x a m p l e 5.1.

Let $\mathcal{H}$ be a separable Hilbert space and $\phi_1, \phi_2, \ldots$ an orthonormal basis in $\mathcal{H}$. By $\mathfrak{D}$ we denote the set of all linear combinations of finite many of the basis vectors $\phi_1, \phi_2, \ldots$. Every $A \in \mathcal{L}_+(\mathfrak{D})$ is uniquely determined by a matrix $A = (a_{\mu\nu})$, defined by

$$A\,\phi_\mu = \sum_\nu a_{\mu\nu}\,\phi_\nu ,$$

and for $A^+ = (a^+_{\mu\nu})$ it holds $a^+_{\mu\nu} = \overline{a}_{\nu\mu}$. Furthermore it is $a_{\mu\nu} = 0$ for $\mu \geqslant \mu_0(\nu)$ or $\nu \geqslant \nu_0(\mu)$. Hence $\mathcal{L}_+(\mathfrak{D})$ is the set of all matrices which have in every row or column only finite many elements different from zero. Let γ_n, $n = 1, 2, \ldots,$ be an arbitrary sequence of positive numbers and $A(\gamma_n) \in \mathcal{L}_+(\mathfrak{D})$ the operator. $a_{\mu\nu} = \delta_{\mu\nu}\gamma_\nu$, then for $\phi = \sum_{n \geqslant 1} x_n \phi_n \in \mathfrak{D}$ it holds (Lemma 3.1)

$$\|\phi\|_{(\gamma_n)} \equiv \|\phi\|_{A(\gamma_n)} = \Big(\sum_{n \geqslant 0} |x_n|^2 \gamma_n^2\Big)^{\frac{1}{2}} .$$

This system of seminorms is equivalent to the system

$$p_{(\gamma_n)}(\phi) = \sum_{n \geqslant 1} |x_n|\,\gamma_n , \quad (\gamma_n) \text{ positive sequence} .$$

The system $p_{(\gamma_n)}(\phi)$ defines in $\mathfrak{D}$ the strongest possible locally convex topology t_{max} and consequently, $t_{\mathcal{L}_+} = t_{max}$.

Statement 1

In the Op^*-algebra $\mathcal{L}_+(\mathfrak{D})$ the uniform topology coincides with the weak topology $\tau_S \equiv \tau_{\mathfrak{S}_{min}}$, $\mathfrak{S}_{min}$ the system of all finite dimensional bounded sets of $\mathfrak{D}$.

$\mathcal{L}_+[\mathfrak{T}_\mathfrak{D}]$ is not complete and since $\mathcal{L}_+(\mathfrak{D})$ is the maximal Op^*-algebra on $\mathfrak{D}$, the $\hat{O}^*$-algebra $\mathcal{L}_+[\mathfrak{T}_\mathfrak{D}]$ cannot enlarged to an O^*-algebra,

It is seen that every bounded set $\mathcal{M} \subset \mathfrak{D}[t_\mathfrak{A}]$ is contained in a certain finite-dimensional space, generated by $\phi_1, \dots, \phi_N$. Consequently, in $\mathcal{L}_+(\mathfrak{D})$ the uniform topology $\mathfrak{T}_\mathfrak{D}$ coincides with the weak topology $\mathfrak{T}_s$. The lineare space of all matrices $(\alpha_{\mu\nu}) \in \mathcal{L}_+(\mathfrak{D})$ regarded as a subspace of the topological product $\prod_{\mu,\nu \geq 1} \mathbb{C}_{\mu\nu}$, $\mathbb{C}_{\mu\nu} = \mathbb{C}$, the complex field, is dense in the direct product and on $\mathcal{L}_+(\mathfrak{D})$ the topology $\mathfrak{T}_\mathfrak{D} = \mathfrak{T}_S$ coincides with the direct product topology. Hence $\mathcal{L}_+(\mathfrak{D})[\mathfrak{T}_\mathfrak{D}]$ is not complete.

Statement 2.

In $\mathcal{L}_+(\mathfrak{D})$ the involution $A \to A^+$ is not continuous with respect to the topology $\mathfrak{T}^\mathfrak{D}$.

We regard the sequence A_n, $n = 1, 2, \dots$, of operators

$$A_n \phi_\nu = \phi_{\nu-n}, \quad \phi_{\nu-n} = 0 \quad \text{for} \quad \nu - n \leq 0 .$$

Then it holds

$$A_n^+ \phi_\nu = \phi_{\nu+n} .$$

Since every bounded $\mathcal{M} \subset \mathfrak{D}[t_{\mathcal{L}_+}]$ is contained in a $L[\phi_1, \dots, \phi_N]$, we have for $B \in \mathcal{L}_+(\mathfrak{D})$

$$p_{B,\mathcal{M}}(A_n) = 0 \quad \text{if} \quad n \geq N .$$

Thus A_n converges to zero with respect to $\mathfrak{T}^\mathfrak{D}$. But

$$p_{I,\mathcal{M}}(A_n^+) = \sup_{\phi \in \mathcal{M}} \| A_n^+ \phi \| = \sup_{\phi \in \mathcal{M}} \| \phi \| \neq 0,$$

and hence A_n^+ does not converge to zero.

Next we regard an Op -algebra which is not an Op^*-algebra.

E x a m p l e 5.2.

Let $\mathcal{H}$ be a Hilbert space with the orthonormal basis $\{\phi_1, \phi_2, \dots, \psi_1, \psi_2, \dots\}$ and $\mathcal{D}$, as in the foregoing example, the algebraic linear hull of the basis vectors. For $\mathcal{A}(\mathcal{D})$ we take the Op-algebra generated by the operators $A_n, n = 1, 2, \dots,$ and B (and the identity operator I), defined by

$$A_n \phi_m = \delta_{nm} \phi_m \quad , \quad n, m = 1, 2, \dots$$

$$A_n \psi_m = 0$$

$$B \phi_m = m \psi_m$$

$$B \psi_m = 0 \quad .$$

Since A_n and B are operators of $\mathcal{L}_+(\mathcal{D})$, the Op-algebra $\mathcal{A}(\mathcal{D})$ is a subalgebra of $\mathcal{L}_+(\mathcal{D})$, but not a sub-$*$--algebra.

Statement 1

The linear mapping $A \to BA$ of $\mathcal{A}(\mathcal{D})$ is not continuous with respect to the topology $\mathcal{T}_{\mathcal{D}}$, i.e. $\mathcal{A}[\mathcal{T}_{\mathcal{D}}]$ is not a LK -algebra.

It is easy to see that the topology $t_{\mathcal{A}}$ is defined by the single norm $\|\phi\|_1 = (\|\phi\|^2 + \|B\phi\|^2)^{\frac{1}{2}}, \ \phi \in \mathcal{D},$

and consequently the topology $\mathfrak{T}_{\mathfrak{D}}$ of $\mathcal{A}(\mathfrak{D})$ can be defined by the single norm

$$\|A\|_0 = \sup_{\phi,\psi \in \mathcal{K}} |\langle \phi, A\psi \rangle| \quad , \quad A \in \mathcal{A}(\mathfrak{D}),$$

where $\mathcal{K} = \{ \phi \in \mathfrak{D} : \|\phi\|_1 \leq 1 \}$ is the unit sphere with respect to the norm $\| \ \|_1$. $\mathcal{K}$ is exactly the set of all $\phi = \sum_{n \geq 1} (x_n \phi_n + y_n \psi_n) \in \mathfrak{D}$ with

$$\sum_{n \geq 1} (|x_n|^2 (n^2+1) + |y_n|^2) \leq 1 .$$

Now let $\psi = \sum_{n \geq 1} (x_n' \phi_n + y_n' \psi_n)$ be another element of $\mathfrak{D}$ then it holds

$$|\langle \phi, A_n \psi \rangle| = |x_n x_n'| \leq (n^2+1)^{-1} ,$$

i.e. $\|A_n\|_0 \leq (n^2+1)^{-1}$. Consequently, A_n converges to zero for $n \to \infty$.

For $\chi_n = (n^2+1)^{-\frac{1}{2}} \phi_n \in \mathcal{K}$ and ψ_n we get

$$\|BA_n\|_0 \geq |\langle \psi_n, BA_n \chi_n \rangle| = \frac{n}{\sqrt{n^2+1}} \geq \frac{1}{2} ,$$

and thus BA_n does not converge to zero for $n \to \infty$.

From the foregoing considerations we get yet the following

Statement 2

The unbounded operator $B \in \mathcal{L}_+(\mathfrak{D})$ is continuous with respect to the norm $\| \ \|_1$ and the topology $\mathfrak{T}_{\mathfrak{D}}$ is given by a norm, although the algebra contains unbounded operators.

This statement shows that the assumption of Theorem 2.1 A to be symmetric and the assumption of Theorem 4.3 $\mathcal{A}(\mathfrak{D})$ to be an Op^*-algebra are essential.

6. Realizations of Algebras of Functions

In this section we prove for a large class of commutative multiplicatively-convex topological $*$-algebras[9] that they are AO^*-algebras.

Let $\mathfrak{M}$ be a topological Hausdorff space and $C(\mathfrak{M})$ the $*$-algebra of all continuous functions on $\mathfrak{M}$ with the usual involution $f \to f^* = \overline{f(x)}$, $x \in \mathfrak{M}$, the complex conjugate function.

We equipped $C(\mathfrak{M})$ with the so-called bicompact-open topology defined by the system of seminorms

$$\|f\|_K = \sup_{x \in K} |f(x)| ,$$

where K is an arbitrary bicompact subset of $\mathfrak{M}$.

In[10] it is proved that a commutative multiplicatively-convex topological $*$-algebra $\mathfrak{R}$ is topologically and algebraically $*$-isomorphic with a $C(\mathfrak{M})$ if $\mathfrak{R}$ is a complete bornological space. We prove the

Theorem 6.1.

If $\mathfrak{M}$ is a locally bicompact Hausdorff space in which any compact subset is bicompact so $C(\mathfrak{M})$ with the bicompact- open topology is an AO^*-algebra.

Before we prove this theorem we show that it does not hold for every locally bicompact space $\mathfrak{M}$. Let W denote the space of ordinals less than the first uncountable ordinal. It is well-known that W is locally bicompact and that every function $f \in C(W)$ is bounded. Hence, the norm $\|f\|_0 = \sup_{x \in W} |f(x)|$ defines a stronger topology than the bicompact-open topology and since W is not bicompact, the bicompact- open topology of $C(W)$ is not a norm-topology. Consequently, we obtain from Theorem 5.3 that $C(W)$ is not a AO^*-algebra.

Proof of Theorem 6.1

Let $\mathfrak{F}$ be the system of all positive measures μ on $\mathfrak{M}$ and T_μ the support of μ. Let be further $\mathcal{H}_\mu = L_2(T_\mu, \mu)$ and $\mathfrak{D}_\mu$ the space of all $\phi_\mu \in \mathcal{H}_\mu$ with bicompact support T_{ϕ_μ}. We define $\mathcal{H} = \sum_{\mu \in \mathfrak{F}} \oplus \mathcal{H}_\mu$ (Hilbert direct sum) and $\mathfrak{D} = \sum_{\mu \in \mathfrak{F}} \mathfrak{D}_\mu$ (algebraic direct sum).

Now we define a $*$-representation of $C(\mathfrak{M})$ into $\mathcal{L}_+(\mathfrak{D})$, $f \to A(f)$, by

$$A(f)\phi = \sum_{\mu \in \mathfrak{F}} A(f)\phi_\mu \, , \quad (A(f)\phi_\mu)(x) = f(x)\, \phi_\mu(x),$$

where $f = f(x) \in C(\mathfrak{M})$. Let $\mathcal{A}(\mathfrak{D})$ be the set of all so defined operators $A(f)$. It is easy to see that $C(\mathfrak{M})$ and $\mathcal{A}(\mathfrak{D})$ are algebraically $*$-isomorphic. It remains to prove that the $*$-isomorphism $f \leftrightarrow A(f)$ is a homomorphism between $C(\mathfrak{M})$ and $\mathcal{A}[\tau_\mathfrak{D}]$. $C(\mathfrak{M})$ is complete, because $\mathfrak{M}$ is assumed to be locally bicompact (completely regular).

Let K be a bicompact subset of $\mathfrak{M}$ and $\mathcal{M}_K$ the set of all $\phi = \sum_{\mu \in \mathfrak{F}} \phi_\mu \in \mathfrak{D}$ with $T_{\phi_\mu} \subset K$ and $\|\phi\| \leq 1$. Then it holds

Proposition 1

$\mathcal{M}_K$ is a bounded set in $\mathfrak{D}[t_{\mathfrak{A}}]$ and

$$\|f\|_K = \sup_{x \in K} |f(x)| = \|A(f)\|_{\mathcal{M}_K} . \qquad (6.I)$$

For $\phi = \sum_\mu \phi_\mu \in \mathcal{M}_K$ it holds

$$\|A(f)\phi\| \leq \sup_{x \in K} |f(x)| = \|f\|_K < \infty \qquad (6.2)$$

because for every component ϕ_μ

$$\|A(f)\phi_\mu\|^2 = \int_{T\phi_\mu} |f(x)|^2 |\phi_\mu(x)|^2 d\mu \leq \|f\|_K^2 \|\phi_\mu\|^2 .$$

Consequently, $\mathcal{M}_K$ is bounded in $\mathfrak{D}[t_{\mathfrak{A}}]$.
By application of (6.2) one shows immediately

$$\|A(f)\|_{\mathcal{M}_K} \leq \|f\|_K .$$

To prove the converse inequality let $x_0 \in K$ be a point with $|f(x_0)| = \|f\|_K$, μ_{x_0} the point-measure $\mu_{x_0}(x_0) = 1$ and $\phi_{\mu_{x_0}}(x_0) = 1$ the vector of the (one-dimensional) space $L_2(T_{\mu_{x_0}}, \mu_{x_0}) = \mathfrak{D}_{\mu_{x_0}}$. Then it holds

$$\|A(f)\|_{\mathcal{M}_K} \geq |\langle \phi_{\mu_{x_0}}, A(f)\phi_{\mu_{x_0}} \rangle| = |f(x_0)| = \|f\|_K ,$$

and proposition 1 is proved. This proposition proves that $A(f) \to f$ is continuous from $\mathfrak{A}[\mathfrak{T}_{\mathfrak{D}}]$ into $\mathcal{C}(\mathfrak{M})$.
To end the proof of the Theorem we prove

Proposition 2

For a bounded set $\mathcal{M}$ of $\mathfrak{D}[t_{\mathcal{A}}]$ there exists a bicompact set $M \subset \mathfrak{M}$ with

$$\| A(f) \|_{\mathcal{M}} \leq c \| f \|_{M} , \quad c > 0 .$$

With all $\phi = \sum_{\mu \in \mathfrak{F}} \phi_{\mu} \in \mathcal{M}$ we define

$$K = \{ \bigcup T\phi_{\mu} : \phi \in \mathcal{M}, \mu \in \mathfrak{F} \}$$

and show that $M = \overline{K}$ is a compact and therefore a bicompact set. Suppose that M is not compact, then in K exists a sequence of mutually different points $x_1, x_2, \dots$ which has not an adherent point in $\mathfrak{M}$. Let ϕ_n, $n = 1, 2, \dots$ be such a sequence of vectors of $\mathcal{M}$ that the support $T\phi_{n\mu_n}$ of the μ_n-th component of ϕ_n contains the point x_n. Now we apply the

Statement:

There exists a sequence $W_1, W_2, \dots$ of mutually disjoint closed neighbourhoods of $x_1, x_2, \dots$ so that for any sequence $\kappa_1, \kappa_2, \dots$ of positive numbers there is a continuous function $f(x)$ on $\mathfrak{M}$ with $f(x) = \kappa_n$ for $x \in W_n$.

Since $\mathfrak{M}$ is a normal space, one can construct the W_n in such a way that $\bigcup_n W_n$ is closed in $\mathfrak{M}$. We omit details of this construction. The existence of $f(x)$ follows then by the Urysohn's lemma.

Now it is

$$\beta_n = \Big(\int_{W_n \cap T\phi_{n\mu_n}} | \phi_{n\mu_n}(x) |^2 \, d\mu_n \Big)^{\frac{1}{2}} \neq 0$$

since x_n is a point of the support $T\phi_{n\mu_n}$.
Let $f(x)$ be the function of the statement with $\varkappa_n = n\,\beta_n^{-1}$, then it follows

$$\| A(f)\,\phi_n \| \geq \| A(f)\,\phi_{n\mu_n} \| \geq n \,, \quad n = 1,2,\cdots,$$

in contradiction to the boundness of $\mathcal{M}$. Therefore $M = \overline{K}$ is bicompact. Now one proves as in proposition 1

$$\| A(f) \|_{\mathcal{M}} \leq c\, \| f \|_K \,,$$

where $c = \sup_{\phi \in \mathcal{M}} \| \phi \|^2$, q.e.d.

7. Algebra of Polynoms

In this section we investigate with respect to which topology $\mathfrak{T}$ the $*$-algebra $\mathcal{P}$ of all polynoms $p = p(x) = \sum_{n\geq 0} \alpha_n x^n$, $p^* = \overline{p(x)}$, becomes an $A\hat{O}^*$-algebra $\mathcal{P}[\mathfrak{T}]$. The problem can be reformulated in the following way: Let $\mathfrak{D}$ (dense in $\mathcal{H}$) be an unitary space and $T = T^+$ a symmetric operator of $\mathcal{L}_+(\mathfrak{D})$. T is called infinite if the $*$-homomorphism $\mathcal{P} \ni p \to p(T) = \sum_{n\geq 0} \alpha_n T^n \in \mathcal{L}_+(\mathfrak{D})$ is a $*$-isomorphism. Let $\mathcal{P}(T)$ be the $*$-algebra of all operators $p(T)$. If T is infinite, then the uniform topology $\mathfrak{T}_{\mathfrak{D}}$ of $\mathcal{P}(T)$ induces in $\mathcal{P}$ a locally convex topology which we denote by $\mathfrak{T}_T$. The problem is to determine all topologies $\mathfrak{T}_T$. If T is a bounded operator, then it is infinite if and only if the spectrum $\sigma(T)=\sigma(\bar{T})$, $\bar{T}$ the unique extension of T to $\mathcal{H}$, contains infinite many points. The topology $\mathfrak{T}_T$ in $\mathcal{P}$ is then the normtopology

$$\mathfrak{T}_T' : \qquad \|p\|_T = \sup_{x\in\sigma(T)} |p(x)| \quad . \tag{7.1}$$

Hence the LK^*-algebras $\mathcal{P}[\mathfrak{T}_T]$ with the topologies (1) are $\hat{AO}^*$-algebras. They are not complete. The question is now which topologies $\mathfrak{T}_T$ one obtains for unbounded T. I conjecture that for any unbounded T the topology $\mathfrak{T}_T$ is equal to the topology defined by all seminorms

$$\mathfrak{T}_\infty : \qquad \|p\|_{(\gamma_n)} = \sum_{n\geq 0} \gamma_n |\alpha_n| \;, \quad (\gamma_n)\in\Gamma_\infty \;, \tag{7.2}$$

where Γ_∞ is the system of all positive sequences and $p=\sum_{n\geq 0}\alpha_n x^n$. We prove $\mathfrak{T}_T=\mathfrak{T}_\infty$ for a big class of operators, namely:

<u>Theorem 7.1.</u>

Let $T=T^+\in\mathcal{L}_+(\mathcal{D})$ be an operator for which exists a sequence of $\phi_n\in\mathcal{D}$, $n=1,2,\dots$, a monoton sequence $\lambda_0\leq\lambda_1\leq\dots$ of positive numbers and a sequence of positive numbers t_n, $n=1,2,\dots$, with $3\leq t_n+2\leq t_{n+1}$ so that hold

a) $\langle T^k\phi_n, T^l\phi_m\rangle = 0$ for $n\neq m$, $k,l=0,1,\dots$

b) $\lambda_m(t_n-1)^m \leq \|T^m\phi_n\| \leq M\lambda_m(t_n)^m$

$m=0,1,2,\dots$; $n=1,2,\dots$; $M\geq 1$.

For an operator T, satisfying this assumption, it is $\mathfrak{T}_T=\mathfrak{T}_\infty$.

Before we prove this theorem we show for two important cases that the assumptions of the theorem are satisfied.

Example 7.1.

If the operator $T = T^+ \in \mathcal{L}_+(\mathcal{D})$ has a spectral decomposition $T\phi = \int \lambda \, dE(\lambda)\phi$, $\phi \in \mathcal{D}$ so that $\mathcal{D}$ is invariant for $E(\chi_1,\chi_2) = E(\chi_2) - E(\chi_1)$, $-\infty < \chi_1 < \chi_2 < +\infty$, i.e. $E(\chi_1,\chi_2)\mathcal{D} \subset \mathcal{D}$, then T satisfies the assumptions a) and b) of the theorem.

Proof:

Because T is unbounded, we can suppose that $E(\lambda) \neq I$ for all $\lambda < \infty$ and consequently there is a sequence $t_1, t_2, \ldots$ $3 \leq t_n + 2 \leq t_{n+1}$ so that $E_n = E(t_n) - E(t_n - 1) \neq 0$ and hence $E_n \mathcal{D} \neq \{0\}$. Let $\phi_1, \phi_2, \ldots$ be a sequence of vectors with $\phi_n \in E_n \mathcal{D}$, $\|\phi_n\| = 1$, then the assumptions a) and b) are satisfied with $\lambda_0 = \lambda_1 = \ldots = 1$.

Example 7.2.

Let Ω be a domain of Euclidean s-space R^s, $\mathcal{H} = L_2(\Omega)$ and $\mathcal{D} = C_0^\infty(\Omega)$ the space of all functions with compact support in Ω, which have derivatives of any order and $T = \sum_{i_1,\ldots,i_s} a_{i_1\ldots i_s} \frac{\partial^{i_1+\ldots+i_s}}{\partial x_1^{i_1}\ldots\partial x_s^{i_s}}$ a symmetric differential operator with constant coefficients then the assumptions a) and b) of the theorem are satisfied.

Proof:

T can be written in the form $T = T_p + S$ where

T_p is a homogeneous differential operator of the degree $p \geq 1$ and S a differential operator of an order less than p. Let $f(x)$ be a function of $C^\infty(R^s)$ with the support in the unit ball around the origin of R^s. We put

$\nu_m = \| T_p{}^m f \|$ and

$$\mu_m = \sup_{k=0,\dots,m-1} m \binom{m}{k} \| T_p{}^k S^{m-k} f \|$$

The function f can be chosen in such a way, that

$\nu_m \geq 2 \mu_m$, $\nu_{m+1} \geq \nu_m$. Now let $K_1, K_2, \dots$ be a sequence of mutually disjoint balls contained in Ω with centres in t_n and the radius $\varrho_n \leq 1$, $n = 1, 2, \dots$.

We define

$$\phi_n = \phi_n(x) = \varrho_n^{-\frac{s}{2}} f\left(\frac{x - t_n}{\varrho_n}\right) \qquad (7.3)$$

$\phi_n = \phi_n(x)$ are functions with support in K_n. Since the balls are mutually disjoint the assumption a) is satisfied for the ϕ_n, $n = 1, 2, \dots$.

Further it holds

$$\| T_p{}^m \phi_n \| - \| \sum_{k=0}^{m-1} \binom{m}{k} T^k S^{m-k} \phi_n \| \leq \qquad (7.4)$$

$$\leq \| T^m \phi_n \| \leq \| T_p{}^m \phi_n \| + \| \sum_{k=0}^{m-1} \binom{m}{k} T^k S^{m-k} \phi_n \| .$$

Since $\| T_\rho^{\,m} \phi_n \| = (\varrho_n^{-\rho})^m \| T_\rho^{\,m} f \| = t_n^m \nu_m$,

$t_n = \varrho_n^{-\rho}$, $\quad \| \sum_{k=0}^{m-1} \binom{m}{k} T^k S^{m-k} \phi_n \| \leq t_n^m \mu_m \leq$

$\leq t_n^m \frac{\nu_m}{2}$ we obtain from (7.2)

$$\frac{1}{2} \nu_m (t_n)^m \leq \| T^m \phi_n \| \leq \frac{3}{2} \nu_m (t_n)^m . \tag{7.5}$$

Consequently, also the assumption b) is satisfied with $\lambda_m = \frac{1}{2} \nu_m$, $M = 3$; since the ϱ_n can be chosen in such a way that the t_n have the required properties.

<u>Proof of Theorem 7.1</u>.

Let (s_n) be an arbitrary strongly monotom sequence of naturals $s_0 < s_1 < s_2 \ldots$. We put

$$\begin{aligned} \gamma_i &= \lambda_{2i}^{-1} (t_{s_i})^{-i-\alpha_i} \\ x_i &= \lambda_i^{-1} (t_{s_i})^{-\alpha_i} ; \quad \alpha_i = \frac{2i-1}{4}, \ i = 0, 1, \ldots \end{aligned} \tag{7.6}$$

and construct the elements

$$\psi_j = \sum_{i=0}^{j} \gamma_i T^i \phi_{s_i}$$

$$\phi_j(\varepsilon) = \sum_{i=0}^{j} \varepsilon_i x_i \phi_{s_i} , \tag{7.7}$$

where $(\varepsilon) = (\varepsilon_0, \varepsilon_1, \ldots)$ is an arbitrary sequence with $|\varepsilon_i| = 1$.

Statement 1

The set $\mathcal{M}_{(s_n)}$ of all $\psi_j, \phi_j(\varepsilon)$ is a bounded set of $\mathfrak{D}[t_{\mathfrak{F}(T)}]$

To prove statement 1 we have to show that $\|T^n \phi\|$ is bounded for $\phi \in \mathcal{M}_{(s_n)}$ (Lemma 3.3). It holds for $n = 0, 1, 2, \dots$

$$\|T^n \psi_j\| \leq \sum_{i=0}^{j} \gamma_i \|T^{n+i} \phi_{s_i}\|$$

$$\leq M \sum_{i=0}^{\infty} \frac{\lambda_{n+i}}{\lambda_{2i}} (t_{s_i})^{n-\alpha_i} = \delta(n) < \infty$$

This series converges, because for $i \geq n$ $\lambda_{n+i} \leq \lambda_{2i}$, $t_{s_i} \geq s_i \geq i$ and $\alpha_i \to \infty$. Consequently, it holds $\sup_j \|T^n \psi_j\| \leq \leq \sigma(n)$. For the $\phi_j(\varepsilon)$ we obtain

$$\|T^n \phi_j(\varepsilon)\| \leq M \sum_{i=0}^{\infty} \frac{\lambda_n}{\lambda_i} (t_{s_i})^{n-\alpha_i}$$

and also this series converges. Next we prove

Statement 2

The numbers (3) x_n, y_n (depending on s_n) are chosen in such a way that

$$\lim_{s_n \to \infty} x_n y_n \|T^n \phi_{s_n}\|^2 = +\infty, \quad n = 0, 1, \dots$$

$$\lim_{s_n \to \infty} x_n y_n \|T^n \phi_{s_n}\| \|T^m \phi_{s_n}\| = 0 \quad \text{for } m \leq n-1 \tag{7.8}$$

Applying b) we obtain

$$x_n y_n \|T^n \phi_{s_n}\|^2 \geq \frac{\lambda_n}{\lambda_{2n}} \frac{(t_{s_n}-1)^{2n}}{(t_{s_n})^{n+2\alpha_n}} \geq \frac{2^{-2n} \lambda_n}{\lambda_{2n}} t_{s_n}^{n-2\alpha_n}$$

$$\geq 2^{-2n} \lambda_n (\lambda_{2n})^{-1} (t_{s_n})^{\frac{1}{2}}$$

and for $i \leq n-1$

$$x_n y_n \|T^n \phi_{s_n}\| \|T^i \phi_{s_n}\| \leq M^2 (t_{s_n})^{n-2\alpha_n-1} = M^2 (t_{s_n})^{-\frac{1}{2}}$$

and from this follows the statement, since $t_{s_n} \to \infty$ for $s_n \to \infty$.

After this preparation we can prove the

<u>Statement 3</u>

For every sequence of positive number (γ_ν), $\nu = 0, 1, 2, \dots$ there exist a sequence (s_n), $n = 0, 1, \dots$ of natural numbers such that

$$\|p\|_{(\gamma_n)} = \sum_{\nu \geq 0} |\alpha_\nu| \gamma_\nu \leq \|p(T)\|_{\mathcal{M}(s_n)},$$

$$p(T) = \sum_{\nu \geq 0} \alpha_\nu T^\nu$$ and consequently the topology $\mathcal{T}_T$ is stronger than $\mathcal{T}_\infty$.

We construct the sequence (s_n), $n = 0, 1, 2, \dots$, in the following way: Suppose that we have already chosen s_i for $i \leq n-1$, then we choose s_n so big that

$$x_n y_n \|T^n \phi_{s_n}\|^2 \geq 1 + \gamma_n + \sum_{i \leq n-1} x_i y_i \|T^i \phi_{s_i}\| \|T^n \phi_{s_i}\| \tag{7.9}$$

and for $m \leq n-1$

$$x_n y_n \|T^n \phi_{s_n}\| \|T^m \phi_{s_n}\| \leq \frac{1}{2^n} \tag{7.10}$$

This is possible in consequence of (7.8). Let $\mathcal{M}(s_n)$ be the so constructed bounded set of the vectors (4), then it holds

$$\|p(T)\|_{\mathcal{M}(s_n)} \geq \left|\left\langle \psi_j , \sum_{n\geq 0} \alpha_n T^n \phi_j(\varepsilon)\right\rangle\right| \geq$$

$$\geq \left| \sum_{n\geq 0} \alpha_n \sum_{i=0}^{j} \varepsilon_i x_i y_i \langle T^i \phi_{s_i}, T^n \phi_{s_n} \rangle \right| , \quad (7.\mathrm{II})$$

where we have applied the property a). Now for $p(T) = \sum_{n=0}^{N} \alpha_n T^n$, N the degree of p, we take $j = N$, $\varepsilon_i = \overline{\alpha_i} |\alpha_i|^{-1}$ $(= 1$ for $\alpha_i = 0)$.
Then follows from (7.II)

$$\|p(T)\|_{\mathcal{M}(s_n)} \geq \sum_{n\geq 0} |\alpha_n| x_n y_n \|T^n \phi_{s_n}\|^2$$

$$- \sum_{n\geq 0} |\alpha_n| \sum_{i\leq n-1} x_i y_i \|T^i \phi_{s_i}\| \|T^n \phi_{s_i}\|$$

$$- \sum_{n\geq 0} |\alpha_n| \sum_{i\geq n+1} x_i y_i \|T^i \phi_{s_i}\| \|T^n \phi_{s_i}\|$$

and from (6) and (7) we obtain finally

$$\|p(T)\|_{\mathcal{M}(s_n)} \geq \sum_{n\geq 0} |\alpha_n| \left(\gamma_n + 1 - \sum_{i\geq n+1} \frac{1}{2^i} \right)$$

$$\geq \sum_{n\geq 0} |\alpha_n| \gamma_n = \|p\|_{(\gamma_n)} .$$

Since $\mathfrak{T}_\infty$ is the strongest possible locally convex topology in $\mathcal{P}$, from Statement 3 it follows $\mathfrak{T}_T = \mathfrak{T}_\infty$, and the Theorem is completely proved.

Acknowledgement

I wish to thank Professors A.Uhlmann and H.Beckert for their interest and conversations on the subject. I am indebted to Professors D.I.Blockhinzev and A.N.Tavkhelidze for their hospitality at the Joint Institute for Nuclear Research, Dubna.

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Received by Publishing Department

on July 16, 1969.