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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR RETARDED DIFFERENTIAL EQUATION

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1 Introduction

In the recent papers [1]-[11] the oscillatory and asymptotic properties of various types of differential equations

$$\left(|u'(t)|^{\alpha-1}u'(t)\right)' + p(t)f[u(\tau(t))] = 0 \tag{1}$$

have been considered. In this paper we shall study thòse properties under the following hypothesis (H):

(H1) $\alpha > 0$ is a real constant;

(H2) $p \in C[t_0, \infty)$, $p(t) > 0$;

(H3) $\tau \in C^1[t_0, \infty)$, $\tau'(t) > 0$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$;

(H4) $f \in C(-\infty, \infty)$, f is nondecreasing, $f \in C^1(M)$ $M = (-\infty, 0) \cup (0, \infty)$, $uf(u) > 0$ for $u \neq 0$.

By a solution of (1) we mean a function $u \in C^1[T_u, \infty)$, $T_u \geq t_0$, which has the property $|u'(t)|^{\alpha-1}u'(t) \in C^1[T_u, \infty)$ and satisfies (1) on $[T_u, \infty)$. We consider only those solutions of (1) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for all $T \geq T_u$. We assume that (1) possesses such a solution. A nontrivial solution of (1) is said to be oscillatory if it has arbitrarily large zeros: otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory. It is known that the condition $\int^\infty p(s)\,ds = \infty$ is enough for oscillation of (1). In our paper we are concerned the case when $\int^\infty p(s)\,ds < \infty$. The aim of this paper is to present some new oscillatory criteria which are new also for $\alpha = 1$, namely for the second order nonlinear differential equation

$$u''(t) + p(t)f[u(\tau(t))] = 0.$$

As is customary all functional inequalities are assumed to hold eventually, that is they are satisfied for all sufficiently large t.

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2 Main results

Theorem 2.1 *Let $f'(u)$ be nondecreasing on $(-\infty, -t^*)$ and nonincreasing on (t^*, ∞), $t^* \geq 0$. Let $\alpha \geq 1$. Further assume that*

$$\int^{\infty} p(s)|f[c\tau(s)]|\, \mathrm{ds} = \infty \qquad \textit{for all } c \neq 0 \tag{2}$$

and moreover for some $\lambda > 0$

$$\int^{\infty} \left(\tau^{\alpha}(s)p(s) - \frac{\alpha^2 \tau^{\alpha-2}(s)\tau'(s)}{4f'[\pm\lambda\tau(s)]} \right) \mathrm{ds} = \infty. \tag{3}$$

Then Eq.(1) is oscillatory.

Proof. Assume on the contrary that Eq.(1) has an eventually positive solution $u(t)$. (The case $u(t) < 0$ can be treated similarly.) Then

$$\left(|u'(t)|^{\alpha-1}u'(t) \right)' = -p(t)f[u(\tau(t))] < 0.$$

Hence, the function $|u'(t)|^{\alpha-1}u'(t)$ is decreasing. Therefore, either

(i) $u'(t) > 0$, eventually or

(ii) $u'(t) < 0$, eventually.

Since

$$0 > \left(|u'(t)|^{\alpha-1}u'(t) \right)' = \alpha|u'(t)|^{\alpha-1}u''(t),$$

we see that $u''(t) < 0$. Condition (ii) now yields $u(t) \to -\infty$ as $t \to \infty$. This is a contradiction. Therefore we conclude that $u(t) > 0$, $u'(t) > 0$, $u''(t) < 0$, eventually and

$$\left[(u'(t))^{\alpha} \right]' = -p(t)f[u(\tau(t))]. \tag{4}$$

Define

$$w(t) = \tau^{\alpha}(t)\frac{[u'(t)]^{\alpha}}{f[u(\tau(t))]}. \tag{5}$$

Then $w(t) > 0$ and

$$\begin{aligned} w'(t) &= \alpha\tau^{\alpha-1}\tau'(t)\frac{[u'(t)]^{\alpha}}{f[u(\tau(t))]} + \tau^{\alpha}(t)\frac{[(u'(t))^{\alpha}]'}{f[u(\tau(t))]} \\ &\quad -\tau^{\alpha}(t)\frac{[u'(t)]^{\alpha} f'[u(\tau(t))]u'(\tau(t))\tau'(t)}{f^2[u(\tau(t))]} \\ &= \alpha\frac{\tau'(t)}{\tau(t)}w(t) - \tau^{\alpha}(t)p(t) \\ &\quad -w(t)\frac{f'[u(\tau(t))]u'(\tau(t))\tau'(t)}{f[u(\tau(t))]}. \end{aligned} \tag{6}$$

We claim that $u'(t) \to 0$ as $t \to \infty$. To prove it assume the contrary, that is $u'(t) \to 2c$ as $t \to \infty$, $c > 0$. Then $u'(t) \geq 2c$ which on integration from t_1 to t implies

$$u(t) \geq u(t_1) + 2c(t - t_1) \geq ct, \tag{7}$$

eventually. Integrating (4) from t_1 to t and using (7) one gets

$$-[u'(t)]^{\alpha} + [u'(t_1)]^{\alpha} = \int_{t_1}^{t} p(s)f[u(\tau(s))]\, \mathrm{ds} > \int_{t_1}^{t} p(s)f[c\tau(s)]\, \mathrm{ds}.$$

Letting $t \to \infty$ we have

$$\int_{t_1}^{\infty} p(s)f[c\tau(s)]\, \mathrm{ds} < \infty.$$

This contradiction shows that $u'(t) \to 0$ as $t \to \infty$. Therefore, for any $\lambda > 0$ there exists a t_1 such that $\lambda/2 > u'(t)$, $t \geq t_1$. Integrating the last functional inequality from t_1 to t we get

$$u(t) \leq u(t_1) + \frac{\lambda}{2}(t - t_1) \leq \lambda t, \qquad t \geq t_2 \geq t_1$$

and so for any $\lambda > 0$ and t large enough

$$f'[u(\tau(t))] \geq f'[\lambda\tau(t)]. \tag{8}$$

On the other hand, since $u'(t)$ is decreasing and $u'(t) \to 0$ as $t \to \infty$ it follows that

$$u'(\tau(t)) \geq u'(t) \geq (u'(t))^{\alpha}, \tag{9}$$

eventually. Combining (8) and (9) together with (6) we see that

$$\begin{aligned} w'(t) &\leq -\tau^{\alpha}(t)p(t) + \alpha\frac{\tau'(t)}{\tau(t)}w(t) - \frac{\tau'(t)f'[\lambda\tau(t)]}{\tau^{\alpha}(t)}w^2(t) \\ &= -\tau^{\alpha}(t)p(t) \\ &- \frac{\tau'(t)f'[\lambda\tau(t)]}{\tau^{\alpha}(t)}\left[\left(w(t) - \frac{\alpha\tau^{\alpha-1}(t)}{2f'[\lambda\tau(t)]}\right)^2 - \frac{\alpha^2\tau^{2\alpha-2}(t)}{4\left(f'[\lambda\tau(t)]\right)^2}\right] \\ &\leq -\tau^{\alpha}(t)p(t) + \frac{\alpha^2\tau^{\alpha-2}(t)\tau'(t)}{4f'[\lambda\tau(t)]}. \end{aligned} \tag{10}$$

Integrating the above inequality from t_2 to t we conclude in view of (3) that $w(t) \to -\infty$ as $t \to \infty$. This is a contradiction and the proof is complete now. □

Remark 1 *There has been usually the condition*

$$f(u)\,\mathrm{sgn}\,u \geq |u|^{\beta}\mathrm{sgn}\,u$$

imposed on the function f. In this sense Theorem 2.1 extends and complements Theorem 1 in [6] and Theorem 2.3 in [1].

For $\alpha = 1$ Theorem 2.1 gives

Corollary 2.1 *Let $f'(u)$ be nondecreasing on $(-\infty, -t^*)$ and nonincreasing on (t^*, ∞), $t^* \geq 0$. Further assume that (2) holds for any $c \neq 0$ and for some $\lambda > 0$*

$$\int^{\infty}\left(\tau(s)p(s) - \frac{\tau'(s)}{4\tau(s)f'[\pm\lambda\tau(s)]}\right)\mathrm{d}s = \infty. \tag{11}$$

Then equation

$$u''(t) + p(t)f[u(\tau(t))] = 0 \tag{12}$$

is oscillatory.

Remark 2 *Corollary 2.1 generalizes and extends the results presented in [2] and [12].*

Example 1 *Consider the second order nonlinear equation*

$$u''(t) + p(t)\ln\left(1 + |u[\tau(t)]|\right)\mathrm{sgn}\,u[\tau(t)] = 0. \tag{13}$$

By Corollary 2.1, Eq.(13) is oscillatory provided that

$$\int^{\infty} p(s)\ln(1 + c\tau(s))\,\mathrm{d}s = \infty \qquad \text{for any } c > 0$$

and for some $\lambda > 0$

$$\int^{\infty}\left(\tau(s)p(s) - \frac{1+\lambda\tau(s)}{4}\frac{\tau'(s)}{\tau(s)}\right)\mathrm{d}s = \infty.$$

Corollary 2.2 *Let $0 < \beta < 1$, $\alpha > 1$. Assume that*

$$\int^{\infty}\tau^{\beta}(s)p(s)\,\mathrm{d}s = \infty \tag{14}$$

and for some $M > 0$

$$\int^{\infty}\left(\tau^{\alpha}(s)p(s) - M\tau^{\alpha-\beta-1}(s)\tau'(s)\right)\mathrm{d}s = \infty \tag{15}$$

Then the equation

$$\left(|u'(t)|^{\alpha-1}u'(t)\right)' + p(t)|u(\tau(t))|^{\beta-1}u(\tau(t)) = 0 \tag{16}$$

is oscillatory.

Proof. It is easy to verify that (2) and (3) reduce to (14) and (15), respectively for Eq.(16).

Theorem 2.2 *Let $f'(u)$ be nonincreasing on $(-\infty, -t^*)$ and nondecreasing on (t^*, ∞), $t^* \geq 0$. Let $\alpha \geq 1$. Further assume that (2) holds for any $c \neq 0$. If for some $M > 0$*

$$\int^{\infty} \left(\tau^{\alpha}(s)p(s) - M\tau^{\alpha-2}(s)\tau'(s)\right) ds = \infty. \qquad (17)$$

Then Eq.(1) is oscillatory.

Proof. Assume that $M > 0$ is such that (17) holds. Admit that $u(t)$ is a positive solution of (1). Proceeding exactly as in the proof of Theorem 2.1 we can verify that $u'(t) > 0$, $u''(t) < 0$ and $u'(t) \to 0$ as $t \to \infty$. Then there exists $c > 0$ such that $u[\tau(t)] > c$, eventually. Let $w(t)$ be defined by (5), then $w(t) > 0$ and (6) is fulfilled. It is easy to see that

$$f'[u(\tau(t))]u'(\tau(t)) \geq f'(c)\,u'(t) = f'(c)\Big(u'(t)\Big)^{1-\alpha}\Big(u'(t)\Big)^{\alpha}. \quad (18)$$

Since $u'(t) \to 0$ then for any $\lambda > 0$ we have $u'(t) < \lambda$, eventually. It follows from (18) that

$$f'[u(\tau(t))]u'(\tau(t)) \geq f'(c)\lambda^{1-\alpha}\Big(u'(t)\Big)^{\alpha} = K\Big(u'(t)\Big)^{\alpha},$$

where λ is chosen such that $f'(c)\lambda^{1-\alpha} = \alpha^2/(4M)$. Then

$$\begin{aligned} w'(t) &\leq -\tau^{\alpha}(t)p(t) + \alpha\frac{\tau'(t)}{\tau(t)}w(t) - K\frac{\tau'(t)}{\tau^{\alpha}(t)}w^2(t) \\ &= -\tau^{\alpha}(t)p(t) \\ &\quad -K\frac{\tau'(t)}{\tau^{\alpha}(t)}\left[\left(w(t) - \frac{\alpha\tau^{\alpha-1}(t)}{2K}\right)^2 - \frac{\alpha^2\tau^{2\alpha-2}(t)}{4K^2}\right] \\ &\leq -\tau^{\alpha}(t)p(t) + \frac{\alpha^2}{4K}\tau^{\alpha-2}(t)\tau'(t). \end{aligned} \qquad (19)$$

Integrating the obtained inequality from t_1 to t, (t_1 large enough) and then letting $t \to \infty$ we get desirable contradiction. The proof is complete now. □

Corollary 2.3 *Let $\beta > 1$, $\alpha > 1$. Assume that (14) and (17) are satisfied. Then Eq. (16) is oscillatory.*

The following considerations are intended to relax the monotonicity conditions imposed onto $f'(u)$ in Theorems 2.1 and 2.2.

Let us consider the following differential equation

$$\left(|u'(t)|^{\alpha-1}u'(t)\right)' + p(t)h[u(\tau(t))] = 0 \qquad (20)$$

subject to conditions (H1)-(H3) and

(H5) $h \in C(-\infty, \infty)$, $uh(u) > 0$ for $u \neq 0$.

Theorem 2.3 *Assume that*

$$h(u)\,\mathrm{sgn}\,u \geq f(u)\,\mathrm{sgn}\,u, \qquad u \neq 0 \qquad (21)$$

and (H4) holds. If all assumptions of Theorem 2.1 are satisfied then Eq.(20) is oscillatory.

Proof. Assume that $u(t)$ is a positive solution of (20). Then $u'(t) > 0$, $u''(t) < 0$ and

$$\left([u'(t)]^{\alpha}\right)' = -p(t)h[u(\tau(t))] \leq -p(t)f[u(\tau(t))].$$

Let $w(t)$ be defined by (5). Then $w(t) > 0$ and

$$w'(t) \leq \alpha\frac{\tau'(t)}{\tau(t)}w(t) - \tau^{\alpha}(t)p(t) - w(t)\frac{f'[u(\tau(t))]u'(\tau(t))\tau'(t)}{f[u(\tau(t))]}.$$

The rest of the proof is similar to the proof of Theorem 2.1 and so it can be omitted. □

Theorem 2.4 *Let (H4) and (21) holds. Assume that all assumptions of Theorem 2.2 are satisfied. Then Eq.(20) is oscillatory.*

It remains an open problem to obtain the oscillatory criteria similar to Theorem 2.1 and 2.2 for (1) with $0 < \alpha < 1$. The following theorem provides a partial answer.

Theorem 2.5 *Assume that*

$$\int_{t_0}^{\infty} \frac{\mathrm{du}}{|f(\pm u)|^{1/\alpha}} < \infty$$

and

$$\int_{t_0}^{\infty} \tau'(s) \left(\int_s^{\infty} p(x)\, \mathrm{dx} \right)^{1/\alpha} \mathrm{ds} = \infty.$$

Then Eq.(1) is oscillatory.

Proof. Assume that $u(t)$ is a positive solution of (1). Similarly as in the proof of Theorem 2.1 it can be shown that $u'(t) > 0$ and $u''(t) < 0$. Integrating (1) from t to s ($\geq t$) we obtain

$$\begin{aligned} -[u'(s)]^{\alpha} + [u'(t)]^{\alpha} &= \int_t^s p(x) f[u(\tau(x))]\, \mathrm{dx} \\ &\geq f[u(\tau(t))] \int_t^s p(s)\, \mathrm{ds}. \end{aligned}$$

Using properties of $u'(t)$ and letting $s \to \infty$ we have

$$(u'[\tau(t)])^{\alpha} \geq (u'(t))^{\alpha} \geq f[u(\tau(t))] \int_t^{\infty} p(s)\, \mathrm{ds}. \qquad (22)$$

(Now it is easy to see that the case $\int^{\infty} p(x)\, \mathrm{dx} = \infty$ leads to a contradiction.) It follows from (22) that

$$\frac{u'[\tau(t)]\tau'(t)}{f^{1/\alpha}[u(\tau(t))]} \geq \tau'(t) \left(\int_t^{\infty} p(x)\, \mathrm{dx} \right)^{1/\alpha}$$

which on integration from t_1 to t gives

$$\int_{u[\tau(t_1)]}^{u[\tau(t)]} \frac{\mathrm{ds}}{f^{1/\alpha}(s)} \geq \int_{t_1}^{t} \tau'(s) \left(\int_s^{\infty} p(x)\, \mathrm{dx} \right)^{1/\alpha} \mathrm{ds}. \qquad (23)$$

The left side of (23) is bounded, however the right side of (23) tends to ∞ as $t \to \infty$. The proof is complete. □

Remark 3 *Theorem 2.5 cannot be applied to Eq.(1) with $f(u) = u$ and $\alpha \geq 1$. In this case Theorem 2.1 may be successful.*

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