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# A METHOD FOR TESTING COMPLEX LINEAR HYPOTHESES AND ITS USE IN THE PHASE - SHIFT ANALYSIS

ЛАБОРАТОРИЯ ВЫЧИСЛИТЕЛЬНОЙ ТЕХНИКИ
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# A METHOD FOR TESTING COMPLEX LINEAR HYPOTHESES AND ITS USE IN THE PHASE - SHIFT ANALYSIS

Р е з ю м е

В работе приводится новый критерий проверки сложной линейной гипотезы против сложной линейной альтернативы. Показано, что величина $r$, на которой основан этот критерий, имеет нормальное распределение с параметрами 0,1, откуда вытекает простой способ вычисления ошибки первого рода. Критерий является близким критерию отношения правдоподобия и при большом числе экспериментов совпадает с оптимальным критерием для проверки простых гипотез.

Критерий полезен для выбора правильного решения при неоднозначном фазовом анализе экспериментов по рассеянию элементарных частиц. В сравнении с $\chi^2$-критерием он дает выигрыш в числе экспериментов, необходимых для определения истинного набора фазовых сдвигов.

## Introduction

The problem of testing hypotheses basing on the results of performed physical experiments often occurs in experimental physics. The testing problem which is to be discussed here occurs for instance in the NN-scattering experiments when the phase - shift analysis is ambiguous, because of lack of experimental results. In such a case an efficient statistical test helps to define reliably the true set of phase shifts.

The $\chi^2$-test which has been often used to discriminate the false set of phase shifts is not optimal as it was shown by mathematicians/1/ and by physicists/2,3/. Instead of the $\chi^2$-test the likelihood ratio test has been proposed /1,3,4/, but the computation of the characteristics of the test (for instance, the probability of the Type I error) is cumbersome and it can be done only approximately/3/. The test proposed further has good asymptotic properties of the likelihood ratio test, but on the other hand, the computation of the Type I error is as simple as for the $\chi^2$-test.

In this paper we shall mainly discuss the mathematical theory of this new test. The reader who is interested only in the applications of the test in the phase-shift analysis is refered directly to the paragraph 4.

## I. A General Formulation

Let us denote $y_1, \dots, y_n$ the results of (independent) experiments which are distributed following the normal distribution with variances $\sigma_1^2, \dots, \sigma_n^2$, respectively.

We have to test the hypothesis $\mathcal{H}_0$ that the expected values of the measured quantities $y_i$ are equal to

$$\mathcal{E}\{y_i\} = \eta_i(\vec{\theta}), \quad i = 1, \dots, n. \tag{1}$$

The alternative is the hypothesis $\mathcal{H}_1$ that

$$\mathcal{E}\{y_i\} = \nu_i(\vec{\theta}), \quad i = 1, \dots, n. \tag{2}$$

The functions $\eta_i(\ )$ and $\nu_i(\ )$ are known linear functions, but the parameters $\vec{\theta} = (\theta_1, \dots, \theta_m)$ are unknown and they are to be estimated from the experimental results. We shall use the vector notation, for instance:

$$\vec{\eta}(\ ) = \begin{pmatrix} \eta_1(\ ) \\ \vdots \\ \eta_n(\ ) \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \vec{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_m \end{pmatrix} \tag{3}$$

and we use the prime to denote the transposition of a vector. Matrices generally will be denoted by capital letters.

Thus, it can be written:

$$\vec{\eta}(\vec{\theta}) = F\vec{\theta} + \vec{c}_1 \tag{4}$$

$$\vec{\nu}(\vec{\theta}) = G\vec{\theta} + \vec{c}_2, \tag{5}$$

where $F, G$ are known $n \times m$ matrices ($n$ is the number of performed experiments, $m$ is the number of unknown parameters) and $\vec{c}_1, \vec{c}_2$ are constant vectors.

Obviously both the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ are complex and, generally speaking, we are to decide to what of two $m$-dimensional hyperplanes of an $n$-dimensional Euclidean space belongs the theoretical value of the results of the experiments.



In general the testing problems are solved in the following way/5/: Let be $r = r(\vec{y})$ an adequately choosen (measurable) function of the experimental data $\vec{y}$. Usually the function $r(\vec{y})$ is one-dimensional, but this is not necessary. Obviously $r$ is a random variable taking values from an one-dimensional set $\Omega$. We choose a measurable set $W_\alpha \subset \Omega$ (usually $W_\alpha$ is an interval) so that

$$p\{r \in W_\alpha\} \leq \alpha \tag{6}$$

under the assumption that $\mathcal{H}_0$ is true. This can be done by several ways and a statistical test on the $\alpha$-level is given if the function $r(\vec{y})$ and the set $W_\alpha$ are chosen. If the number $r_{ex}$ computed from the experimental results is a point of $W_\alpha$ ($r_{ex} \in W_\alpha$), then the experimenter rejects the hypothesis $\mathcal{H}_0$ (i.e. accepts $\mathcal{H}_1$). If $r_{ex} \notin W_\alpha$ the amount of experiments is not sufficient for accepting $\mathcal{H}_1$.

The value $\alpha$ is the probability of erroneously rejecting the hypothesis $\mathcal{H}_0$ (Type I error) and it can be chosen as small as it is needed by the experimenter. On the other hand, the probability of not rejecting $\mathcal{H}_0$, when $\mathcal{H}_0$ is false (the Type II error), is defined by the chosen test:

$$p\{r \notin W_\alpha / \mathcal{H}_1 \text{ is true}, \vec{\theta} \text{ is true}\} = \beta(\vec{\theta}). \tag{7}$$

It depends not only on the hypothesis $\mathcal{H}_1$ but also on the true value of $\vec{\theta}$. Obviously the optimal test is the test which for a given $\alpha$ gives minimal values of $\beta(\vec{\theta})$.

It is often sufficient to know $\beta(\vec{\theta})$ assuming a large amount of experiments (asymptotically). The region of possible values of $\vec{\theta}$ is then reduced (with probability one) to a single point – the maximal likelihood estimate $\hat{\vec{\theta}}$ /7/, and instead of the function $\beta(\vec{\theta})$ we have only one number $\beta$:

$$\beta = \lim_{n \to \infty} p\{r \in W_\alpha / \mathcal{E}\{\vec{y}\} = \vec{\nu} / \vec{\Phi})\}. \tag{8}$$

It is our task to search a test which for a given (non-asymptotical) $\alpha$ gives the smallest value of $\beta$.

## 2. The Construction of the $r$-Test

In this and the following paragraph we shall assume that $\sigma_i^2=1$, $i=1,\dots,n$. (If it is not true we have only to substitute $y_i$ for $y_i/\sigma_i$, $F_{ij}$ for $F_{ij}/\sigma_i$, and $G_{ij}$ for $G_{ij}/\sigma_i$).

Assuming $\mathcal{H}_0$ is true, the likelihood function is given as

$$L_0(\vec{\theta}/\vec{y})=\frac{1}{(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}[\vec{y}-\vec{\eta}(\vec{\theta})]'[\vec{y}-\vec{\eta}(\vec{\theta})]\right\} \qquad (9)$$

and the maximal likelihood estimate is equal to [6]:

$$\hat{\vec{\theta}}=[F'F]^{-1}F'\vec{y}+\text{const}. \qquad (10)$$

On the other side, if $\mathcal{H}_1$ is true, then:

$$L_1(\vec{\theta}/\vec{y})=\frac{1}{(2\pi)^{n/2}}\exp\left\{-\frac{1}{2}[\vec{y}-\vec{\nu}(\vec{\theta})]'[\vec{y}-\vec{\nu}(\vec{\theta})]\right\}, \qquad (11)$$

and the maximal likelihood estimate is equal to:

$$\hat{\vec{\Phi}}=[G'G]^{-1}G'\vec{y}+\text{const}. \qquad (12)$$

The $\chi^2$-test is based on the function [1] (chpt. 7 § 12)

$$\chi^2_{\hat{\theta}}(\vec{y})=-2\ln L_0(\hat{\vec{\theta}}/\vec{y})=[\vec{y}-\vec{\eta}(\hat{\vec{\theta}})]'[\vec{y}-\vec{\eta}(\hat{\vec{\theta}})]. \qquad (13)$$

Following the $\chi^2$-test $\mathcal{H}_0$ is rejected if the value obtained from the experiments, $\chi^2_{ex}$, is, generally speaking, sufficiently large.

The likelihood ratio test is based on the quantity



$$\lambda=-2\ln\frac{L_0(\hat{\vec{\theta}}/\vec{y})}{L_1(\hat{\vec{\Phi}}/\vec{y})}=[\vec{y}-\vec{\eta}(\hat{\vec{\theta}})]'[\vec{y}-\vec{\eta}(\hat{\vec{\theta}})]-[\vec{y}-\vec{\eta}(\hat{\vec{\Phi}})]'[\vec{y}-\vec{\eta}(\hat{\vec{\Phi}})] \qquad (14)$$

and $\mathcal{H}_0$ is rejected if $\lambda_{ex}$ is larger than a number $k$ which depends on the probability of the Type I error $\alpha$, i.e. $k=k(\alpha)$. However the computing of $k$ for given $\alpha$ is cumbersome, it has to be repeated after each experiment and can be done only approximately [3].

The proposed $r$-test is a variation of the likelihood ratio test. We substitute in (14) for $\hat{\vec{\Phi}}$ another estimate, $\hat{\vec{\Phi}}_{\hat{\theta}}$ which is obtained from the maximal likelihood method (least square method) if substituting insted of the experimental data $y_1,\dots,y_n$ their estimates $\eta_1(\hat{\vec{\theta}}),\dots,\eta_n(\hat{\vec{\theta}})$:

$$L_1(\hat{\vec{\Phi}}_{\hat{\theta}}/\vec{\eta}(\hat{\vec{\theta}}))=\max_{\vec{\theta}} L_1(\vec{\theta}/\vec{\eta}(\hat{\vec{\theta}})). \qquad (15)$$

Following (12) we obtain

$$\hat{\vec{\Phi}}_{\hat{\theta}}=[G'G]^{-1}G'\vec{\eta}(\hat{\vec{\theta}})+\text{const}. \qquad (16)$$

Let be

$$T=-\ln\frac{L_0(\hat{\vec{\theta}}/\vec{y})}{L_1(\hat{\vec{\Phi}}_{\hat{\theta}}/\vec{y})}+\mathcal{E}_{\hat{\theta}}\left\{-\ln\frac{L_0(\hat{\vec{\theta}}/\vec{y})}{L_1(\hat{\vec{\Phi}}_{\hat{\theta}}/\vec{y})}\right\}=[\vec{y}-\vec{\eta}(\hat{\vec{\theta}})]'[\vec{\nu}(\hat{\vec{\Phi}}_{\hat{\theta}})-\vec{\eta}(\hat{\vec{\theta}})] \qquad (17)$$

and let us divide $T$ by the best (unbiased and consistent [7]) estimate of the variance of $T$. We obtain the quantity

$$r=r(\vec{y})=\frac{[\vec{y}-\vec{\eta}(\hat{\vec{\theta}})]'[\vec{\nu}(\hat{\vec{\Phi}}_{\hat{\theta}})-\vec{\eta}(\hat{\vec{\theta}})]}{\sqrt{[\vec{\nu}(\hat{\vec{\Phi}}_{\hat{\theta}})-\vec{\eta}(\hat{\vec{\theta}})]'R[\vec{\nu}(\hat{\vec{\Phi}}_{\hat{\theta}})-\vec{\eta}(\hat{\vec{\theta}})]}}. \qquad (18)$$

The matrix $R$ is equal to

$$R = I - F[F'F]^{-1} F', \tag{19}$$

where $I$ is a unit matrix.

$r$ depends on $\vec{y}$ also through the estimates $\hat{\vec{\theta}}$ and $\hat{\vec{\Phi}}_{\hat{\theta}}$ (see /10/,/12/, and/16/). The statistical properties of $r$ are very simple as we shall show in the sequel, though this dependence is nonlinear.

## 3. The Probability of the Type I Error and the Optimality of the $r$-Test

A basis meaning has the following

Theorem: If $\mathcal{H}_0$ is true, then $r$ is distributed as a normal variable with parameters $\mathcal{E}\{r\} = 0$, $\mathcal{D}\{r\} = 1$, i.e.

$$p\{r > k\} = \frac{1}{\sqrt{2\pi}} \int_k^\infty e^{-\frac{x^2}{2}} dx \tag{20}$$

under the hypothesis $\mathcal{H}_0$.

Proof: Assuming $\mathcal{H}_0$ is true we denote by $\vec{\theta}_0$ the true (unknown) value of the parameters. We obtain from (10):

$$\hat{\vec{\theta}} - \vec{\theta}_0 = [F'F]^{-1} F'(\vec{y} - \vec{\eta}(\vec{\theta}_0)). \tag{21}$$

Multiplying (21) by $F$ and using (4) we have

$$\vec{\eta}(\hat{\vec{\theta}}) - \vec{\eta}(\vec{\theta}_0) = F[F'F]^{-1} F'(\vec{y} - \vec{\eta}(\vec{\theta}_0)). \tag{22}$$

By subtraction of $\vec{y}$ from (22) we get

$$\vec{y} - \vec{\eta}(\hat{\vec{\theta}}) = (I - C)(\vec{y} - \vec{\eta}(\vec{\theta}_0)) = R(\vec{y} - \vec{\eta}(\vec{\theta}_0)), \tag{23}$$



where

$$C = F[F'F]^{-1} F', \quad R = I - C. \tag{24}$$

From (24) we deduce the following useful rules:

$$CC = C, \quad C' = C, \quad RR = R, \quad R' = R \qquad CR = RC = 0. \tag{25}$$

Following the assumption that $\mathcal{H}_0$ is true

$$\mathcal{E}\{\vec{y}\} = \vec{\eta}(\vec{\theta}_0)$$

and the variable

$$\vec{z} = \vec{y} - \vec{\eta}(\vec{\theta}_0)$$

is distributed as an $n$-dimensional normal variable with parameters

$$\mathcal{E}\{\vec{z}\} = 0, \qquad \mathcal{D}\{\vec{z}\} = I.$$

In an analogical way, as we obtained (23) from (10), we can get from (16) and (22) the expession

$$\vec{\nu}(\hat{\vec{\Phi}}_{\hat{\theta}}) - \vec{\eta}(\hat{\vec{\theta}}) = S[\vec{b} - C\vec{z}]. \tag{26}$$

Here

$$S = I - G[G'G]^{-1} G' \qquad S' = S, \quad SS = S \tag{27}$$

and $\vec{b}$ is a constant (non randon) vector

$$\vec{b} = \vec{\nu}(\hat{\vec{\Phi}}_{\theta_0}) - \vec{\eta}(\vec{\theta}_0). \tag{28}$$

The (constant) vector $\hat{\vec{\Phi}}_{\theta_0}$ is derived from (15) substituting $\vec{\theta}_0$ instead of $\hat{\vec{\theta}}$.

Substituting (23) and (26) into the definition of $r$ (formula (18)) we obtain

$$r = \frac{\vec{z}\,' RS[\vec{b} - C\vec{z}\,]}{\sqrt{[\vec{b} - C\vec{z}\,]' SRS[\vec{b} - C\vec{z}\,]}} \qquad (29)$$

Let us compute the $k$-th moment of $r$ : $\mu_k = \mathcal{E}\{r^k\}$.

For this purpose define two random vectors

$$\vec{u} = C\vec{z}$$
$$\vec{v} = SR\vec{z}\,. \qquad (30)$$

In this notation

$$\mu_k = \mathcal{E}\left\{ \frac{[\vec{v}\,'(\vec{b} - \vec{u}\,)]^k}{[(\vec{b} - \vec{u}\,)' SRS(\vec{b} - \vec{u}\,)]^{k/2}} \right\}. \qquad (31)$$

Being linear functions of $\vec{z}$, the variables $\vec{u}$ and $\vec{v}$ are also normally distributed. Moreover, $u_i$ and $v_j$ are independent variables $(i = 1, \dots, n,\ j = 1, \dots, n)$. Indeed,

$$\mathrm{cov}\{u_i, v_j\} = \mathcal{E}\left\{ \sum_{k=1}^{n} C_{ik} z_k \sum_{\ell,h=1}^{n} S_{j\ell} R_{\ell h} z_h \right\} =$$

$$= \sum_{k,\ell,h} C_{ik} S_{j\ell} R_{\ell h} \delta_{hk} = \sum_{k,\ell} S_{j\ell} R_{\ell k} C_{ki} = 0 \qquad (32)$$

because $C_{ik} = C_{ki}$ and $\sum_k R_{\ell k} C_{ki} = 0$ (see (25)).

From (31) and from the independence of $\vec{w}$ and $\vec{v}$ we obtain

$$\mu_k = \sum_{j_1,\dots,j_k=1}^{n} \mathcal{E}\left\{ \frac{(b_{j_1} - u_{j_1}) \dots (b_{j_k} - u_{j_k})}{([\vec{b} - \vec{u}\,]' SRS[\vec{b} - \vec{u}\,])^{k/2}} \right\} \times \mathcal{E}\{v_{j_1} \dots v_{j_k}\} \qquad (33)$$



The following equality is valid:

$$\mathcal{E}\{v_{j_1} \dots v_{j_k}\} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum\limits_{\text{pairs}} \left(\sum\limits_{\ell h} S_{j_1 \ell} R_{\ell h} S_{h j_2} \dots \sum\limits_{\ell h} S_{j_{k-1} \ell} R_{\ell h} S_{h j_k}\right) & \end{cases} \qquad (34)$$

if $k$ is even, where the first sum is taken over all possible orderings of the indices $j_1, \dots, j_k$ into pairs.

To prove (34) we shall use the characteristic function $\phi(\vec{t}\,)$ of the multidimensional normal variable $\vec{v}$ /7/. We have

$$\mathcal{E}\{\vec{v}\} = 0, \qquad \mathfrak{D}\{\vec{v}\} = \mathfrak{D}\{SR\vec{z}\} = SRS,$$

and /7/

$$\phi(\vec{t}\,) = \exp\left\{ -\frac{1}{2} \vec{t}\,' SRS\vec{t} \right\}.$$

The $k$-th mixed moment of $\vec{v}$ is obtained from the relation /8/:

$$\mathcal{E}\{v_{j_1} \dots v_{j_k}\} = \frac{1}{i^k} \frac{\partial^k \phi(\vec{t}\,)}{\partial t_{j_1} \dots \partial t_{j_k}} \bigg|_{\vec{t} = 0}. \qquad (35)$$

Computing (35) we obtain (34) and substituting into (33) we get for $k$ even

$$\mu_k = \sum_{\text{pairs}} \mathcal{E}\left\{ \frac{(\vec{b} - \vec{u}\,)' SRS(\vec{b} - \vec{u}\,)}{(\vec{b} - \vec{u}\,)' SRS(\vec{b} - \vec{u}\,)} \right\}^{k/2}.$$

There are $(k-1)!!$ possibilities of ordering $k$ indices into pairs. Thus we get

$$\mu_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (k-1)!! = 1.3.5 \dots (k-1) & \text{if } k \text{ is even} \end{cases} \qquad (36)$$

what coincides with the moments of a one-dimensional normal variable with parameter 0 and I /7/. The theorem is proved.

The Type I error. To a given $\alpha$ we choose $k = k(\alpha)$ from the equation

$$\int_k^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \alpha . \quad (37)$$

If the experimental value

$$r_{ex} > \alpha$$

we can reject $H_0$ with a probability of error equal to $\alpha$ .

The optimality. If $n \to \infty$ , then $\hat{\vec{\theta}} \to \vec{\theta}_0$ and $\hat{\vec{\Phi}}_{\hat{\theta}} \to \hat{\vec{\Phi}}_{\theta_0}$ (convergence in probability /7/ ). We have thus two fixed vectors

$$\vec{\eta}_I = \vec{\eta}(\vec{\theta}_0) , \quad \vec{\eta}_{II} = \vec{\eta}(\hat{\vec{\Phi}}_{\theta_0}) \quad (38)$$

and the testing problem consists of testing $\vec{\eta}_I$ against $\vec{\eta}_{II}$ .

If $n \to \infty$ , then

$$R \to I . \quad (39)$$

Proof: Taking the limit of (23) we get

$$\vec{y} - \vec{\eta}(\vec{\theta}_0) \to R(\vec{y} - \vec{\eta}(\vec{\theta}_0))$$

for every $\vec{y}$ , i.e. $R \to I$ .

If $n \to \infty$ then $r$ converges (in probability) to

$$r_{as} = \frac{[\vec{y} - \vec{\eta}_I]'[\vec{\eta}_{II} - \vec{\eta}_I]}{\sqrt{[\vec{\eta}_{II} - \vec{\eta}_I]'[\vec{\eta}_{II} - \vec{\eta}_I]}} . \quad (40)$$

When multiplying $r_{as}$ by a constant number or subtracting a constant number from $r_{as}$ the test based on $r_{as}$ remains invariant. So we can obtain from (40) an identical test based on the variable



$$2|\vec{\eta}_{II} - \vec{\eta}_I| r_{as} - (|\vec{\eta}_{II} - \vec{\eta}_I|)^2 =$$
$$= (\vec{y} - \vec{\eta}_I)'(\vec{y} - \vec{\eta}_I) - (\vec{y} - \vec{\eta}_{II})'(\vec{y} - \vec{\eta}_{II}) . \quad (41)$$

But (41) is the likelihood ratio for simple hypothesis and the test based on it (the Neyman - Pearsen test) is the optimal test /1/, i.e. the test with the smallest probability of Type II error $\beta$ . Thus

a) the $r$ -test is optimal in the sence defined in paragraph I.

b) for large $n$ the $r$ -test and the likelihood ratio test (see (14) ) coincide.

## 4. The Phase Shift Analysis

The described method of testing can be advantageously used in the phase-shift analysis, if there is an ambiguity of the phase shifts. Still there is a slight difference between the formulation of the problem in the paragraph I and in the phase-shift analysis.

1. Let be $y_1, \dots, y_n$ the experimental data (the results of measuring the cross section, the polarization etc.) and let be $\eta_1(\vec{\theta}), \dots, \eta_n(\vec{\theta})$ the theoretical values of $y_1, \dots y_n$ depending on the phase shifts $\theta_1, \dots, \theta_m$ . The functions $\eta_i(\vec{\theta})$ are substantially nonlinear and the amount of performed experiments, though large, is not sufficient to obtain a single solution of the maximal likelihood method. Say, there are two solutions, two sets of phase-shifts, $\hat{\vec{\theta}}$ and $\hat{\vec{\Phi}}$ . On the other hand, the amount of performed experiments is usually sufficient to suppose that $y_i$ are normally distributed and that in the neighbourhood of $\hat{\vec{\theta}}$ and $\hat{\vec{\Phi}}$ (in the ellipsoid of concentration /7/ around $\hat{\vec{\theta}}$ and $\hat{\vec{\Phi}}$ ), the functions $\eta_i(\vec{\theta})$ are approximatively linear (Taylor's formula):

$$\eta_i(\vec{\theta}) = \sum_{k=1}^m F_{ik}(\theta_k - \hat{\theta}_k) + \eta_i(\hat{\vec{\theta}})$$

and

$$\eta_i(\vec{\theta}) = \sum_{k=1}^{m} G_{ik}(\theta_k - \hat{\Phi}_k) + \eta_i(\vec{\hat{\Phi}}), \qquad (42)$$

where

$$F_{ik} = \frac{\partial \eta_i(\vec{\theta})}{\partial \theta_k}\Big|_{\vec{\theta} = \vec{\hat{\theta}}}, \quad G_{ik} = \frac{\partial \eta_i(\vec{\theta})}{\partial \theta_k}\Big|_{\vec{\theta} = \vec{\hat{\Phi}}}. \qquad (43)$$

The expression (42) coincides with (4) and (5) if we denote $\vec{\eta}(\vec{\theta})$ by $\vec{\nu}(\vec{\theta})$ in the neighbourhood of $\vec{\hat{\Phi}}$.

The assumption of quasilinearity (42) is needed if using the $\chi^2$-test or the likelihood ratio test/3/ as well.

2. The quantity $r$ is computed in the following way. Let be $\vec{\hat{\Phi}}$ the better of the two sets of phase shifts, i.e.

$$\chi^2_{\hat{\theta}} > \chi^2_{\hat{\Phi}}.$$

An auxiliary set of phase shifts $\vec{\hat{\Phi}}_{\hat{\theta}}$ is computed from the least square program/9/ substituting in it the $\eta_1(\vec{\hat{\theta}}), \dots, \eta_n(\vec{\hat{\theta}})$ instead of $y_1, \dots, y_n$ and performing only the first iterative step of the program. Then

$$r_{ex} = \frac{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}[y_i - \eta_i(\vec{\hat{\theta}})]\, x_i}{\sqrt{\sum_{i=1}^{n} \frac{x_i^2}{\sigma_i^2} - \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{m} \frac{x_i}{\sigma_i^2} F_{ik} D_{k\ell} F_{j\ell} \frac{x_j}{\sigma_j^2}}}, \qquad (44)$$

where

$\sigma_i^2$ is the variance of $y_i$,

$$x_i = \eta_i(\vec{\hat{\Phi}}_{\hat{\theta}}) - \eta_i(\vec{\hat{\theta}})$$

and $D$ is the error matrix of $\vec{\hat{\theta}}$. If $r_{ex} > k$ where $k$ is defined from the tables of normal distribution as

$$\alpha = \int_k^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$



then the phase shift set $\vec{\hat{\theta}}$ can be rejected with a probability of error equal to $\alpha$.

3. We note that the probalility of Type I error is even smaller than the computed one, because we test every time only the worse set $\vec{\hat{\theta}}$ ($\chi^2_{\hat{\theta}} > \chi^2_{\hat{\Phi}}$) and

$$p\{r > k \text{ and } \chi^2_{\hat{\theta}} > \chi^2_{\hat{\Phi}}\} \le p\{r > k\} = \alpha.$$

But it can be shown that for small value of $\alpha$ the difference is negligible.

4. The meaning of the Type II error is obvious. $\beta$ is the probability that we make superfluous experiments. The optimal test is, in certain sence, the most economical test.

## 5. Generalizations

The quantity $r$ in (18) depends only on some scalar products in an $n$-dimensional vector space. Evidently the $r$-test can be used also if the experimental results $y_1, \dots, y_n$ are mutually dependent, or if a continuous function of time $y(t)$ is measured ($y(t)$ being an element of a Hilbert space).

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