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ЛАБОРАТОРИЯ ВЫЧИСЛИТЕЛЬНОЙ ТЕХНИКИ
И АВТОМАТИЗАЦИИ

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ON THE SPECTRUM OF A SINGULAR INTEGRAL OPERATOR

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Let Ω be a bounded domain in the n-dimensional Euclidean space with a sufficiently smooth boundary $\partial\Omega$. By $G(x;y)$ $x=(x_1 \dots x_n)$, $y=(y_1 \dots y_n)$ is denoted the Green function of the problem

$$\begin{aligned} \Delta u &= f \\ u/\partial\Omega &= 0 \end{aligned} \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \dots\dots + \frac{\partial^2}{\partial x_n^2}.$$

In this paper we shall study the spectrum of the operator

$$Tu = \int_\Omega G(x,y) \frac{\partial^2 u}{\partial y_1^2} dy. \tag{1}$$

Our problem is coming from the investigation of S.L. Sobolev /1/ on the movement of a body with cavity filled by perfect liquid. It has been considered already in /2,3/.

§1. $H(\Omega)$ are, as usual, the Sobolev's spaces which consist of all the functions having square summable generalized derivatives of order S in Ω. Let us denote by $\overset{0}{H}_2(\Omega)$ the set of functions of $H_2(\Omega)$ vanishing on the boundary $\partial\Omega$.

Expression (1) defines a linear operator T mapping continuously $\overset{0}{H}_2(\Omega)$ into itself. One can easily see that (1) may be written in the following form

$$Tu = \int_\Omega \frac{\partial^2 G}{\partial y_1^2} u(y)\,dy , \qquad u \in \overset{0}{H}_2(\Omega) \tag{2}$$

We used the fact that $u(y)$ vanishes on the boundary. Now it is clear from (2) that T is a singular integral operator.

Let us remember on some definitions [4,5]:

Suppose A is a closed operator in a Banach space. A is called an <u>operator of Fredholm's type</u> if the following conditions are fulfilled:

1. The dimension of $\operatorname{Ker} A$, the latter being the set of zeros of A, is finite.

2. $\operatorname{Im} A$, i.e. the range of A, is closed.

3. The codimension of $\operatorname{Im} A$ is finite.

<u>The essential spectrum</u> of A is defined as the set of points λ on the complex plane for which $A-\lambda$ is not an operator of Fredholm's type.

<u>*Theorem 1. The spectrum of the operator T is identical to its essential spectrum both coinciding with the interval [0,1].*</u>

<u>Proof.</u> First of all there is no point of the spectrum besides of the interval [0,1]. Indeed, let for instance $\lambda < 0$. We perform the following transformation

$$x_1' = \frac{1}{\sqrt{1-\lambda}} x_1 , \quad x_2' = \frac{1}{\sqrt{-\lambda}} x_2 , \ldots , x_n' = \frac{1}{\sqrt{-\lambda}} x_n . \tag{3}$$

In this way Ω is transformed into a new domain Ω_λ, whereas the function $u(x_1, \ldots, x_n) \in H_2(\Omega)$ into $\hat{u}_\lambda(x_1', \ldots, x_n') \in H_2(\Omega_\lambda)$.
We denote by θ_λ the operator transforming u into $\hat{u}_\lambda$.

Let

$$v = (T-\lambda)u , \quad u, v \in \overset{0}{H}_2(\Omega).$$

Applying to both sides the operator Δ, we obtain

$$\Delta v = \frac{\partial^2 u}{\partial x_1^2} - \lambda \Delta u .$$

By means of (3) the last equation is transformed into

$$\frac{1}{\lambda(1-\lambda)} \hat{v}_\lambda - \frac{1}{\lambda} \Delta \hat{v}_\lambda = \Lambda \hat{u}_\lambda .$$

Inverting the operator Δ one obtains

$$\hat{u}_\lambda = \theta_\lambda (T-\lambda)^{-1}\theta_\lambda^{-1}\hat{v}_\lambda = \frac{1}{\lambda(1-\lambda)}\int_{\Omega_\lambda} G_\lambda(x',y')\frac{\partial^2 \hat{v}_\lambda}{\partial y_1'^2}dy' - \frac{1}{\lambda}\hat{v}_\lambda , \qquad (4)$$

where $G_\lambda(x',y')$ is the Green function of the Dirichlet's problem for the Laplace's operator in Ω_λ. From (4) it can be seen that $(T-\lambda)^{-1}$ is bounded. Hence λ don't belong to the spectrum.

Now we shall prove that every point $\lambda \in [0,1]$ belongs to the essential spectrum of T. For this purpose we consider the boundary problem

$$\frac{\partial^2 u}{\partial x_1^2} - \lambda \Delta u = F$$
$$u/\partial\Omega = f \qquad (5)$$

and define the operator $\mathcal{P}_\lambda$ mapping $H_2^0(\Omega)$ into $H_0(\Omega)$

$$\mathcal{P}_\lambda u = \frac{\partial^2 u}{\partial x_1^2} - \lambda\Delta u , \qquad u \in H_2^0(\Omega).$$

It is easy to see that problem (5) is not an elliptical one/6/, hence the operator $\mathcal{P}_\lambda$ is not of Fredholm's type (ellipticity is necessary for $\mathcal{P}_\lambda$ to be of Fredholm's type/7/). Moreover from this follows that $T-\lambda$ is also not of Fredholm's type. Let us prove for instance that if $\mathrm{Im}\,\mathcal{P}_\lambda$ is not closed then $\mathrm{Im}(T-\lambda)$ is not closed too. Indeed, let $f_n \in \mathrm{Im}\,\mathcal{P}_\lambda$, $f_n \xrightarrow{H_0} f$. We denote

$$\phi_n = \Delta^{-1} f_n = \int_\Omega G(x,y) f_n(y)dy .$$

Then $\phi_n \in \mathrm{Im}(T-\lambda) \subset H_2^0$ and $\phi_n \xrightarrow{H_2^0} \Delta^{-1} f = \phi$ . . As $\mathrm{Im}(T-\lambda)$ is closed, so $\phi \in \mathrm{Im}(T-\lambda)$. Hence

$$f = \Delta\phi \in \Delta\,\mathrm{Im}(T-\lambda) = \mathrm{Im}\,\Delta(T-\lambda) = \mathrm{Im}\,\mathcal{P}_\lambda$$

and this was to be proved.

Thus every point of the interval [0,1] belongs to the essential spectrum of T.

§2. In §1 we saw that the essential spectrum of the operator T does not depend on the domain Ω. The more detailed structure of the spectrum and particularly the existence of eigenvalues, depend on the domain. It is proved in /2,3/ that if Ω is an ellipsoid or a cylinder, the spectrum consists of an everywhere dense set of eigenvalues of infinite multiplicity. Now we are going to costruct some domains for which the operator has not eigenvalues at least on some subinterval of $[0,1]$.

Suppose that for some $0 < \lambda < 1$ and some $u \in H^0_2(\Omega)$

$$Tu - \lambda u = 0 .$$

Then it follows

$$\frac{\partial^2 u}{\partial x_1^2} - \lambda \Delta U = 0 \qquad u/\partial\Omega = 0 . \tag{6}$$

Let us perform the substitution

$$x_1' = \frac{1}{\sqrt{2(1-\lambda}} x_1 , \quad x_2' = \frac{1}{\sqrt{2\lambda}} x_2 , \ldots , x_n' = \frac{1}{\sqrt{2\lambda}} x_n . \tag{7}$$

In this way the function $u(x_1, \ldots x_n)$ is transformed into $\hat{u}_\lambda(x_1', \ldots x_n')$, the domain Ω into a new Ω_λ and (6) into

$$\frac{\partial^2 \hat{u}_\lambda}{\partial x_1'^2} - \frac{\partial^2 \hat{u}_\lambda}{\partial x_2'^2} - \ldots\ldots - \frac{\partial^2 \hat{u}_\lambda}{\partial x_n'^2} = 0, \quad u_\lambda/\partial\Omega_\lambda = 0 . \tag{8}$$

Thus $\hat{u}_\lambda$ satisfies the wave equation and vanishes on the boundary of Ω_λ. In order to prove that λ is not an eigenvalue it is sufficient to prove that from (8) $\hat{u}_\lambda \equiv 0$ follows. This latter means the uniqueness of the solution of the Dirichlet's problem for the wave equation in $H_2(\Omega)$, called in what follows shortly "uniqueness".

First of all we shall consider the two-dimensional case, moreover let Ω be a triangle. The domain Ω_λ is also a triangle for every $\lambda \in [0, 1]$. We shall show the uniqueness for arbitrary triangle, from which follows that T has no eigenvalues.

So, let Ω be an arbitrary triangle. At least one of the angles of Ω must lie inside of a characteristic angle (an angle determined by two characteristics of the wave equation). Indeed, let us draw the three straight lines parallel to the sides of the triangle through the origin of the coordinate system (Fig.1.). As there are two characteristic angles whereas three straight lines, at least two of these latters must be inside of one of the characteristic angles.

Let P be an arbitrary point in the triangle. We draw through P the characteristics as shown in Fig.2. Suppose now $u \in H^0_2$ satisfies the wave equation. Then the following relations hold

$$u(P) + u(P_1) = u(A_1) + u(B_1)$$
$$u(P_1) + u(P_2) = u(A_2) + u(B_2) \qquad (9)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u(P_{n-1}) + u(P_n) = u(A_n) + u(B_n) .$$

The right-hand sides of (9) are equal to zero because u vanishes on the boundary of Ω. Let us multiply the first equation of (9) by +1, the second one by −1, the third one again by +1, and so on and add them together. We then obtain

$$u(P) + (-1)^{n-1} u(P_n) = 0 ; \qquad (10)$$

Obviously P_n converges to A and as u is continuos by virtue of the Sobolev's Imbedding Theorem from (10) follows $u(P) = 0$. Thus the uniqueness is proved.

We can proceed in a similar way in the three dimensional case. Let us consider the cone Ω enclosed by the surfaces (Fig.3)

$$\omega_1 : x_1^2 - (1 + a^2)(x_2^2 + x_3^2) = 0$$
$$\omega_2 : x_1 = h ,$$

where a and h are arbitrary constants. Ω lies inside of the characteristic cone of the wave equation

$$x_1^2 - x_2^2 - x_3^2 = 0 .$$

We shall prove the uniqueness for such arbitrary cones.

Let P be a point of Ω. Let this point coincide with the vertex of a characteristic cone. We denote by ω_1' the part of ω_1 cut off by the characteristic cone and reversely by σ the part of this latter cut off by ω_1 (Fig.3). Then the following relations are fulfilled:

$$\nu_1^2 - \nu_2^2 - \nu_3^2 = 0 \qquad \sigma$$

$$\nu_1^2 - \frac{1}{1+a^2}(\nu_2^2 + \nu_3^2) = 0 \qquad \omega_1',$$

where ν_k are the direction cosines of the normal of the surfaces in question.

Next we use the identity

$$0 = 2\frac{\partial u}{\partial x_1}\Box u = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right)^2 + \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_3}\right)^2 + \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)^2 -$$

$$-2\frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right) - 2\frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}\right).$$

Integrating over the domain Ω' enclosed by σ and ω_1' one obtains

$$0 = \int_{\Omega'} 2\frac{\partial u}{\partial x_3}\Box u \, d\Omega = \int_{\omega_1'+\sigma} \left\{\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right]\nu_1 - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\nu_1 -\right.$$

$$\left. - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}\right\} d\omega = \int_\sigma \frac{1}{\nu_1}\left\{\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right]\nu_1^2 + \left(\frac{\partial u}{\partial x_1}\right)^2(\nu_2^2+\nu_3^2) -\right.$$

$$\left. - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\nu_1\nu_2 - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}\nu_1\nu_3\right\} d\omega + \int_{\omega_1'} \frac{1}{\nu_1}\left\{\left[\left(\frac{\partial u}{\partial x_1}\right)^2 +\right.\right.$$

$$\left.\left. + \left(\frac{\partial u}{\partial x_3}\right)^2\right]\nu_1^2 + \left(\frac{\partial u}{\partial x_1}\right)^2(\nu_2^2+\nu_3^2) - \beta^2\left(\frac{\partial u}{\partial x_1}\right)^2(\nu_2^2+\nu_3^2) - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\nu_1\nu_2 -\right.$$

$$\left. -2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}\nu_1\nu_3\right\} d\omega =$$

$$= \int_{\sigma} \frac{1}{\nu_1} \left[\left(\frac{\partial u}{\partial x_2} \nu_1 - \frac{\partial u}{\partial x_1} \nu_2 \right)^2 + \left(\frac{\partial u}{\partial x_3} \nu_1 - \frac{\partial u}{\partial x_1} \nu_3 \right)^2 \right] d\omega$$

$$+ \beta^2 \int_{\omega'_1} \frac{1}{\nu_1} \left(\frac{\partial u}{\partial x_1} \right)^2 (\nu_2^2 + \nu_3^2) \, d\omega$$

$$+ \int_{\omega'_1} \frac{1}{\nu_1} \left[\left(\frac{\partial u}{\partial x_2} \nu_1 - \frac{\partial u}{\partial x_1} \nu_2 \right)^2 + \left(\frac{\partial u}{\partial x_3} \nu_1 - \frac{\partial u}{\partial x_1} \nu_3 \right)^2 \right] d\omega .$$

We used the notation

$$\frac{1}{1 + \alpha^2} = 1 - \beta^2$$

Since $u \in H_2(\Omega)$, all integrals exist in these formulae. The last **integral** vanishes because of the presence of the tangential derivatives of u on ω' in the integrand. Since Ω lies inside of the characteristic cone, $\nu_1 > 0$ holds on σ and $\nu_1 < 0$ on ω'_1. From this follows that

$$\int_{\sigma} \frac{1}{\nu_1} \left[\left(\frac{\partial u}{\partial x_2} \nu_1 - \frac{\partial u}{\partial x_1} \nu_2 \right)^2 + \left(\frac{\partial u}{\partial x_3} \nu_1 - \frac{\partial u}{\partial x_1} \nu_3 \right)^2 \right] d\omega = 0$$

thus both of the linear independent inner derivatives $\frac{\partial u}{\partial x_2} \nu_1 - \frac{\partial u}{\partial x_1} \nu_2$ and $\frac{\partial u}{\partial x_3} \nu_1 - \frac{\partial u}{\partial x_1} \nu_3$ are equal to zero on σ. Therefore the function u is a constant on σ and as it vanishes along the intersection of σ and ω_1, it is zero on the entire σ. Hence $u(P) = 0$ and the uniqueness is proved.

By means of substitution (7) Ω is transformed into a cone Ω_λ determined by the surface

$$x'^2_1 - (1 + \alpha^2) \frac{1 - \lambda}{\lambda} (x'^2_2 + x'^2_3) = 0.$$

Obviously Ω_λ lies inside of the characteristic cone if and only if

$$0 \leqq \lambda \leqq \frac{1+\alpha^2}{2+\alpha^2} \tag{11}$$

consequently there are no eigenvalues in the interval (11).

Yu.M. Berezansky [8] proposed a method to construct some domains where the problem of Dirichlet for wave equations has a weak solution, the solvability being stable against little deformations of the domain. It is easy to see that for such domain the spectrum of T contains interval without eigenvalues. In what follows we describe briefly this method applying to our case.

Let us introduce the notations

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \dots - \frac{\partial^2 u}{\partial x_n^2} = \sum_{j=1}^{n} c_j \frac{\partial^2 u}{\partial x_j^2}, \quad c_1 = 1,\ c_2 = \dots c_n = -1.$$

Suppose that $u \in H_2(\Omega)$ satisfies the equation $\mathcal{L}u = 0$ and vanishes on the boundary. Let $A_1(x), \dots, A_n(x)$, $A(x)$ be real sufficiently smooth functions. Integrating by part the following identities may be proved [8]:

$$0 = \int_\Omega \mathcal{L}u \left(\sum_{k=1}^{n} A_k \frac{\partial u}{\partial x_k} + Au \right) dx =$$

$$= \int_\Omega \left\{ \sum_{j=1}^{n} c_j \left(\sum_{k=1}^{n} \frac{\partial A_k}{\partial x_k} - 2\frac{\partial A_j}{\partial x_j} - 2A \right) \left(\frac{\partial u}{\partial x_j} \right)^2 + \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \left(-c_j \frac{\partial A_k}{\partial x_j} - c_k \frac{\partial A_j}{\partial x_k} \right) \times \right.$$

$$\left. \times \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial k_k} \right\} dx + \int_\Omega \left(\sum_{j=1}^{n} c_j \frac{\partial^2 A}{\partial x_j^2} \right) u^2 dx + \int_{\partial\Omega} N_u^2 \sum_{k=1}^{n} A_k \nu_k \sum_{j=1}^{n} c_j \nu_j^2 \tag{12}$$

where ν_k are the direction cosines of the surface $\partial\Omega$, N_u, is a function such that $\frac{\partial u}{\partial x_j} = N_u \nu_j$ holds on $\partial\Omega$. /As $u/\partial\Omega = 0$, $N_u(x)$ exists/. Let us choose the functions $A_k(x)$ and $A(x)$ so that the quadratic form

$$\sum_{j=1}^{n} c_j \left(\sum_{k=1}^{n} \frac{\partial A_k}{\partial x_k} - 2\frac{\partial A_j}{\partial x_j} - 2A\right)\left(\frac{\partial u}{\partial x_j}\right)^2 + \sum_{j,k=1}^{n} \left(-c_j \frac{\partial A_k}{\partial x_j} - c_k \frac{\partial A_j}{\partial x_k}\right)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k} \qquad (13)$$

is strictly positively definite and the inequality

$$\sum_{j=1}^{n} c_j \frac{\partial^2 A}{\partial x_j^2} \geqq 0 \qquad (14)$$

is fulfilled. Furthermore we determine the domain Ω so that on $\partial\Omega$

$$\sum_{k=1}^{n} A_k \nu_k \sum_{j=1}^{n} c_j \nu_j^2 \geqq 0. \qquad (15)$$

As a consequence of (12–15) for such a domain $u \equiv 0$ holds. Moreover if little alteration of Ω and $A_k(x)$ are made the positive definitness of (13) and the inequaties (14), (15) keep true so consequentely the uniqueness remains valid.

As seen from (7) $\Omega_\lambda = \Omega$ when $\lambda = \frac{1}{2}$, so when λ lies in a sufficiently small neighbourhood of $\frac{1}{2}$, Ω_λ differs little from Ω and the uniqueness holds. This means that there are no eigenvalues in a sufficiently small neighbourhood of $\lambda = \frac{1}{2}$.

As an example let us now describe a domain constructed in /8/ in the way we have explained. Take $A_1 = -x_1$, $A_k = x_k$ $(k = 2, \dots, n)$. Consider the four branches of hyperboles $x_1 x_2 = h$ on the $x_1 0 x_2$ plane (Fig.4) and close them with curves AA_1, AA_2, BB_1 .... These latters can be arbitrary, the only restriction on them is the following. Place a hyperbola $x_1 x_2 = h$ through an arbitrary point M of these curves. We then obtain two angles α or β, both determined by the coordinate axe (see Fig.4) and by the tangent of the curve or by that of the hyperbola, respectively. α and β obey the relation $\beta < \alpha < \frac{\pi}{4}$.

In this way one receives a bounded domain $\tilde{\Omega}_1$ on the $x_1 0 x_2$ plane. Consider now a cylinder $\Omega_1 = \tilde{\Omega}_1 \times E_{n-2}$, where E_{n-2} means an $n-2$ dimensional space built up of all coordinate axes with the exception of ox_1 and ox_2. Similarly, we construct $\tilde{\Omega}_2$ on the $x_1 ox_3$ plane with the corresponding cylinder Ω_2, etc. Let Ω be the intersection of all these cylinders constructed, Ω is bounded, its boundary consists of smooth surfaces of finite number. The boundness of Ω follows from the fact that x_1 and x_2 are bounded in Ω_1, x_1 and x_3 are bounded in Ω_2 etc. Choosing the constant h in the equations of the hyperbola sufficiently small in modulus and the points A, B, C, D sufficiently near to 0 we receive a domain for which the quadratic formula (13) is strictly positive definite and inequalities (14, 15) are valid, consequently the uniqueness holds. At the same time this property remains unchanged in case of small deformations of the domain chosen. As a consequence, for this domain the spectrum has no eigenvalues in some neighbourhood of the point $\lambda = \frac{1}{2}$.

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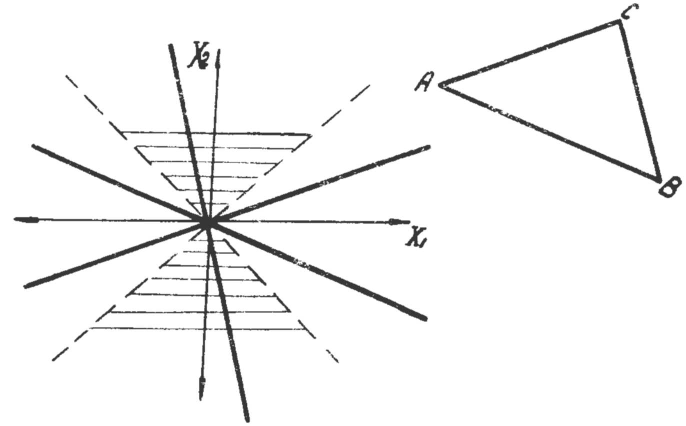

Fig 1.

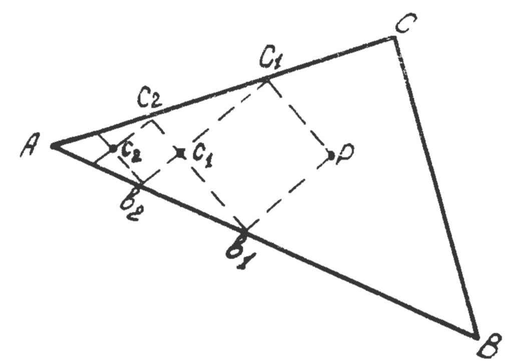

Fig 2.

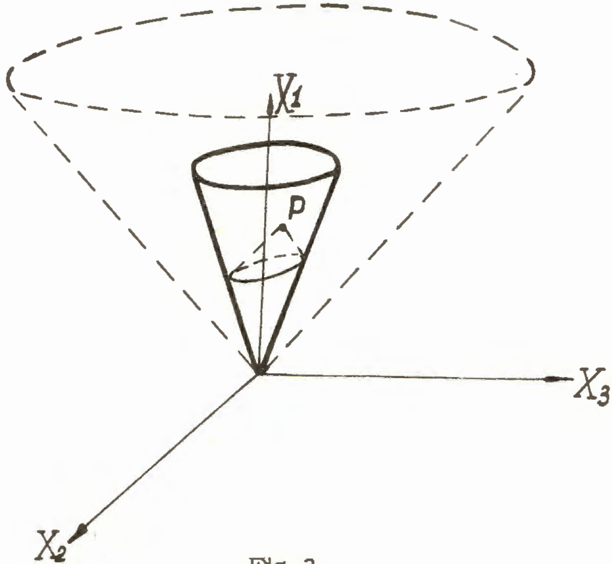

Fig 3.

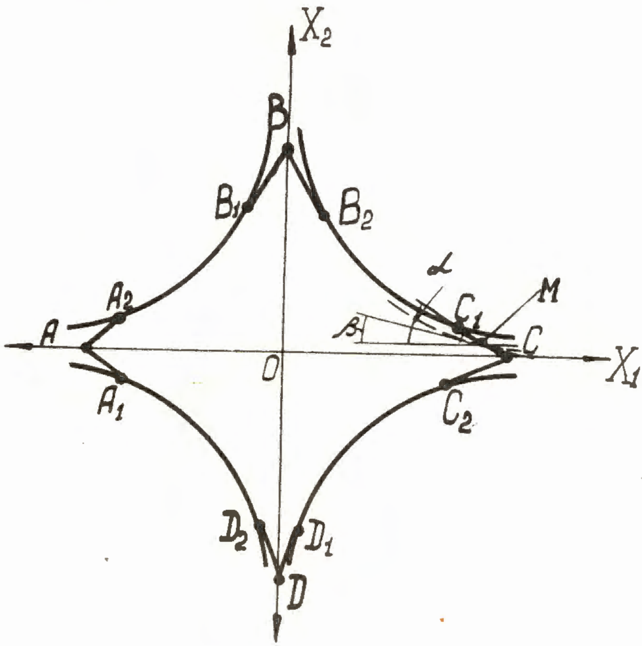

Fig 4.