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LANGEVIN EQUATION WITH SCALE-DEPENDENT NOISE

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The Langevin equation is the most general approximation for a large variety of dynamical systems affected by thermal fluctuations. It arises in the description of magnetic at the presence of magnetic field fluctuations, in the description of interface growth, in hydrodynamic turbulence theory, in stochastic quantization of gauge theories and in a large variety of other problems [1, 2, 3, 4]. Let us consider the Langevin equation for a scalar field in d-dimensional Euclidean space

$$\frac{\partial\phi(t,\boldsymbol{x})}{\partial t} = U[\phi(t,\boldsymbol{x})] + \eta(t,\boldsymbol{x}), \quad \langle\eta(x)\eta(x')\rangle = D(x,x'), \tag{1}$$

where $U[\phi]$ is the nonlinear interaction potential, $\eta(t,\boldsymbol{x})$ is the Gaussian random noise, which accounts for the fluctuations of the system environment. The Minkovski-like $(d+1)$ dimensional notation $x \equiv (t,\boldsymbol{x})$ is used hereafter.

The standard way to solve the Langevin equation (1) is to separate the interaction potential $U[\phi]$ into a linear part $L = G_0^{-1}$ and a nonlinear part taken with the small parameter $\lambda V[\phi]$, and then solve the equation in each order of the perturbative expansion. The procedure is simplified by the assumption of the Gaussian statistics of the random noise which allows one to take into account only even order correlators of the random noise: all terms containing the odd number of η are equal to zero. The diagram technique for the iterative solution of the Langevin equation, often called the Wyld diagram technique [5], is identical to the Feynman diagram technique in quantum field theory. Similarly to that in quantum field theory, it requires elimination of loop divergences by renormalization group methods [2, 3].

The degree of divergences arising in the perturbative solution of the Langevin equation depends on a particular type of the random force correlation function. Most approximations use random force δ-correlated in time

$$\langle\eta(k_1)\eta(k_2)\rangle = (2\pi)^{d+1}\delta(k_1+k_2)D(\boldsymbol{k}_2). \tag{2}$$

The spatial part of the correlation function $D(\boldsymbol{k})$ is assumed either to be a constant or to have a power behavior. This is an artifact of the perturbation solution and is done for the sake of analytical evaluation of the loop integrals.

From a physical standpoint, in contrast, a random force acting in a limited range of scales is often desirable. for instance, a limited band forcing is the case for the stirred hydrodynamic turbulence and magnetic systems.

In this paper we propose a novel method to describe the limited band stochastic forcing. It is shown, that for a narrow band forcing, an appropriate chose of the random force correlation functions yield a theory which is free of loop divergences and does not require renormalization. The proposed method preserves the whole structure of the perturbation expansion, but the

decomposition with respect to the affine group is used instead of the Fourier transform. This change of the functional space provides cancellation of the loop divergences. In the limiting case of scale independent forcing all common results are preserved.

The idea of the method is to study the dynamical system described by the Langevin equation separately at each scale. Following [6], instead of the usual space of the random functions $f(x,\cdot) \in (\Omega, \mathcal{A}, P)$, where $f(x) \in \mathrm{L}^2(\mathrm{R}^n)$ for each given realization of the random process, we go to the multi-scale representation provided by the continuous wavelet transform

$$W_\psi(a,\boldsymbol{b},\cdot) = \int |a|^{-\frac{d}{2}}\overline{\psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)} f(\boldsymbol{x},\cdot)d^dx. \tag{3}$$

Since the structure of divergences and the localization of the solution are determined by the spatial part of the random force correlator (see *e.g.* [3]), the wavelet transform is performed only in the *spatial* argument of the dynamical variable ϕ, but not in its temporal argument.

The existence by the inverse wavelet transform

$$f(\boldsymbol{x},\cdot) = C_\psi^{-1}\int |a|^{-\frac{n}{2}}\psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right) W_\psi(a,\boldsymbol{b},\cdot)\frac{dad\boldsymbol{b}}{a^{d+1}} \tag{4}$$

is provided by the admissibility condition

$$C_\psi = S_d^{-1}\int \frac{|\hat\psi(\boldsymbol{k})|^2}{|k|^d}d^dk = \int \frac{|\hat\psi(a\boldsymbol{k})|^2}{a}da < \infty, \tag{5}$$

which constrains the choice of the basic wavelet $\psi \in \mathrm{L}^2(\mathrm{R}^n)$. The extra factor S_d, the area of a unit sphere in d dimensions, stands for the rotationally symmetric wavelet $\psi(\boldsymbol{x}) = \psi(|\boldsymbol{x}|)$.

The use of the wavelet coefficients $W_\psi(a,\boldsymbol{b},\cdot)$ instead of the original stochastic process provides an extra analytical flexibility of the method: there exist more than one set of random functions $W(a,\boldsymbol{b},\cdot)$ the images of which have coinciding correlation functions in the space of $f(x,\cdot)$. It is easy to check that a random process generated by wavelet coefficients with the correlation function

$$\langle\widehat{W}(a_1,k_1)\widehat{W}(a_2,k_2)\rangle = C_\psi^{-1}(2\pi)^{d+1}\delta^d(k_1+k_2)a_1^{d+1}\delta(a_1-a_2)D_0,$$

has the same correlation function as white noise

$$\begin{aligned}\langle\hat f(k_1)\hat f(k_2)\rangle &= (2\pi)^dD_0\delta^d(k_1+k_2)\\ \langle\widehat{W}(a_1,k_1)\widehat{W}(a_2,k_2)\rangle &= (2\pi)^dD_0\delta^d(k_1+k_2)(a_1a_2)^{d/2}\overline{\hat\psi(a_1k_1)\psi(a_2k_2)}.\end{aligned}$$

Therefore, starting from a given random process in the space of scale-dependent functions $W(a,\boldsymbol{b},t)$ rather than in a common space of square integrable functions $f(\boldsymbol{x},t)$, we can construct a narrow band forcing with no contradictions to other physical constraints on forcing. This can be done by applying the requirement $W(a,\boldsymbol{b},t)\to 0$ for all a outside a certain domain $[a_{min},a_{max}]$.

As an example, let us consider the well known Kardar-Parisi-Zhang model of interface growth [3]:

$$\dot{Z}-\nu\Delta Z=\frac{\lambda}{2}(\nabla Z)^2+\eta. \tag{6}$$

Substituting the wavelet transform

$$Z(x)=C_\psi^{-1}\int \exp(\imath(\boldsymbol{k}\boldsymbol{x}-k_0t))a^{\frac{d}{2}}\hat{\psi}(a\boldsymbol{k})\hat{Z}(a,k)\frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{da}{a^{d+1}} \tag{7}$$

where $k\equiv(\boldsymbol{k},\omega), x\equiv(\boldsymbol{x},t)$, into the equation (6), with the random force of the form

$$\begin{array}{ll}\langle\hat{\eta}(a_1,k_1)\hat{\eta}(a_2,k_2)\rangle &= (2\pi)^{d+1}\delta^{d+1}(k_1+k_2)a_1^{d+1}\delta(a_1-a_2)D(a_2,k_2),\\ \langle\hat{\eta}(a,k)\rangle &= 0,\end{array} \tag{8}$$

after straightforward calculations, we get an integral equation

$$\begin{array}{l}(-\imath\omega+\nu\boldsymbol{k}^2)\hat{Z}(a,k)=\eta(a,k)-\frac{\lambda}{2}a^{\frac{d}{2}}\overline{\hat{\psi}(a\boldsymbol{k})}C_\psi^{-2}\int(a_1a_2)^{\frac{d}{2}}\hat{\psi}(a_1\boldsymbol{k}_1)\hat{\psi}(a_2(\boldsymbol{k}-\boldsymbol{k}_1))\\ \qquad \boldsymbol{k}_1(\boldsymbol{k}-\boldsymbol{k}_1)\hat{Z}(a_1,k_1)\hat{Z}(a_2,k-k_1)\frac{d^{d+1}k_1}{(2\pi)^{d+1}}\frac{da_1}{a_1^{d+1}}\frac{da_2}{a_2^{d+1}}\ .\end{array}$$

In one loop approximation this gives the contribution to the Green function $G(k)=G_0(k)+\lambda^2G_2(k)+O(\lambda^4)$:

$$\begin{array}{l}G(k)=G_0(k)-\lambda^2G_0^2(k)\int\frac{d^{d+1}k_1}{(2\pi)^{d+1}}\Delta(k_1)\\ \qquad \boldsymbol{k}_1(\boldsymbol{k}-\boldsymbol{k}_1)|G_0(k_1)|^2\boldsymbol{k}\boldsymbol{k}_1G_0(k-k_1)+O(\lambda^4),\end{array} \tag{9}$$

where $G_0^{-1}(k)=-\imath\omega+\nu\boldsymbol{k}^2$ is the zero-th order approximation of the Green function. The difference from the standard result obtained by Fourier transform [3] is in the form of the effective force correlator, which is scale dependent in our approach

$$\Delta(k)\equiv C_\psi^{-1}\int\frac{da}{a}|\hat{\psi}(a\boldsymbol{k})|^2D(a,\boldsymbol{k}) \tag{10}$$

and has the meaning of the effective force averaged over all scales.

The Green function (9) obtained with the random force (8), does not depend on scale explicitly

$$\hat{Z}(a,k)=G(k)\hat{\eta}(a,k).$$

Similarly to (9), for the correlation function $\langle ZZ\rangle$ in one loop approximation $C(k)=C_0(k)+\lambda^2C_2(k)+O(\lambda^4)$, we get

$$\begin{array}{l}C_2(a_i,a_f,k)=\frac{1}{2}|G_0(k)|^2\overline{\hat{\psi}(a_i\boldsymbol{k})\hat{\psi}(-a_f\boldsymbol{k})}\\ \qquad \int\frac{d^{d+1}k_1}{(2\pi)^{d+1}}|G_0(k_1)|^2|G_0(k-k_1)|^2[\boldsymbol{k}_1(\boldsymbol{k}-\boldsymbol{k}_1)]^2\Delta(k_1)\Delta(k-k_1).\end{array} \tag{11}$$

For the random stirring which does not depend on scale, the integration over a in equation (10), after substitution $k_1=k_1'+\frac{k}{2}$ in both equations (9,11), leads to the known result [3].

For example, let us consider a single band forcing

$$D(a,\boldsymbol{k})=\delta(a-a_0)D(\boldsymbol{k}) \tag{12}$$

and the "Mexican hat" as a basic wavelet

$$\hat{\psi}(k)=(2\pi)^{d/2}(-\imath\boldsymbol{k})^2\exp(-\boldsymbol{k}^2/2),\quad C_\psi=(2\pi)^d. \tag{13}$$

Substituting (12) and (13) into the equation (9), after integration over the frequency, in the leading order in the small parameter $x=|\boldsymbol{k}|/|\boldsymbol{k}_1|$, we get the contribution to the Green function:

$$G(k)=G_0(k)+\lambda^2G_0^2(k)\frac{S_d}{(2\pi)^d}\frac{a_0^3k^2}{\nu^2}\frac{d-2}{8d}\int_0^\infty D(\boldsymbol{q})\exp\left(-a_0^2\boldsymbol{q}^2\right)q^{d+1}dq+O(\lambda^4), \tag{14}$$

For constant $D(\boldsymbol{q})=D_0$ the obtained contribution to the Green function is finite and does not require any further renormalization.

Absolutely in the same way we can evaluate other polynomial interactions in the Langevin equation. For instance, for the simplest Langevin equation with square interaction $\frac{\lambda^2}{2}Z^2$, which is often used in hydrodynamics, the corresponding equations for (9) and (11) differ from that obtained above only by the sign and the absence of the scalar products of the wave vectors in each vertex.

References

[1] P.C. Martin, E.D. Sigia, and H.A. Rose. Statistical dynamics of classical systems. *Phys. Rev.*, A8:423–437, 1973.

[2] L.Ts. Adzhemyan, A.N. Vasil'ev, and Yu. M. Pis'mak. Renormalization group approach in turbulence theory. *Theoretical and Mathematical Physics*, 57:268–281 Russian; 1131–1143 English, 1983.

[3] M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56(9):889–892, 1986.

[4] J. Zinn-Justin. Renormalization and stochastic quantization. *Nuclear Physics B*, 275(FS17):135–159, 1986.

[5] Wyld H. W. Formulation of the theory of turbulence in an incompressible fluid. *Annals of Physics*, 14(2):143–165, 1961.

[6] Altaisky M.V. Scale-dependent function in statistical hydrodynamics: a functional analysis point of view. *European Journal of Physics B*, 8(4), 1999.

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