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THE q-BOSON-FERMION REALIZATIONS OF THE QUANTUM SUPERALGEBRA $U_q(\mathrm{gl}(m/n))$

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1 Introduction

Quantum groups or q-deformed Lie algebras imply some specific deformations of classical Lie algebras. From a mathematical point of view, it is a noncommutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively staring from works by Jimbo [1] and Drinfeld, [2] and many authors.

In the course of studying quantum algebras, a lot of attention has been paid to the case of quantum superalgebras [3], [4]. These algebras provide solutions to the Yang-Baxter equations and there may serve as a source of new exactly solvable models in statistical mechanics [5]. The applications in field theory are connected with the WZW models [6] and we can find some application of quantum superalgebra $U_q(\mathrm{gl}(m/n))$ for construction of the Alexander-Conway polynomial in the knot theory [7].

The Weyl and Clifford algebras also admit quantum deformation [8] with q-analogues of the Bose, and respectively, Fermi oscillator operators as generators [8, 9, 10]. These quantized algebras have been used to construct oscillator realizations of the quantum algebras that correspond to all classical Lie algebras [8]. These realizations are of the Jordan–Schwinger type [11]. The realizations of this type for quantum superalgebras were constructed in [12]. In the literature there exist Dyson [13] and Holstein-Primakoff [14] realizations which were first written for the algebra sl(2).

The first "quantum" version of Holstein-Primakoff was worked out for $U_q(\mathrm{sl}(2))$ [15] and then for $U_q((\mathrm{sl}(3))$ [16]. These realizations found immediate applications [17-22].

In our paper [23] we formulated the method starting from the Verma modules for obtaining boson realizations and in [24] we obtained explicitly a braid class of realizations which generalized the results from [25, 26].

Later the idea of boson–fermion realizations was extended to the Lie superalgebra, and the Dyson type boson-fermion realizations were explicitly given in [27], generalizing the results to sl(2/1) ([28], [29]).

In our papers [30, 31, 32] we studied the Dyson realizations of the series algebras $U_q(\mathrm{sl}(2))$, $U_q(\mathrm{gl}(n))$, $U_q(B_n)$, $U_q(C_n)$ and $U_q(D_n)$. There is some special case [31] for which the realization of the subalgebra $U_q(\mathrm{gl}(n-1))$ in the recurrence is trivial. Such special realizations of the quantum algebra $U_q(\mathrm{sl}(n))$ of Dyson type were studied in [33].

The aim of the present paper is to show that it is possible to generalize our method [23] to deriving the boson fermion realization, too. This will be exemplified by the quantum superalgebra $U_q(\mathrm{gl}(m/n))$. This superalgebra can be applied to physical problems such as strongly correlated electron systems [34, 35, 36]. We explicitly see the recurrence with respect to $U_q(\mathrm{gl}(m-1/n))$ and consequently we will show that again it is a generalization of the result from [37].

2 Preliminaries

We will use the definition of the quantum superalgebra $U_q(\mathrm{gl}(m/n))$ which can be found in [37].

Let q be an independent variable, $\mathcal{A} = C[q, q^{-1}]$ and $\mathcal{C}(q)$ be a division field of $\mathcal{A}$. The superalgebra $U_q(\mathrm{gl}(m/n))$ is the associative superalgebra over $\mathcal{C}(q)$ generated by even generators K_i, K_i^{-1}, $i = 1, 2, \dots, m+n$, E_k, F_k, $k = 1, \dots, m+n-1$, $k \neq m$, and odd generators E_m, F_m, which satisfy the following relations:

$$
\begin{aligned}
&K_iK_j = K_jK_i\,, \qquad && K_iK_j^{-1} = K_j^{-1}K_i\,,\\
&K_iK_i^{-1} = 1\,, && K_i^{-1}K_j^{-1} = K_j^{-1}K_i^{-1}\\
&K_iE_i = qE_iK_i\,, && K_i^{-1}E_i = q^{-1}E_iK_i^{-1}\,,\\
&K_iF_i = q^{-1}F_iK_i\,, && K_i^{-1}F_i = qF_iK_i^{-1}\\
&K_iE_{i-1} = q^{-1}E_{i-1}K_i\,, && K_i^{-1}E_{i-1} = qE_{i-1}K_i^{-1}\,,\\
&K_iF_{i-1} = qF_{i-1}K_i\,, && K_i^{-1}F_{i-1} = q^{-1}F_{i-1}K_i^{-1}\\
&[E_i, E_j] = 0\,, \quad [F_i, F_j] = 0 \quad \text{for } j \neq i,\, i \pm 1\\
&[E_i, F_j] = 0 \quad \text{for } i \neq j\\
&[E_i, F_i] = \frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q - q^{-1}} \quad \text{for } i \neq m\\
&E_mF_m + F_mE_m = \frac{K_mK_{m+1} - K_m^{-1}K_{m+1}^{-1}}{q - q^{-1}}\\
&E_i^2E_{i\pm 1} - (q + q^{-1})E_iE_{i\pm 1}E_i + E_{i\pm 1}E_i^2 = 0 \quad \text{for } i \neq m\\
&F_i^2F_{i\pm 1} - (q + q^{-1})F_iF_{i\pm 1}F_i + F_{i\pm 1}F_i^2 = 0 \quad \text{for } i \neq m\\
&E_m^2 = F_m^2 = 0\,,\\
&\{E_{m-1}E_m - qE_mE_{m-1}, E_mE_{m+1} - qE_{m+1}E_m\} = 0\,,\\
&\{F_{m-1}F_m - qF_mF_{m-1}, F_mF_{m+1} - qF_{m+1}F_m\} = 0\,,
\end{aligned}
\tag{1}
$$

where $[X,Y] = XY - YX$ is the commutator and $\{X,Y\} = XY + YX$ is the anticommutator of two elements.

The Hopf structure of this superalgebra is defined by the following operations:

1. *Coproduct* Δ ($k = 1, 2, \ldots, m+n$, $r < m$ and $s > m$)

$$\begin{array}{ll}
\Delta(1) = 1 \otimes 1 & \Delta(K_k^{\pm 1}) = K_k^{\pm 1} \otimes K_k^{\pm 1} \\
\Delta(E_r) = E_r \otimes 1 + K_r K_{r+1}^{-1} \otimes E_r & \Delta(F_r) = F_r \otimes K_r^{-1} K_{r+1} + 1 \otimes F_r \\
\Delta(E_m) = E_m \otimes 1 + K_m K_{m+1} \otimes E_m & \Delta(F_m) = F_m \otimes K_m^{-1} K_{m+1}^{-1} + 1 \otimes F_m \\
\Delta(E_s) = E_s \otimes 1 + K_s^{-1} K_{s+1} \otimes E_s & \Delta(F_s) = F_s \otimes K_s K_{s+1}^{-1} + 1 \otimes F_s \, .
\end{array}$$

2. *Counit* ε

$$\begin{aligned}
&\varepsilon(1) = \varepsilon(K_r) = \varepsilon(K_r^{-1}) = 1 \\
&\varepsilon(E_r) = \varepsilon(F_r) = 0 \, .
\end{aligned}$$

3. *Antipode* S

$$\begin{array}{lll}
S(1) = 1 & S(K_r) = K_r^{-1} & S(K_r^{-1}) = K_r \\
S(E_r) = -K_r^{-1} K_{r+1} E_r & S(F_r) = -F_r K_r K_{r+1}^{-1} & r < m \\
S(E_m) = -K_m^{-1} K_{m+1}^{-1} E_m & S(F_m) = -F_m K_m K_{m+1} & \\
S(E_s) = -K_s K_{s+1}^{-1} E_s & S(F_s) = -F_s K_s^{-1} K_{s+1} & s > m \, .
\end{array}$$

However we do not use these operations for construction of the realization.

The method of construction used is the same as in the case of the Lie algebras [23] or quantum algebra [32] and is based on using the induced representation. The difference from quantum algebra is that together with q-deformed boson operators [9], [10] we also use fermion operators.

The algebra $\mathcal{B}$ of the q-deformed boson operators is the associative algebra over the field $\mathcal{C}(q)$ generated by the elements of a^+, $a^- = a$, q^x and q^{-x}, satisfying the commutation relations

$$\begin{array}{lll}
q^x q^{-x} = q^{-x} q^x = 1 \, , & q^x a^+ q^{-x} = q a^+ , & q^x a q^{-x} = q^{-1} a, \\
a a^+ - q^{-1} a^+ a = q^x , & a a^+ - q a^+ a = q^{-x} . &
\end{array} \tag{2}$$

The algebra $\mathcal{B}$ has faithful representation on vector space with the basic elements $|n\rangle$, where $n = 0, 1, \ldots,$ of the form

$$q^x |n\rangle = q^n |n\rangle \, , \quad a^+ |n\rangle = |n+1\rangle \, , \quad a|n\rangle = [n]|n-1\rangle \, , \tag{3}$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$.

Because of odd generators E_m and F_m we construct realization by means of the algebra $\mathcal{B}$ for even elements, and by fermion algebra $\mathcal{F}$ with elements b^+ and b for odd ones. These fermion elements commute with the elements of $\mathcal{B}$ and together fulfil the relations

$$bb = b^+b^+ = 0\,, \quad bb^+ + b^+b = 1\,. \tag{4}$$

The algebra $\mathcal{F}$ has faithful representation on vector space with the basis $|0\rangle$ and $|1\rangle$ of the form

$$b|M\rangle = \frac{1-(-1)^M}{2}|M-1\rangle\,, \qquad b^+|M\rangle = \frac{1+(-1)^M}{2}|M+1\rangle\,. \tag{5}$$

We use superalgebra $\mathcal{H}(r,s)$ for realization superalgebra. We mean the superalgebra of r copies of the algebras $\mathcal{B}$ of q-deformed bosons and s copies of the algebras $\mathcal{F}$ of the fermions. We suppose that the elements from $\mathcal{B}_i$ commute with the elements $\mathcal{F}_k$ for all i, k, and for elements $x_i \in \mathcal{B}_i$ and $y_i \in \mathcal{F}_i$ the relations $[x_i, x_k] = \{y_i, y_k\} = 0$ for $i \neq k$ hold.

As in the case of the Lie algebras or quantum groups our realizations contain elements of quantum sub-superalgebra of $\mathcal{A}_0 \subset U_q(\mathrm{gl}(m/n))$ namely, quantum superalgebra $U_q(\mathrm{gl}(m-1/n))$. The element x of this subalgebra commutes with the elements from $\mathcal{B}_i$, and for the fermion elements $b^\pm$ the relation

$$xb^\pm = (-1)^{\deg x} b^\pm x \tag{6}$$

holds. This superalgebra is denoted by $\mathcal{W}(r,s,\mathcal{A}_0)$.

In general, we define

Definition. Realization of the quantum superalgebra $\mathcal{A}$ is called the homomorphism ρ of $\mathcal{A}$ to the superalgebra $\mathcal{W}(r,s,\mathcal{A}_0)$, where $\mathcal{A}_0$ is the sub-superalgebra of $\mathcal{A}$.

3 Construction of the realization

First, for construction of the realization we find the induced representation of $U_q(\mathrm{gl}(m/n))$. As subalgebra $\mathcal{A}_0$ of $U_q(\mathrm{gl}(m/n))$ we choose a quantum superalgebra generated by E_k, $k > 1$, F_r, $r = 1, \ldots, m+n-1$, K_i and

K_i^{-1}, $i = 1, \ldots, m+n$. Let φ be a representation of $\mathcal{A}_0$ on vector space V. Let λ be the left regular representation on $U_q(\mathrm{gl}(m/n)) \otimes V$, i.e. for $x, y \in U_q(\mathrm{gl}(m/n))$ and $v \in V$ the representation λ is defined by

$$\lambda(x)(y \otimes v) = xy \otimes v\,. \tag{7}$$

Let $\mathcal{I}$ be subspace of $U_q(\mathrm{gl}(m/n)) \otimes V$ generated by the relations

$$xy \otimes v = x \otimes \varphi(y)v\,,$$

for all $x \in U_q(\mathrm{gl}(m/n))$, $y \in \mathcal{A}_0$ and $v \in V$. It is easy to see that the subspace $\mathcal{I}$ is λ invariant. Therefore, (7) gives the representation on the factor space $W = [U_q(\mathrm{gl}(m/n)) \otimes V]/\mathcal{I}$.

To find the induced representation of $U_q(\mathrm{gl}(m/n))$ explicitly we define the elements

$$\begin{aligned} X_1 &= E_1 \\ X_k &= X_{k-1}E_k - q^{-1}E_kX_{k-1} \qquad \text{for } k = 2, \ldots, m \\ X_k &= X_{k-1}E_k - qE_kX_{k-1} \qquad \text{for } k = m+1, \ldots, m+n-1\,. \end{aligned} \tag{8}$$

It follows from (8) that X_k for $k < m$ are even and for $k \geq m$ are odd elements. We can derive from generating relations (1) that the elements X_k fulfil the relations

$$\begin{aligned} X_rX_s &= q^{-1}X_sX_r\,, && \text{for } s < r \text{ and } s < m\,; \\ X_k^2 &= 0\,, && \text{for } r \geq m\,; \\ X_rX_s &= -q^{-1}X_sX_r\,, && \text{for } m \leq r < s\,. \end{aligned} \tag{9}$$

Let $|N_1, N_2, \ldots, N_{m+m-1}\rangle = |N\rangle = X_1^{N_1}X_2^{N_2}, \ldots, X_{m+n-1}^{N_{m+n-1}}$. Due to the Poincaré Birkhoff Witt theorem the space W of the induced representation is generated by the elements $|N\rangle \otimes v$, where $N_k = 0, 1, 2, \ldots$ for $k < m$, $N_k = 0, 1$ for $k \geq m$ and $v \in V$.

To obtain the explicit form of the induced representation, we give some relations. They can be proved by mathematical induction from relations (1).

Lemma 1. For any $n = 0, 1, 2, \ldots$ the following formulae hold:

$$\begin{aligned} E_rX_{r-1}^N &= q^NX_{r-1}^NE_r - q[N]X_{r-1}^{N-1}X_r && \text{for } r \leq m \\ E_rX_r^N &= q^{-N}X_r^NE_r && \text{for } r < m \\ E_mX_m^N &= (-q)^NX_m^NE_m && \\ E_mX_r^N &= (-1)^NX_r^NE_m && \text{for } r > m \\ E_rX_{r-1}^N &= q^{-N}X_{r-1}^nE_r - \frac{1-(-1)^N}{2}\,q^{-N}X_{r-1}^{N-1}X_r && \text{for } r > m \end{aligned}$$

$$E_r X_r^N = q^N X_r^N E_r \qquad \text{for } r > m$$
$$K_1 X_r^N = q^N X_r^N K_1$$
$$K_r X_{r-1}^N = q^{-N} X_{r-1}^N K_r \qquad \text{for } r > 1$$
$$F_r X_r^N = X_r^N F_r - q^{-1}[N] X_{r-1} X_r^{N-1} K_r K_{r+1}^{-1} \qquad \text{for } r < m$$
$$F_m X_m^N = (-1)^N X_m^N F_m + \frac{1-(-1)^N}{2} q^{-N} X_{m-1} X_m^{N-1} K_m K_{m+1}$$
$$F_r X_r^N = X_r^N F_r - \frac{1-(-1)^N}{2} q^{-N+2} X_{r-1} X_r^{N-1} K_r^{-1} K_{r+1} \qquad \text{for } r > m$$
$$F_1 X_1^N = X_1^N F_1 - \frac{[N]}{q-q^{-1}} X_1^{N-1}(q^{N-1} K_1 K_2^{-1} - q^{-N+1} K_1^{-1} K_2)$$
$$F_1 X_r^N = X_r^N F_1 + q^{-N+1}[N] X_r^{N-1} Y_r K_1^{-1} K_2 \qquad \text{for } 1 < r < m$$
$$F_1 X_s^N = X_s^N F_1 + \frac{1-(-1)^N}{2} X_s^{N-1} Y_s K_1^{-1} K_2 \qquad \text{for } s > m$$

where

$$Y_2 = E_2$$
$$Y_k = Y_{k-1} E_k - q^{-1} E_k Y_{k-1} \qquad \text{for } k = 3, \ldots, m$$
$$Y_k = Y_{k-1} E_k - q E_k Y_{k-1} \qquad \text{for } k = m+1, \ldots, m+n-1\,.$$

We omit the details of the calculations and write the result for the action of the induced representation on the basis elements $|N\rangle \otimes v$.

Theorem 1. Let $1 < r < m$, $s > m$, $t > 1$ and $S_k = \sum_{i=k}^{m+n-1} N_i$. Then the formulae

$$E_1|N\rangle \otimes v = |N + 1_1\rangle \otimes v$$
$$E_r|N\rangle \otimes v = -q[N_{r-1}]|N - 1_{r-1} + 1_r\rangle \otimes v + q^{N_{r-1}-N_r}|N\rangle \otimes \varphi(E_r)v$$
$$E_m|N\rangle \otimes v = -q[N_{m-1}]|N - 1_{m-1} + 1_m\rangle \otimes v + (-1)^{S_m} q^{N_{m-1}+N_m}|N\rangle \otimes \varphi(E_m)v$$
$$E_s|N\rangle \otimes v = -\frac{1-(-1)^{N_{s-1}}}{2} q^{-N_{s-1}}|N - 1_{s-1} + 1_s\rangle \otimes v + q^{N_s - N_{s-1}}|N\rangle \otimes \varphi(E_s)v$$
$$K_1|N\rangle \otimes v = q^{S_1}|N\rangle \otimes \varphi K_1 v$$
$$K_t|N\rangle \otimes v = q^{-N_{t-1}} \otimes \varphi K_t v$$
$$F_r|N\rangle \otimes v = -q^{-1}[N_r]|N + 1_{r-1} - 1_r\rangle \otimes (K_r K_{r+1}^{-1})v + |N\rangle \otimes \varphi(F_r)v$$
$$F_m|N\rangle \otimes v = \frac{1-(-1)^{N_m}}{2} q^{-N_m}|N + 1_{m-1} - 1_m\rangle \otimes \varphi(K_m K_{m+1})v +$$
$$+(-1)^{S_m}|N\rangle \otimes \varphi(F_m)v$$

$$F_s|N\rangle\otimes v = -\frac{1-(-1)^{N_s}}{2}q^{-N_s+2}|N+1_{s-1}-1_s\rangle\otimes\varphi(K_s^{-1}K_{s+1})v + |N\rangle\otimes\varphi(F_s)v$$

$$\begin{aligned}F_1|N\rangle\otimes v = &-\frac{[N_1]q^{S_1-1}}{q-q^{-1}}|N-1_1\rangle\otimes\varphi(K_1K_2^{-1})v+\\ &+\frac{[N_1]q^{-S_1+1}}{q-q^{-1}}|N_1-1\rangle\otimes\varphi(K_1^{-1}K_2)v+\\ &+\sum_{r=2}^{m-1}q^{-S_r+1}[N_r]|N-1_r\rangle\otimes\varphi(Y_rK_1^{-1}K_2)v+\\ &+\sum_{s=m}^{m+n-1}\frac{1-(-1)^{N_s}}{2}(-q)^{-S_s+1}|N-1_s\rangle\otimes\varphi(Y_sK_1^{-1}K_2)v+|N\rangle\otimes\varphi(F_1)v\end{aligned}$$

give the induced representation of the quantum superalgebra $U_q(\mathrm{gl}(m/n))$. We use the notation $|N\pm 1_r\rangle = |N_1,\dots,N_r\pm 1,\dots,N_{m+n-1}\rangle$.

To obtain the realization of quantum superalgebra $U_q(\mathrm{gl}(m/n))$ we choose the representation φ, for which $\varphi(F_1)v = 0$ and $\varphi(K_1)v = q^{\lambda}v$ and rewrite the induced representation given in Theorem 1 by means of the elements from $\mathcal{W}(m-1,n,\mathcal{A}_0)$. It follows from (3) and (5) that we substitute

$$\begin{aligned}
&q^{\pm N_k}\to q^{\pm x_k} && \text{for } k<m\\
&q^{\pm N_k}\to b_kb_k^+ + q^{\pm 1}b_k^+b_k \equiv q^{\pm x_k} && \text{for } k\geq m\\
&|N+1_k\rangle\to a_k^+ && \text{for } k<m\\
&[N_k]\,|N-1_k\rangle\to a_k && \text{for } k<m\\
&(-1)^{N_m+\dots+N_{k-1}}|N+1_k\rangle\to b_k^+ && \text{for } k\geq m\\
&(-1)^{N_m+\dots+N_{k-1}}\frac{1-(-1)^{N_k}}{2}|N-1_k\rangle\to b_k && \text{for } k\geq m\\
&\varphi(F_1)v\to 0 &&\\
&\varphi(K_1^{\pm1})v\to q^{\pm\lambda} &&\\
&\varphi(K_r^{\pm1})v\to k_r^{\pm1} && \text{for } k\geq 2\\
&\varphi(E_k)v\to e_k && \text{for } k\neq m\\
&\varphi(F_k)v\to f_k && \text{for } k\neq m\\
&(-1)^{S_m}\varphi(E_m)v\to e_m &&\\
&(-1)^{S_m}\varphi(F_m)v\to f_m &&\\
&\varphi(Y_k)v\to y_k && \text{for } k<m\\
&(-1)^{S_m}\varphi(Y_k)v\to y_k && \text{for } k\geq m\,.
\end{aligned}\tag{10}$$

The factors $(-1)^{N_m+\ldots+N_{k-1}}$ reflect the fact that the corresponding elements are fermions.

By substitutions (10) we obtain the realization of the quantum superalgebra $U_q(\mathrm{gl}(m/n))$, which is given in the following theorem.

Theorem 2. Let be $r = 2, \ldots, m-1$ and $s = m+1, \ldots, m+n-1$. The mapping $\rho : U_q(\mathrm{gl}(m/n)) \to \mathcal{W}\big(m-1, n, U_q(\mathrm{gl}(m-1/n))\big)$ defined by the formulae

$$
\begin{aligned}
\rho(E_1) &= a_1^+ \\
\rho(E_r) &= -qa_{r-1}a_r^+ + q^{x_{r-1}-x_r}e_r \\
\rho(E_m) &= -qa_{m-1}b_r^+ + q^{x_{m-1}+x_m}e_m \\
\rho(E_s) &= q^{-1}b_{s-1}b_s^+ + q^{x_s-x_{s-1}}e_s \\
\rho(K_1) &= q^{\lambda+x_1+x_2+\ldots+x_{m+n-1}} \\
\rho(K_r) &= q^{-x_{r-1}}k_r \\
\rho(K_m) &= q^{-x_{m-1}}k_m \\
\rho(K_s) &= q^{-x_{s-1}}k_s \\
\rho(F_r) &= -q^{-1}a_{r-1}^+a_rk_rk_{r+1}^{-1} + f_r \\
\rho(F_m) &= q^{-1}a_{m-1}^+b_mk_kk_{m+1} + f_m \\
\rho(F_s) &= -qb_{s-1}^+b_sk_s^{-1}k_{s+1} + f_s \\
\rho(F_1) &= -\frac{a_1}{q-q^{-1}}\Big(q^{\lambda+x_1+\ldots+x_{m+n-1}-1}k_2^{-1} - q^{-\lambda-x_1-\ldots-x_{m+n-1}+1}k_2\Big)+ \\
&\quad + \sum_{r=2}^{m-1} a_rq^{-\lambda-x_r-\ldots-x_{m+n-1}+1}y_rk_2 + \sum_{r=m}^{m+n-1} b_rq^{-\lambda-x_{r+1}-\ldots-x_{m+n-1}}y_sk_2
\end{aligned}
$$

is the realization of the quantum superalgebra $U_q(\mathrm{gl}(m/n))$.

In the formulae we used the aberrations $q^{\pm x_k} = b_kb_k^+ + q^{\pm 1}b_k^+b_k$ for $k \geq m$ and $q^{x_r+x_s} = q^{x_r}q^{x_s}$ for simplicity.

Proof: Since the representations of $\mathcal{B}$ and $\mathcal{F}$ given in (3) and (5) are faithful, the representation $\mathcal{W}\big(m-1, n, U_q(\mathrm{gl}(m-1/n))\big)$ is faithful on the vector space W generated by $|N\rangle \otimes x$, where $x \in U_g(\mathrm{gl}(m-1/n))$. Theorem follows from the fact that the representation $U_q(\mathrm{gl}(m/n))$ on W, which we obtain by means of the inverse formulae to (10), is the representation given in Theorem 1.

4 Conclusion

In this paper we gave the method of construction of the q–boson–fermion realization of quantum superalgebras and applied it to the quantum superalgebra $U_q(\mathrm{gl}(m/n))$. One of the advantages of this method, in comparison with [37], is that we automatically obtain a realization and we do not need to verify the generating relation.

The other advantage we see in the fact that our realization is expressed by means of polynomials of q–deformed bosons and fermions. On the other hand, we can easily obtain the Dyson realization of quantum superalgebra. For this purpose, it is sufficient to choose a realization of the generators of the algebra $\mathcal{B}$ in the form

$$a^+ = A^+ \,, \quad a = \frac{[N+1]}{N+1} A \,, \quad q^x = q^N \,, \tag{11}$$

where $[A, A^+] = 1$ and $N = A^+ A$. It is easy to verify that the realization of $U_q(\mathrm{gl}(m/n))$ from Theorem 2 with realization (11) of the algebra $\mathcal{B}$ and with a trivial realization of the subalgebra $U_q(\mathrm{gl}(m-1/n))$ leads, after homomorphism of $U_q(\mathrm{gl}(m/n))$, to the realization given in [37]. In this case, the realization is of course expressed by means of a series in the operators A^+ and A. Therefore, we prefer our form of realizations.

Finally, our realizations contain, in contrast with those in [37], quantum sub superalgebras. Various forms of realizations of this sub–superalgebra give various realizations of the quantum superalgebra. For instance, by the presented method we can find realization of $U_q(\mathrm{gl}(m-1/n))$ on $\mathcal{W}\big(m-2, n, U_q(\mathrm{gl}(m-2,n))\big)$ and use this realization in our formulae. In this case we obtain the realization of $U_q(\mathrm{gl}(m/n))$ on $\mathcal{W}\big(2m-3, n, U_q(\mathrm{gl}(m-2/n))\big)$. On the other hand, we can construct the realization of $U_q(\mathrm{gl}(m-1/n))$ on $\mathcal{W}\big(n-1, m-1, U_q(\mathrm{gl}(m-1, n-1))\big)$. Using this realization in our formulae we obtain realization $U_q(\mathrm{gl}(m/n))$ on $\mathcal{W}\big(m+n-1, m+n-1, U_q(\mathrm{gl}(m-1, n-1))\big)$. The possibility of using similar recurrence is, in our opinion, one of the most important advantages of presented construction.

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