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SMALL SAMPLE TEST OF COMPOSITE HYPOTHESES IN HILBERT SPACE

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Introduction. A test is proposed for testing composite linear hypothesis against composite linear alternatives on the mean of a normal process. A simple special case of the studied problem can be stated as follows: A normal random sample $y=(y_1, \dots, y_r)$ is observed, where y_i are independent normal variables with known variances σ_i^2 (up to a constant factor). We have to test the hypothesis

$$E y_i = \sum_{k=1}^{m} \theta_k f_{ki} + f_{0i}, \quad i = 1, \dots, r$$

against the alternative

$$E y_i = \sum_{k=1}^{n} \phi_k g_{ki} + g_{0i}, \quad i = 1, \dots, r,$$

where f_{ki}, g_{ji} $(k=0, \dots, m,\ j=0, \dots, n,\ i=1, \dots, r)$ are known numbers, θ_k, ϕ_j $(k=1, \dots, m,\ j=1, \dots, n)$ are unknown parameters. In other words, we have to test that $E y \subset \mathfrak{F}_f$ against $E y \subset \mathfrak{G}_g$, where $\mathfrak{F}_f, \mathfrak{G}_g$ are two linear varieties contained in the sample space. In the general case the standard likelihood ratio test based on the variable

$$\lambda = \frac{\sup_{E y \in \mathfrak{G}_g} \ln p_y(y / E y)}{\sup_{E y \in \mathfrak{F}_f} \ln p_y(y / E y)}$$

gives reasonable results, only asymptotically, for $r \to \infty$.

In this paper another test is proposed which is convenient also for finite samples (in the case of a continuous process we are speaking about small samples). Let x be the maximal likelihood (m.l.) estimate for Ey if $Ey \in \mathcal{F}_f$ ($p_y(y/x) = \max_{m \subset \mathcal{F}_f} p_y(y/m)$) and let z be the m.l. estimate for the orthogonal projection of Ex onto $\mathcal{G}_g$ ($p_y(x/z) = \max_{m \in \mathcal{G}_g} p_y(x/m)$). Denote D_{kl} ($k, l = 1, \dots, m$) the covariance matrix of the m.l. estimate for $\theta = (\theta_1, \dots, \theta_m)$ under the hypothesis $Ey \in \mathcal{F}_f$ and denote F the matrix $F_{ij} = \Sigma_{k,l=1}^{m} (1/\sigma_i) f_{ki} D_{kl} f_{lj} (1/\sigma_j)$ $(i, j = 1, \dots, r)$. Let be

$$r = \frac{\sum_{i=1}^{r} (1/\sigma_i^2)(y_i - x_i)(z_i - x_i)}{\left[\sum_{i,j=1}^{r} (1/\sigma_i)(z_i - x_i)(I - F)_{ij}(z_j - x_j)(1/\sigma_j)\right]^{1/2}},$$

where I is the unit matrix. If $Ey \in \mathcal{F}_f$ then $r \subset N(0,1)$ (theorem 5), The one side test based on r is similar, unbiassed, consistent and "almost the most powerful" (theorem 6). The explicit expression for the power function of the test is given in theorem 6.

The test can be used for any normal process with a continuous covariance and for any nonorthogonal linear varieties $\mathcal{F}_f$ and $\mathcal{G}_g$ such that $\mathcal{F}_f \not\subset \mathcal{G}_g, \mathcal{G}_g \not\subset \mathcal{F}_f$ and the elements of which satisfy a certain convergence condition (see theorem 7, expression (44)).

A Hilbert space (Euclidean space) technique is used. However, the emphasis is not on the space of random variables with bounded variances/5/ but on the sample space of the process/2/ .

1. Preliminaries. A probability space $(\Omega, \mathcal{A}, P)$ and a closed bounded subset T of a finite-dimensional Euclidean space are given (e.g. T is a finite set or a closed interval etc.). Consider a real normal process $y(t,\omega)$, $t \in T$, $\omega \in \Omega$. By $y(t,\omega)$ we denote also a sample of the process if ω is fixed (denoted also as $y(t)$ or simply y).

Let $K(t_1, t_2)$, $t_1, t_2 \in T$, be the covariance and $m(t)$, $t \in T$ – the mean of the process. $K(t_1, t_2)$ is symmetric and semidefinite positive/4/, chpt.10). Suppose that $K(t_1, t_2)$ is continuous on $T \times T$.

We shall use a well known result from the theory of integral equations with symmetric kernels.

<u>Lemma 1.</u> (theorem of Mercer,/7/, chpt.3, §17). For every $t_1 \in T$, $t_2 \in T$

$$K(t_1, t_2) = \sum_{i=1}^{\infty} \lambda_i u_i(t_1) u_i(t_2), \qquad (1)$$

where $\lambda_i > 0$, $u_i(t)$ (continuous on T) are the eigenfunctions of the kernel $K(t_1, t_2)$, $\int_T K(t_1, t_2) u_i(t_2)\, dt_2 = \lambda_i u_i(t_1)$, $\int_T u_i(t) u_j(t)\, dt = \delta_{ij}$, and the series (1) coverges absolutely and uniformly on $T \times T$.

<u>Lemma 2.</u> (/4/, chpt. 10). The random variables

$$w_i(\omega) = \int_T [y(t, \omega) - m(t)]\, w_i(t)\, dt, \quad i = 1, 2, \ldots, \qquad (2)$$

are normal $N(0, \lambda_n)$, independent,

$$y(t, \omega) - m(t) = \sum_{i=1}^{\infty} w_i(\omega)\, u_i(t) \qquad (3)$$

and the series (3) converges with probability one (a.e. in Ω) uniformly on T.

Proof. According to the Kolmogoroff's inequality (/4/, chpt.5) we may write for every $\epsilon > 0$

$$P \bigcup_{i=1}^{\infty} [\,|\sum_{r=s+1}^{s+i} w_r(\omega) u_r(t)| \geq \epsilon] = \lim_{k \to \infty} P[\max_{1 \leq i \leq k} |\sum_{r=s+1}^{s+i} w_r(\omega) u_r(t)| \geq$$

$$\geq \epsilon] \leq \lim_{k \to \infty} \frac{1}{\epsilon^2} \sum_{i=s+1}^{s+k} \lambda_i u_i^2(t) = \frac{1}{\epsilon^2} \sum_{i=s+1}^{\infty} \lambda_i u_i^2(t).$$

From lemma 1 it follows $\lim_{s \to \infty} \sum_{i=s+1}^{\infty} \lambda_i u_i^2(t) = 0$ uniformly on T. Hence $\lim_{s \to \infty} P \bigcup_{i=1}^{\infty} [\,|\sum_{r=s+1}^{s+i} w_r(\omega) u_r(t)| \geq \epsilon] = 0$

uniformly on T, i.e. (/4/, chpt. 2, § 6.2), $\lim_{s,i \to \infty} \sum_{r=s+1}^{s+i} w_r(\omega) u_r(t) = 0$ a.e. in Ω, uniformly on T. Q.E.D.

Denote $\mathcal{L}_2$ the (complete) Hilbert space of square integrable functions on T with the usual inner product

$$\langle a, b \rangle = \int_T a(t)\, b(t)\, dt. \qquad (4)$$

Let $\mathcal{H} \subset \mathcal{L}_2$ be the closed subspace generated by $\{u_i\}_{i=1}^{\infty}$. The orthogonal complement $\mathcal{H}^{\perp}$ is also closed, since if $b^{(i)} \in \mathcal{H}^{\perp}$, i.e. $\langle a, b^{(i)} \rangle = 0$, $a \in \mathcal{H}$, and if $\lim_{i\to\infty} \langle b^{(i)} - b, b^{(i)} - b \rangle = 0$, $b \in \mathcal{L}_2$ then $0 \le \langle a, b \rangle^2 = \langle a, b - b^{(i)} \rangle \le \langle a, a \rangle \langle b - b^{(i)}, b - b^{(i)} \rangle \to 0$.
Hence $b \in \mathcal{H}^{\perp}$. Following the lemma 2, $y - m \in \mathcal{H}$ a.e. in Ω.

We define for $a, b \in \mathcal{L}_2$

1. the component of a : $a_i = \langle a, u_i \rangle / (\lambda_i)^{1/2}$,
2. the operator U^n :

$$U_a^n = (a_1, \dots, a_n) \tag{5}$$

$$U^k U_a^n = (a_1, \dots, a_{\min(k, n)})$$

and denote $a^n \equiv U^n a$,

3. the inner product and the norm

$$(a^n, b^n) = \sum_{i=1}^{n} a_i b_i, \quad \| a^n \|^2 = (a^n, a^n) \tag{6}$$

The space $\mathcal{A}^n = \{ a^n \mid a^n = U_a^n, a \in \mathcal{L}_2 \}$, with the inner product (6) is the n-dimensional Euclidean space.

2. The hypotheses and the statistic. Two linear varieties are given: $\mathcal{F}_f = \{ a \mid a \in \mathcal{L}_2, a - f \in \mathcal{F} \}$, $\mathcal{G}_g = \{ a \mid a \in \mathcal{L}_2, a - g \in \mathcal{G} \}$, where $\mathcal{F}, \mathcal{G}$ are closed subspaces of $\mathcal{L}$ and $f \in \mathcal{L}_2, g \in \mathcal{L}_2$ are two points. We shall suppose that

$$\mathcal{F}_f \not\subset \mathcal{G}_g, \ \mathcal{G}_g \not\subset \mathcal{F}_f \tag{7}$$

and that

$$\lim_{n\to\infty} | (a^n, b^n) | > 0 \tag{8}$$

for some $a \in \mathcal{F}, b \in \mathcal{G}$.

The normal process $y(t, \omega)$ having a known covariance (up to a constant factor) is observed. We have to test the hypothesis

$$H_f : m \subset \mathcal{F}_f$$

against the alternative

$$H_g : m \in \mathcal{G}_g - \mathcal{F}_f \quad ,$$

where $m = Ey$.

Denote $\mathcal{F}^* = \mathcal{F} \cap \mathcal{H}^{\perp}$. Evidently $\mathcal{F}^*$ is closed. There exist the orthogonal projections $F^*(H)$ onto $\mathcal{F}^*(\mathcal{H}^{\perp})$, since $\mathcal{L}_2$ is complete (/8/, theorem 4.82-A). Suppose that H_f is true i.e. $m-f \in \mathcal{F}$ Then we may write $H(y-f) = H(m-f) = F^*(m-f) = F^*(y-f)$, since $y-m \in \mathcal{H}$ If $F^*(m-f) \neq 0$, then the validity of the equation $H(y-f) = F^*(y-f) \neq 0$ proves the hypothesis with probability one. We can restrict the investigation to the case when $F^*(m-f) = 0$ for every $m \in \mathcal{F}_f$.

Thus we shall consider in this paper only the regular case when $\mathcal{F} \subset \mathcal{H}$, $\mathcal{G} \subset \mathcal{H}$, $f \in \mathcal{H}$, $g \in \mathcal{H}$.

We define $\mathcal{F}^n = U^n \mathcal{F}$, $\mathcal{G}^n = U^n \mathcal{G}$,

$$\mathcal{F}_f^n = \{ a^n \mid a^n - f^n \in \mathcal{F}^n \} , \quad \mathcal{G}_g^n = \{ a^n \mid a^n - g^n \in \mathcal{G}^n \} .$$

Lemma 3. An integer n_0 exists, such that for every $n \geq n_0$:

1. $\mathcal{F}^n$ and $\mathcal{G}^n$ are not orthogonal. i.e. $(a^n, b^n) \neq 0$ for some $a^n \in \mathcal{F}^n$, $b^n \in \mathcal{G}^n$;
2. The dimensions of $\mathcal{F}^n$ and $\mathcal{G}^n$ are less than n .
3. $\mathcal{F}_f^n \not\subset \mathcal{G}_g^n$, $\mathcal{G}_g^n \not\subset \mathcal{F}_f^n$.

Proof. 1. is a direct consequence of (8).

2. There is a one-to-one correspondence between every $a^n = (a_1, \dots, a_n)$ and the series $\sum_{i=1}^{n} a_i (\lambda_i)^{1/2} u_i \subset \mathcal{H}$. We may therefore write $\mathcal{H} = \lim_{n\to\infty} U^n \mathcal{H} = \lim_{n\to\infty} \mathcal{R}^n$ and $\mathcal{F} = \lim_{n\to\infty} U^n \mathcal{F} = \lim_{n\to\infty} \mathcal{F}^n$. Suppose that $\dim \mathcal{F}^n = \dim \mathcal{R}^n$ for every n . Then $\mathcal{F}^n = \mathcal{R}^n$ and $\mathcal{F} = \mathcal{H}$, but it must be $\mathcal{F} \subset \mathcal{H}$. Thus $\dim \mathcal{F}^{n_0} < n_0$ for some n_0 which implies $\dim \mathcal{F}^n < n$ for $n > n_0$.

3. If $\mathcal{F}_f^n \subset \mathcal{G}_g^n$, i.e. $\mathcal{F}_f^n = \mathcal{F}_f^n \cap \mathcal{G}_g^n$, then $\mathcal{F}_f = \lim_{n\to\infty} \mathcal{F}_f^n \cap \mathcal{G}_g^n = \lim_{n\to\infty} U^n (\mathcal{F}_f \cap \mathcal{G}_g) = \mathcal{F}_f \cap \mathcal{G}_g$. Hence $\mathcal{F}_f^n \subset \mathcal{G}_g^n$ for some n_0 and thus also for all $n > n_0$. Q.E.D . In the sequel we shall always suppose that $n \geq n_0$.

Denote F^n, G^n the orthogonal projections ($n \times n$ matrices) from $\mathcal{R}^n$ onto $\mathcal{F}^n, \mathcal{G}^n$ (the orthogonality with respect to the inner product (6)) . Further denote $Q^n = I - F^n$, $S^n = I - G^n$. We note that a linear operator (matrix) F^n is an orthogonal projection if and only if [8]

$$F^n F^n = F^n, \ (F^n a^n, b^n) = (a^n, F^n b^n), \ a^n, b^n \in \mathcal{R}^n, \tag{9a}$$

etc. for G^n, Q^n, S^n. If $a \in \mathcal{L}_2$ we define $F^n a = F^n U^n a$, etc. From (9a) it follows

$$(F^n, Q^n b) = 0, \ (G^n a, S^n b) = 0 . \tag{9b}$$

Let us define

$$x^{(n)} = F^n y + Q^n f, \ z^{(n)} = G^n y + S^n g . \tag{10}$$

Evidently $x^{(n)} \subset \mathcal{F}_f^n$, $z^{(n)} \subset \mathcal{G}_g^n$. If H_f is true then $x^{(n)}$ is the m.l. estimate for the mean Ey^n and $z^{(n)}$ is the m.l. estimate for the projection on Ey^n onto $\mathcal{G}_g^n$.

We define the statistic

$$r^{(n)} = \frac{(y^n - x^{(n)}, z^{(n)} - x^{(n)})}{\| Q^n (z^{(n)} - x^{(n)}) \|} , \tag{11}$$

where $y^n = U^n y$. We note that $z^{(n)} - x^{(n)} = 0$ only when $z^{(n)} \subset \mathcal{F}_f^n \cap \mathcal{G}_g^n$. However, for almost every sample y, $x^{(n)} = F^n y + Q^n f$ is from $\mathcal{F}_f^n$ but not from $\mathcal{F}_f^n \cap \mathcal{G}_g^n$, since $\dim (\mathcal{F}_f^n \cap \mathcal{G}_g^n) < \dim \mathcal{F}_f^n$ (lemma 3, statement 3). Hence $z^{(n)} - x^{(n)} \neq 0$ with probability one. Further, if $z^{(n)} - x^{(n)} \neq 0$ then $Q^n(z^{(n)} - x^{(n)}) \neq 0$. Indeed, $0 = Q^n(z^{(n)} - x^{(n)}) = Q^n S^n (g - x^{(n)})$ implies $Q^n S^n = 0$ which contradicts to the statement 1 of lemma 3. Thus $r^{(n)}$ is well defined a.e. in Ω.

We further note that $r^{(n)}$ is related to the likelihood ratio (but is not equal to it). Denote $\bar{z}^{(n)}$ the m.l. estimate for Ey^n under H_g. Then $\lambda = (1/2)\{ \| y^n - x^{(n)} \|^2 - \| y^n - \bar{z}^{(n)} \|^2 \}$ is the logarithm of the likelihood ratio. Setting (formally) $z^{(n)}$ instead of $\bar{z}^{(n)}$ we obtain $\lambda^* = (1/2)\{ \| y^n - x^{(n)} \|^2 - \| y^n - z^{(n)} \|^2 \}$. The consistent unbiassed estimates for the mean and the variance of λ^* under H_g are $\epsilon = (-1/2) \| z^{(n)} - x^{(n)} \|^2$ and $\delta = \| Q^n (z^{(n)} - x^{(n)}) \|^2$. Evidently $r^{(n)} = (\lambda^* - \epsilon) / \delta^{1/2}$.

3. The properties of the test. Substituting (10) into (11) and using (9) we obtain

$$r^{(n)} = \left(y - f \, , \, \frac{Q^n (z^{(n)} - f)}{\| Q^n (z^{(n)} - f) \|} \right) . \tag{12}$$

Let us denote

$$r_1^{(n)} = \left(y - m \, , \, \frac{Q^n (z^{(n)} - f)}{\| Q^n (z^{(n)} - f) \|} \right)$$

$$r_2^{(n)} = \left(m - f \, , \, \frac{Q^n (z^{(n)} - f)}{\| Q^n (z^{(n)} - f) \|} \right) , \tag{13}$$

Evidently $r^{(n)} = r_1^{(n)} + r_2^{(n)}$. Further denote $q^{(n)} = Q^n (z^{(n)} - f)$, $v^{(n)} = Q^n (y - m)$. Using (9) we have $r_1^{(n)} = (v^{(n)}, q^{(n)}) / \| q^{(n)} \|$.

Lemma 4. The k -th moment of $v^{(n)}$ can be expressed as

$$E \prod_{r=1}^{k} v_{j_r}^{(n)} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{\text{pairs}} Q^n_{j_1 j_2} \cdots Q^n_{j_{k-1} j_k} & \text{if } k \text{ is even,} \end{cases} \tag{14}$$

where $v_{j_r}^{(n)}$ is the j_r -th component of the vector $v^{(n)}$, and the summation runs over all possible groupings of the subscripts $j_1, \ldots, j_k$ into pairs..

Proof. The normal variable $v^{(n)} = Q^n (y - m)$ has zero mean and the covariance matrix $Q^n Q^n = Q^n$. We write/1/:

$$E \prod_{r=1}^{k} v_{j_r}^{(n)} = (1/i^k) \frac{\partial^k \Psi(0)}{\partial p_{j_1} \ldots \partial p_{j_k}} , \tag{15}$$

where $i = \sqrt{-1}$, $\Psi(p) = \exp(-\gamma)$, $\gamma = (1/2) \sum_{r,s=1}^{k} p_r Q_{rs} p_s$. For k being odd, (15) contains only terms multiplyed by $\partial \gamma / \partial p_j$. If k is even then $\partial^k \Psi(p) / \partial p_{j_1} \ldots \partial p_{j_k} = \sum_{\text{pairs}} \partial^2 \gamma / \partial p_{j_1} \partial p_{j_2} \cdots \partial^2 \gamma / \partial p_{j_{k-1}} \partial p_{j_k}$ + terms containing $\partial \gamma / \partial p_j$. Thus (14) follows.

Theorem 5. For any $m \in \mathcal{H}$, $r_1^{(n)}$ is normal $N(0,1)$ and $r_1^{(n)}$ and $r_2^{(n)}$ are independent variables.

Proof. $F^n y$ and $Q^n y$ are orthogonal, hence independent components of y^n, and $v^{(n)}$ and $g^{(n)}$ are therefore also independent. The k-th moment of $r_1^{(n)}, \mu_k$ can be expressed as

$$\mu_k = \sum_{j_1,\dots,j_k=1}^{n} E\{\prod_{i=1}^{k} v_{j_i}^{(n)}\} E\{\prod_{i=1}^{k} q_{j_i}^{(n)}/\|q^{(n)}\|\} . \tag{16}$$

Setting (14) into (16) we obtain after simple computation that $\mu_k=0$ if k is odd, $\mu_k=(k-1)!!$ if k is even, since there are $(k-1)!!$ ways of grouping k terms into pairs. The moments μ_k are the moments of an $N(0,1)$ normal variable, thus $r_1^{(n)} \in N(0,1)$ /9/.

From (13) we obtain

$E\{r_1^{(n)}(q_i^{(n)}-Eq_i^{(n)})\}=(Ev^{(n)},E[q^{(n)}(q_i^{(n)}-Eq_i^{(n)})/\|q^n\|=0, \; i=1,\dots,n$,

$r_2^{(n)}$ depends on y only through $q^{(n)}$, thus $r_1^{(n)}$ and $r_2^{(n)}$ are independent. Q.E.D.

Let us denote

$$\mathcal{A} = \{a \mid a = G^n b, \; b \in \mathcal{F}^n\}, \; \mathcal{B} = \{b \mid b = Q^n a, \; a \in \mathcal{A}\} . \tag{17}$$

Evidently $\mathcal{A} \subset \mathcal{G}^n$, $\mathcal{B} \subset (\mathcal{F}^n)^{\perp}$ and $\mathcal{B} \neq \emptyset$. We assert that the dimension

$$\dim \mathcal{B} \leq 1 . \tag{18}$$

Proof of (18): Suppose $\dim \mathcal{B} \geq 2$. Then there are two-dimensional planes $\mathcal{O}_1 \subset \mathcal{A}, \mathcal{O}_2 \subset \mathcal{B}$ such that $\mathcal{O}_2 = Q^n \mathcal{O}_1$, and the straight line $\mathcal{P}_1 = \mathcal{O}_1 \cap \mathcal{O}_2$ is contained in both $\mathcal{A}$ and $(\mathcal{F}^n)^{\perp}$. However $\mathcal{P}_1 \subset \mathcal{A}$ implies that a straight line $\mathcal{P}_2 \subset \mathcal{F}^n$ exists, such that $\mathcal{P}_1$ is the projection of $\mathcal{P}_2$ ($\mathcal{P}_1 = G^n \mathcal{P}_2$). On the other hand, $\mathcal{P}_1 \subset (\mathcal{F}^n)^{\perp}$ implies $\mathcal{P}_1 \perp \mathcal{P}_2$ which can not be true. Thus $\dim \mathcal{B} \leq 1$.

Let us denote H_f^n the hypothesis that $m^n \in \mathcal{F}_f^n$ and H_g^n the hypothesis that $m^n \in \mathcal{G}_g^n - \mathcal{F}_f^n$. Consider the test of H_f^n against

H^n_g taking the critical region of the test $W^n = [r^{(n)} > \gamma]$ where γ is arbitrary but fixed. Denote

$$\Phi(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-t^2/2} \, dt \, , \tag{19}$$

<u>Theorem 6.</u>

1. W^n is a similar critical region with the level of significance

$$\alpha = \Phi(\gamma) \; .$$

2. The test is unbiassed, i.e. for any $m \in \mathcal{F}_f$, $m^* \in \mathcal{G}_g - \mathcal{F}_f$

$$P[W^n/m] < P[W^n/m^*] \, ,$$

where $P[\cdot/m]$ denotes the probability under the hypothesis that $Ey = m$.
3. The test is consistent , i.e. if $y^{(s)} (s=1,2,\dots)$ are (independent) samples from the process $y(t,\omega)$, $y_{(r)} = (1/r) \sum_{s=1}^{r} y^{(s)}$ is a sample from the averaged process and $r^{(n)}_{(r)}$ is the statistic (11) corresponding to $y_{(r)}$, then $\lim_{r\to\infty} P[r^{(n)}_{(r)} > \gamma / m] = 1$ for every $m \in \mathcal{G}_g - \mathcal{F}_f$.
4. Let be h an arbitrary point from $\mathcal{Q} - \mathcal{F}^n$, $k = F^n h$, $l = G^n k$.
The power function of the test is

$$P[r^{(n)} > \gamma/m] = \int \Phi(\gamma - (\beta\nu + \delta^2)(\nu^2 + \delta^2)^{-1/2}) \, p(\nu) \, d\nu \, , \tag{20}$$

where

$$\beta = \beta(m) = |(m^n - Ex^{(n)}, (h-k)/\|h-k\|)| \tag{21}$$

$$\delta = \min_{b \in \mathcal{F}^n_f} \| Q^n b - Q^n (G^n b + S^n g) \| \tag{22}$$

and $p(\nu)$ is a normal probability density,

$$E\nu = \beta \cos^2\phi \, , \quad D\nu = \cos^2\phi - \cos^4\phi \, , \tag{23}$$

where

$$\cos\phi = \|k - l\| / \|h - k\| . \tag{24}$$

Specially, if $\mathcal{F}_f \cap \mathcal{G}_g \neq \emptyset$ then

$$P[r^{(n)} > \gamma \mid m] = \Phi(\gamma - \beta) - \Phi(\beta \operatorname{cotg} \phi)[\Phi(\gamma - \beta) - \Phi(\gamma + \beta)] \tag{25}$$

5. To every $\epsilon > 0$ it exists β_0 so that any α-level test V satisfies the inequality

$$P[r^{(n)} > \gamma \mid m] \geq P[V/m] - \epsilon$$

for every $m \in \mathcal{A}$ such that $\beta(m) > \beta_0$.

Proof. 1. The proof follows immediately from the theorem 5.
2. If H^n_f is true then $r_2^{(n)} = 0$. Since $r^{(n)} = r_1^{(n)} + r_2^{(n)}$, it is sufficient to show that $P[r_2^{(n)} > 0/m] > P[r_2^{(n)} < 0/m]$, if $m \in \mathcal{G} - \mathcal{F}$, i.e. to prove (see (13)) that $0 < E(m-f, z^{(n)} - x^{(n)}) = (m^{(n)} - Ex^{(n)}, Ez^{(n)} - Ex^{(n)})$. The triangle with verticies $m^n, Ex^{(n)}, Ez^{(n)}$ is rectangular, since $m \in \mathcal{G}^n_g$, $Ez^{(n)} \in \mathcal{G}^n_g$ and $Ez^{(n)}$ is the projection of $Ex^{(n)}$ onto $\mathcal{G}^n_g$. Thus $(m^n - Ex^{(n)}, Ez^{(n)} - Ex^{(n)}) = \| Ez^{(n)} - Ex^{(n)} \|^2 > 0$.

3. The averaged process $y_{(r)}$ has the covariance $(1/r) K(t_1, t_2)$. Hence we obtain $r_2^{(n)}(r)$ setting $y_{(r)}$ instead of y into (10) and (13) and using the inner product $(a,b)_r = r(a,b)$ instead of (a,b). The operators F^n, G^n do not change, since (9) obviously hold also for the new inner product. From (10) it follows that $Q^n(z^{(n)} - x^{(n)})$ converges in probability to $Q^n(Ez^{(n)} - Ex^{(n)})$. Therefore $r_2^{(n)}(r) = \sqrt{r}(m-f, Q^n(z^{(n)} - x^{(n)}))/\| Q^n(z^{(n)} - x^{(n)})\|$ and $r_2^{(n)}(r)$ converges to $+\infty$ as $r \to \infty$, since $(m-f, Q^n(Ez^{(n)} - Ex^{(n)})) = \| Ez^{(n)} - Ex^{(n)} \|^2 > 0$, as follows from part 2 of the proof. This, together with the relations $r^{(n)}(r) = r_1^{(n)}(r) + r_2^{(n)}(r)$, $r_1^{(n)}(r) \in N(0,1)$, prove the consistency.

4. Let us denote

$$\mathcal{A}_g = \{ a \mid a = G^n b + S^n g,\ b \in \mathcal{F}^n_f \},\quad \mathcal{B}_g = \{ b' \mid b = Q^n a,\ a \in \mathcal{A}_g \}. \tag{26}$$

Comparing (26) with (17) we see that $\mathcal{A}_g, \mathcal{B}_g$ are translations of $\mathcal{A}, \mathcal{B}$ in the Euclidean space $\mathcal{R}^n$. Thus $\mathcal{A}_g$ is parallel to $\mathcal{A}$, $\mathcal{B}_g$ is parallel to $\mathcal{B}$ and $\dim \mathcal{B}_g \leq 1$ (see (18)).

Next we prove, that (21) and (24) are independent on the choice of h. Indeed, $h - k = Q^n h \subset \mathcal{B}$ since $h \in \mathcal{A}$. Hence the unit vector

$$e_1 = (h-k)/\| h-k \| \tag{27}$$

is parallel to $\mathfrak{B}$ and is uniquely defined (for any h) up to its sign. Analogically, we can prove that $(k-1)$ is parallel to a straight line.

Denote $d \in \mathfrak{B}_g$ the point satisfying (see (22))

$$\| d - Q^n f \| = \min_{b \in \mathfrak{B}_g} \| b - Q^n f \| = \delta \tag{28}$$

and define

$$e_2 = (d - Q^n f) / \| d - Q^n f \| \tag{29}$$

if $\delta \neq 0$. Let A be the orthogonal projection onto $\mathfrak{A}_g$. Define

$$e_3 = (m^n - A m^n) / \| m^n - A m^n \| \tag{30}$$

if $m^n \neq A m^n$. The three vectors e_1, e_2, e_3 form an orthogonal system in $(\mathcal{F}^n)^{\perp}$. Indeed, $e_1, e_2 \in (\mathcal{F}^n)^{\perp}$. For any $b \in \mathcal{F}^n$ we may write $(m^n - A m^n, b) = (m^n - A m^n, b - G^n b) + (m^n - A m^n, G^n b) = 0$ since $(m^n - A m^n) \perp \mathfrak{A}$, $G^n b \in \mathfrak{A}$, and $(m^n - A m^n) \in \mathcal{G}^n$, $\mathcal{G}^n \perp (b - G^n b)$. Hence $e_3 \in (\mathcal{F}^n)^{\perp}$. From (28) it follows $(e_1, e_2) = 0$. Since $d \in \mathfrak{B}_g$, it exists $b_{(d)} \in \mathcal{F}_f^n$, such that $d = Q^n [G^n b_{(d)} + S^n g]$. Hence, $(m^n - A m^n, d - Q^n f) = (m^n - A m^n, G^n b_{(d)} + S^n g - b_{(d)}) = 0$, thus $(e_2, e_3) = 0$. Analogically, if $a \in \mathfrak{A}$ then $(m - Am, Q^n a) = (m - Am, a) = 0$, hence, $(e_1, e_3) = 0$.

From the theorem 5 it directly follows that

$$P[r^{(n)} > \gamma / m] = \int \Phi(\gamma - u) p_{r_2}(u) du , \tag{31}$$

where $p_{r_2}(\cdot)$ is the probability density of (see (13))

$$r_2^{(n)} = (m^n - E x^{(n)}, Q (z^{(n)} - f)) / \| Q^n (z^{(n)} - f) \| . \tag{32}$$

Taking into account that $Q^n z^{(n)} \in \mathfrak{B}_g$ we may write

$$Q^n (z^{(n)} - f) = \nu e_1 + \delta e_2 , \tag{33}$$

where

$$\nu = (Q^n z^{(n)} - d, e_1) \tag{34}$$

is a normal random variable. According to (21), (22) we may write

$$\beta = |(m^n - Ex^{(n)}, e_1)|, \tag{35a}$$

$$\delta = (m^n - Ex^{(n)}, e_2) . \tag{35b}$$

To prove (35b) we write $(m^n - Ex^{(n)}, e_2) = ((Am^n - Ex^{(n)}) + (m^n - Am^n), e_2) = \delta$ as follows from (28) and from $Q^n Am^n \in \mathfrak{B}_g$.

Equalities (32) - (35) show that

$$r_2^{(n)} = (\beta\nu + \delta^2) / [\nu^2 + \delta^2]^{1/2} , \tag{36}$$

where we have chosen h in the definition of e_1 , so that $\beta = |(m^n - Ex^{(n)}, e_1)| = (m^n - Ex^{(n)}, e_1)$. This is true e.g. if we take

$$h = Am^n - c_{(d)} , \tag{37}$$

where $c_{(d)} = G^n b_{(d)} + S^n g$. Then $k = F^n h = Ex^{(n)} + (d - Q^n f) - c_{(d)}$ and $l = G^n k = Ez^{(n)} - c_{(d)}$. From the definition of e_1 and e_2 we obtain

$$\|h - k\| = (m^n - Ex^{(n)}, e_1) + (Am^n - m^n, e_1) + ((Q^n f - d), e_1) = \beta . \tag{38}$$

We may write

$$\|k - l\| / \|h - l\| = \cos\phi = \|l - F^n l\| / \|k - l\| \tag{39}$$

since $(h - l)$ is parallel to $(l - F^n l)$, and from (37) - (39) we obtain

$$\cos\phi = \|k - l\| / \beta = \|d - EQ^n z^{(n)}\| / \|k - l\| . \tag{40}$$

Excluding $\|k - l\|$ from (40) we obtain using (34) $E\nu = \|Q^n Ez^{(n)} - d\| = \beta\cos^2\phi$. To compute the variance $D\nu$ we write using (10) and (34) $\nu - E\nu = (G^n F^n (y^n - Ey^n), e_1)$. Hence $D\nu = (e_1, G^n F^n G^n e_1) = \|F^n G^n e_1\|^2$ From (24) and (27) we obtain $\|G^n e_1\| = \sin\phi$, since $(k - l) \perp \mathcal{G}^n, G^n e_1 \in \mathcal{G}^n$, and $\|F^n G^n e_1\| = \cos\phi \|G^n e_1\|$, since $F^n G^n e_1 \in \mathcal{F}^n$, $(h - k) \perp \mathcal{F}^n$ and since $(k - l) \perp G^n e_1$. Thus, finally, $D\nu = \cos^2\phi \sin^2\phi$.

Setting (36) into (31) we obtain (20). Specially, if $\mathcal{F}_f^n \cap \mathcal{G}_g^n \neq \emptyset$ then $\delta = 0$ and from (36) it follows : $r_2^{(n)} = \beta$ if $\nu > 0$, $r_2^{(n)} = -\beta$ if $\nu < 0$. Thus $P[r^{(n)} > \gamma / m] = \Phi(\gamma - \beta) - P[\nu < 0]\{\Phi(\gamma - \beta) - \Phi(\gamma + \beta)\}$ and (25) follows from (19) and (23).

5. From (36) we compute $\partial r_2^{(n)} / \partial \nu \gtreqless 0$ if $\nu \lesseqgtr \beta$, $\lim_{\nu \to \infty} r_2^{(n)}(\nu) = \beta$, $r_2^{(n)}(\nu = (\beta^2 - \delta^2)/2\beta) = \beta$, $\max_{\nu} r_2^{(n)}(\nu) = r_2^{(n)}(\beta) = \sqrt{\delta^2 + \beta^2} \equiv \Delta$.

From (20) using (19) we thus obtain

$$\Phi(\gamma - \Delta) \geq P[r^{(n)} > \gamma / m] \geq \Phi(\gamma - \Delta) - \int_{-\infty}^{\kappa} [\Phi(\gamma - \Delta) - \\ - \Phi(\gamma - \nu)] p(\nu) \, d\nu - \int_{\kappa}^{\infty} [\Phi(\gamma - \Delta) - \Phi(\gamma - \beta)] p(\nu) \, d\nu \geq \\ \geq \Phi(\gamma - \Delta) - P[\nu < \kappa] - \{\Phi(\gamma - \Delta) - \Phi(\gamma - \beta)\} , \tag{41}$$

where $\kappa = ((\beta^2 - \delta^2)/2\beta)$. According to (19) and (23)

$$P[\nu < \kappa] = \Phi[(\delta^2 + 2\beta^2 \cos^2\phi - \beta^2)/2\beta \sin\phi \cos\phi] \to 0$$

as $\beta \to \infty$, since from (24) follows that $2\cos^2\phi > 1$. Thus the inequality (41) gives

$$\lim_{\beta \to \infty} P[r^{(n)} > \gamma / m] = \Phi(\gamma - \Delta) . \tag{42}$$

Denote $m^* = F^n m^n + Q^n f$. We have $\|m - m^*\| = \sqrt{\delta^2 + \beta^2} = \Delta$, since $m^n - m^* = Q^n(m - f) = (m^n - E x^{(n)}, e_1) e_1 + (m^n - E x^{(n)}, e_2) e_2 = \beta e_1 + \delta e_2$.

For any α-level test V of H_f^n against H_g^n we may write $P[V/m^*] \leq \alpha$, since $m^* \subset \mathcal{F}_f^n$. V is therefore an α-level test for the simple hypothesis $E y^n = m^*$ against the simple alternative $E y^n = m^n$. Thus the power $P[V/m]$ must be smaller than the power of the likelihood ratio test for simple hypothesis /9/ which in this case is equal to $\Phi(\gamma - \|m - m^*\|) = \Phi(\gamma - \Delta)$. Therefore

$$P[V/m] \leq \Phi(\gamma - \Delta) \tag{43}$$

and the statement of the theorem follows from (42) and (43) Q.E.D .

If H_f^n is false then H_f is also false. The test $W^n = [r^{(n)} > \gamma]$ is therefore an α-level test for testing H_f against H_g. However its power function depends on n. The condition under which the power function converges with $n \to \infty$ can be heuristically stated as follows. Consider the series (3)

$$y(t) = \sum_{i=1}^{\infty} [< m, u_i > + w_i] u_i(t).$$

The terms $< m, u_i >$ are the "signal components", and w_i are the "noise components". We can neglect in (3) the term $\sum_{i=n+1}^{\infty} [< m, u_i > + w_i] u_i(t)$ if $< m, u_i >^2$ is "very small" comparing to the variance of w_i for all $i > n$. More exactly, we need that /5/

$$\lim_{n \to \infty} \sum_{i=n+1}^{\infty} < m, u_i >^2 / \lambda_i = 0.$$

Theorem 7. If

$$\lim_{n \to \infty} \sup_{\substack{m \in \mathcal{F}_f \cup \mathcal{G}_g \\ m \neq 0}} \left(\sum_{i=n+1}^{\infty} m_i^2 / \sum_{i=1}^{\infty} m_i^2 \right) = 0 \tag{44}$$

then the power function $P[r^{(n)} > \gamma / m]$ converges with $n \to \infty$ for every $m \in \mathcal{G}_g$.

Proof. Let us denote $\| a \| = \sum_{i=1}^{\infty} a_i^2$, $a \in \mathcal{H}$, $a_i^2 = < a, u_i >^2 / \lambda_i$. Further denote

$$\xi(n) = \sup_{\substack{m \in \mathcal{F}_f \cup \mathcal{G}_g \\ m \neq 0}} \left(\sum_{i=n+1}^{\infty} m_i^2 / \sum_{i=1}^{\infty} m_i^2 \right).$$

Let $\{A_i^n\}_{i=1}^s$ be a sequence of operators which are equal to F^n or to G^n. For $s < \infty$ and for any a, $\|a\| < \infty$, the limit

$$\lim_{n \to \infty} A_1^n \dots A_s^n a^n = a^* \tag{45}$$

exists, and $\|a^*\| \leq \| a \| < \infty$. If moreover $\sum_{i=n+1}^{\infty} a_i^2 / \|a\|^2 \leq \Psi^2 \xi(n)$ for some $1 \leq \Psi < \infty$ then

$$\lim_{n \to \infty} \frac{\| A_1^n \dots A_s^n a^n - a^* \|}{(s+1) \| a \| \Psi \sqrt{\xi(n)}} \leq 1. \tag{46}$$

Proof of (45) and (46): Let be $s=1$, $A_1^n = G^n$. We write

$$\| G^{n+r} a^{n+r} - G^n a^n \| \leq \| G^{n+r} (a^{n+r} - a^n) \| +$$

$$+ \| G^{n+r} a^n - G^n a^n \| \leq \| a^{n+r} - a^n \| + \| G^{n+r} a^n - G^n a^n \| . \tag{47}$$

Denote $(\mathcal{R}^r)^\perp$ the (r-dimensional) orthogonal complement of $\mathcal{R}^n$ in $\mathcal{R}^{n+r}$ ($\mathcal{R}^{n+r} = \mathcal{R}^n \oplus (\mathcal{R}^r)^\perp$). According to the definition of $\mathcal{G}^n$ we write $\overline{\mathcal{G}^{n+r}} \subset \mathcal{G}^n \oplus (\mathcal{R}^r)^\perp$. Further $(a^n - G^n a^n) \perp \mathcal{G}^n$ and $(a^n - G^n a^n) \in \mathcal{R}^n$, hence $(a^n - G^n a^n) \perp \mathcal{G}^{n+r}$, i.e. $G^{n+r}(a^n - G^n a^n) = 0$. Thus we may write

$$\| G^{n+r} a^n - G^n a^n \| = \| G^{n+r} G^n a^n - G^n a^n \| = \min_{b \in \mathcal{G}^{n+r}} \| b - G^n a^n \| \leq$$

$$\leq \| c^{n+r}(n) - G^n a^n \| = \Big(\sum_{i=n+1}^{n+r} [c_i(n)]^2 \Big)^{1/2} < \Big(\sum_{i=n+1}^{\infty} [c_i(n)]^2 \Big)^{1/2}, \tag{48}$$

where $c(n)$ is one of the points from $\mathcal{G}$ satisfying the equation $U^n c(n) = G^n a^n$ ($c(n)$ exists, according to the definition of $\mathcal{G}^n$, and $c^{n+r}(n) = U^{n+r} c(n) \in \mathcal{G}^{n+r}$).

Suppose that $\sup_n \| c(n) \| = \infty$. Then a subsequence $\{c(n_s)\}_s$ exists such that $\lim \| c(n_s) \| = \infty$. Hence $\lim \sum_{i=n_s+1}^{\infty} [c_i(n_s)]^2 / \| c(n_s) \|^2 = \lim (\| c(n_s) \|^2 - \| G^{n_s} a^{n_s} \|^2) / \| c(n_s) \|^2 = 1$, since $\| G^{n_s} a^{n_s} \| \leq \| a \| < \infty$. On the other hand $\sum_{i=n_s+1}^{\infty} [c_i(n_s)]^2 / \| c(n_s) \|^2 < \xi(n_s) \to 0$. This contradiction proves that $\sup_{n \geq n_0} \| c(n) \| = \gamma_{n_0} < \infty$ for any $n_0 \geq 1$, and from (48) we obtain

$$\| G^{n+r} a^{n+r} - G^n a^n \| \leq \gamma_n \sqrt{\xi(n)} \leq \gamma_1 \sqrt{\xi(n)} . \tag{49}$$

Finally from (47) and (49) we obtain $\lim_{n, r \to \infty} \| G^{n+r} a^{n+r} - G^n a^n \| = 0$. Thus the limit $\lim_{n \to \infty} G^n a^n = a^*$ exists and $\| a^* \| < \infty$, since $\| G^n a^n \| < \| a \| < \infty$. The same can be proved taking $A_1^n = F^n$, i.e. (45) is proved for $s = 1$. Let us suppose that (45) is true for some $s \geq 1$. Then $\| A_1^n \dots A_s^n a^n \| < \infty$ and we can take $A_1^n \dots A_s^n a^n$ instead of a^n in (47). So we obtain (45) for $s+1$. If $\sum_{i=n+1}^{\infty} a_i^2 \leq \| a \|^2 \Psi^2 \xi(n)$ then from (44), (47), (48) it follows that

$$\| G^n a^n - a^* \| \leq 2 \max \{ \Psi \| a \|, \gamma_n \} \sqrt{\xi(n)} . \tag{50}$$

Further $\lim_{n_0\to\infty} \gamma^2_{n_0} = \lim_{n_0\to\infty} \sup_{n\geq n_0} (\| G^n a^n \|^2 + \sum_{i=n+1}^{\infty} [c_i(n)]^2) \leq$

$$< \| a \|^2 + \lim_{n_0\to\infty} \sup_{n\geq n_0} \| c(n) \|^2 \Psi^2 \xi(n) = \| a \|^2 .$$

From (50) we obtain

$$1 \geq \lim_{n\to\infty} \frac{\| G^n a^n - a^* \|}{2\sqrt{\xi(n)} \max \{ \Psi \| a \|, \gamma_n \}} = \lim_{n\to\infty} \frac{\| G^n a^n - a^* \|}{2 \Psi \| a \| \sqrt{\xi(n)}} ,$$

i.e. (46) is true for $s=1$ and induction arguments prove (46) for arbitrary s .

Let b be an arbitrary point from $\mathcal{F}-\mathcal{G}$, $\| b \| < \infty$. Denote

$$h^{(n)} = F^n b , \; k^{(n)} = G^n F^n b , \; l^{(n)} = F^n G^n F^n b . \tag{51}$$

According to (45) there exist limits $\lim h^{(n)}$, $\lim f^{(n)}$, $\lim l^{(n)}$ with finite norms. Therefore also $\cos \phi^{(n)} = \| l^{(n)} - k^{(n)} \| / \| h^{(n)} - k^{(n)} \|$ (see (24)) and $e_1^{(n)} = (h^{(n)} - k^{(n)}) / \| h^{(n)} - k^{(n)} \|$ (see (27)) converge with $n\to\infty$. Further $e_2^{(n)} = (d^{(n)} - Q^n f) / \| d^{(n)} - Q^n f \|$ (see (29)) where $d^{(n)}$ is the projection of $Q^n f$ onto the straight line going through the point $Q^n [G^n f + S^n g]$, and parallel to $e_1^{(n)}$. According to (45) the limits with finite norms

$\lim_{n\to\infty} Q^n [G^n f + S^n g] = \lim_{n\to\infty} (I - F^n)(G^n f + (I - G^n) g)$ and

$\lim_{n\to\infty} Q^n f = f - \lim_{n\to\infty} F^n f$ do exist. Hence $\lim_{n\to\infty} d^{(n)}$ and therefore also $\lim_{n\to\infty} e_2^{(n)}$ exist. Now consider the parameters (see (35))

$$\beta^{(n)} = | (m^n - E x^{(n)} , e_1^{(n)}) | ,$$
$$\delta^{(n)} = (m^n - E x^{(n)} , e_2^{(n)}) . \tag{52}$$

Using (10) and (45) we may write

$$\| E x^{(n+r)} - E x^{(n)} \| = \| F^{n+r} m + Q^{n+r} f -$$
$$- (F^n m + Q^n f) \| = \| F^{n+r} (m-f)^{n+r} - F^n (m-f)^n \| \to 0$$

with $n, r \to \infty$. According to (52) $|\beta^{(n+r)} - \beta^{(n)}| \leq |(m^{n+r} - Ex^{(n+r)}, e_1^{(n+r)} - e_1^{(n)})| + |(Ex^{(n+r)} - Ex^{(n)}, e_1^{(n)})| \leq \| m - f \| \| e_1^{(n-r)} - e_1^{(n)} \| +$

$$+ \| Ex^{(n+r)} - Ex^{(n)} \| \to 0 ,$$

i.e. $\lim_{n\to\infty} \beta^{(n)}$ exists. Analogically we prove the existence of $\lim_{n\to\infty} \delta^{(n)}$.

Thus all the parameters $\cos\phi^{(n)}, \beta^{(n)}, \delta^{(n)}$ defining the power function $P[r^{(n)} > \gamma / m]$ (theorem 6) converge and the limit power function is obtained setting the limit values of the parameters into the expression (20). Q.E.D .

Conclusion

1. The inequality (46) shows that the rate of convergence of a projection of any vector $a \in \mathfrak{F}_f \cup \mathfrak{G}_g$ or $a \in \mathfrak{F} \cup \mathfrak{G}$ is comparable with the rate of convergence of $\xi(n)$ and the same must be therefore valid for the convergence of $\cos\phi^{(n)}$, $\beta^{(n)}$, $\delta^{(n)}$.

2. Computationally the problem of constructing the test consists in finding a finite number of eigenvectors and eigenvalues of the symmetric kernel $K(t_1, t_2)$ (see lemma 1). But from (11) and from the theorem 7 it follows that other methods giving the projections onto $\mathfrak{F}$ and $\mathfrak{G}$ and the limit of $(a,b) = \sum_{i=1}^{n} \int_T a(t) u_i(t) dt \int b(t) u_i(t) dt / \lambda_i$ can be used/5/ . The problem is especially simple if T is a finite set (see the expression for r in the introduction, and /6/) . The last case was used to solve approximately a problem in the analysis of nuclear scattering experiments /6/.

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