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EIGENPROBLEM FOR MONOTONE AND TOEPLITZ MATRICES IN MAX-ALGEBRA

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1 Introduction

Let $G = (G, \otimes, \leq)$ be a linearly ordered, commutative group with neutral element $e = 0$. We suppose that G is radicable, i.e. for every integer $t \geq 1$ and for every $a \in G$, there exists a (unique) element $b \in G$ such that $b^t = a$. We denote $b = a^{1/t}$.
Throughout the paper, $n \geq 1$ is a given integer. The set of $n \times n$ matrices over G is denoted by M_n. We introduce a further binary operation $\oplus$ on G by the formula

$$a \oplus b = \max(a, b) \quad \text{for all } a, b \in G.$$

The triple $(G, \oplus, \otimes)$ is called *max-algebra*. If $G = (G, \otimes, \leq)$ is additive group of real numbers, then $(G, \oplus, \otimes)$ is called *max-plus algebra* (often used in applications).
The operations $\oplus, \otimes$ are extended to the matrix-vector algebra over G by the direct analogy to the conventional linear algebra. For $A = (a_{ij}) \in M_n$, the problem of finding $x \in G^m$, $\lambda \in G$, satisfying

$$A \otimes x = \lambda \otimes x$$

is called an extremal eigenproblem corresponding to the matrix A; here λ and x are usually called an extremal eigenvalue and an extremal eigenvector of A, respectively. Throughout the paper, we will omit the word "extremal". This problem was treated by several authors during the sixties, e.g. [3, 5, 14], survey of the results concerning this and similar eigenproblems can be found in [15, 16]. Below, we summarize and recall some of the main results.

First we introduce the necessary notation. Let $N = \{1, 2, \ldots, n\}$ and let C_n be the set of all cyclic permutations defined on non-empty subsets of N. For a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_l) \in$

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C_n and for $A \in M_n$, we denote l, the length of σ by $l(\sigma)$ and define

$$w_A(\sigma) = a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \ldots \otimes a_{i_l i_1}$$
$$\mu_A(\sigma) = w_A(\sigma)^{1/l(\sigma)}$$
$$\lambda(A) = \sum_{\sigma \in C_n}^{\oplus} \mu_A(\sigma)$$

where $\sum^{\oplus}$ denotes the iterated use of the operation $\oplus$.

Theorem 1.1 [4] *Let $A \in M_n$. Then $\lambda(A)$ is the unique eigenvalue of A.*

The problem of finding the eigenvalue $\lambda(A)$ is also called the maximum-cycle mean problem and it has been studied by several authors [2, 3, 5, 10, 7, 12, 13]. Various algorithms for solving this problem are known, that of Karp [10] having the best worst-case performance $O(n^3)$.

For $B \in M_n$ we denote by $\Delta(B)$ the matrix

$$B \oplus B^2 \oplus \ldots \oplus B^n$$

where B^s stands for the s-fold iterated product $B \otimes B \otimes \ldots \otimes B$.

Let $A_\lambda = \lambda(A)^{-1} \otimes A$. (The upper index -1 denotes the inverse element of $\lambda(A)$ in the sense of the group operation $\otimes$.) It is shown in [4] that the matrix $\Delta(A_\lambda)$ contains at least one column, the diagonal element of which is e. Every such a column is an eigenvector of the matrix A, it is called a *fundamental eigenvector* of the matrix A. The set of all fundamental eigenvectors will be denoted by F_A and its cardinality is denoted by $q = |F_A|$. We say that $x, y \in F_A$ are equivalent if $x = \alpha \otimes y$ for some $\alpha \in G$. In what follows $s(A)$ denotes the set of all eigenvectors of A, so called *eigenspace* of A.

Theorem 1.2 [4] *Let $A \in M_n$. Then*

$$s(A) = \left\{ \sum_{i=1}^{q} \alpha_i \otimes g_i;\ \alpha_i \in G,\ g_i \in F_A,\ i = 1, 2, \ldots, q \right\}.$$

It follows from the definition of equivalent fundamental eigenvectors that the set F_A in *Theorem 1.2* can be replaced by any maximal set F'_A of fundamental eigenvectors such that no two of them are equivalent. Every such set F'_A will be called a complete set of generators (of the eigenspace).

The symbol $D_A = (V, E)$ stands for a complete, arc-weighted digraph associated with A. The node set of D_A is N, and the weight of any arc (i, j) is a_{ij}. Throughout the paper, by a cycle in the digraph we mean an elementary cycle or a loop, and by path we mean a nontrivial elementary path, i.e. an elementary path containing at least one arc. Evidently, we will use the same notation, as well as the concept of weight, for both cycles and cyclic permutations. By a strongly connected component of D_A we mean a subdigraph $\mathcal{G} = (L, E \cap L^2)$ generated by a non-empty subset $L \subseteq V$ such that any two distinct vertices $x, y \in L$ are contained in a common cycle, and L is a maximal subset with this property. The set of all strongly connected components of D_A is denoted by $\mathcal{P}$. A strongly connected component $\mathcal{G}$ is called nontrivial, if there is a cycle of positive length in $\mathcal{G}$. In the opposite case $\mathcal{G}$ is called trivial. A cycle $\sigma \in C_n$ is *optimal*, if $\mu_A(\sigma) = \lambda(A)$, a node in D_A is called an *eigennode* if it is contained in at least one optimal cycle; E_A stands for the set of all eigennodes in D_A.

Theorem 1.3 [4] *Let $g_1, g_2, \ldots, g_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then*

(i) $j \in E_A$ if and only if $g_j \in F_A$

(ii) g_i, g_j are equivalent members of F_A if and only if the eigennodes i, j are contained in a common optimal cycle.

Let be $\Delta(A_\lambda) = (\xi_{ij})$. It follows from the definition of $\Delta(A_\lambda)$ that ξ_{ij} is the weight of a heaviest path from i to j in D_A. Hence, $\Delta(A_\lambda)$ can be computed in $O(n^3)$ operations using the Floyd-Warshall algorithm [11]. By trivial search and comparisons one can then find a complete set of fundamental eigenvectors among the columns of $\Delta(A_\lambda)$, using at most $O(n^3)$ operations.

2 Eigenproblem for monotone matrices

The aim of this section is to investigate the above eigenproblem in case when A is a monotone matrix.

Definition 1 *Matrix* $A = (a_{ij})$ *is called monotone if and only if* $a_{ij} \leq a_{i,j+1}$ *and* $a_{ij} \leq a_{i+1,j}$ *for all* $i, j \in N$, *i.e., the entries in every row and in every column are in non-decreasing order.*

Example 1 *The following* 3×3 *matrix* M *is monotone.*

$$\mathbf{M} = \begin{pmatrix} 0 & 3 & 4 \\ 2 & 3 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

Theorem 2.1 *Let* $A = (a_{ij})$ *be monotone matrix. Then*

$$\lambda(A) = \max_{i,j \in N}\{a_{ij}\}.$$

Proof. It is clear that $\lambda(A) \leq \max_{i,j\in N}\{a_{ij}\}$. Since matrix A is monotone than at least diagonal element (say a_{nn}) is equal to $\max_{i,j\in N}\{a_{ij}\}$ and assertion follows trivially. □

In further part of this section we suggest an algorithm for computing all eigenvectors of a given monotone matrix in $O(n^2)$ time, whereby w.l.o.g. we suppose that $G = R$. It is shown in [4] that under the assumption $\lambda(A) = 0$, the matrix $\Delta(A)$ is a metric matrix of the digraph D_A, i.e. any element ξ_{ij} is equal to the heaviest path from i to j, for $i \neq j$. Since the eigenvalue of a monotone matrix is equal to maximal entry of a matrix (*Theorem 2.1*), elements of the metric matrix $\Delta(A_\lambda)$ are non-positive. For computing of an eigenvector we will use small adapted known Dijkstra algorithm, which computes the shortest path problem

from node 1 to i, for all i, $1 \leq i \leq n$ in $O(n^2)$ steps. But at first some property of monotone matrices.

Let D_A be a complete, arc weighted digraph associated with a given matrix A. The symbol D_A^h stands for digraph (called *threshold digraph*) with vertex set $V = \{1, 2, \ldots, n\}$ and (i,j) is from edge set E if $a_{ij} \geq h$. In case when the eigenvalue $\lambda(A)$ of given matrix is equal to maximal element of matrix (denote it by d) the corresponding subdigraph D_A^d of D_A contains the set $\mathcal{P}$ of all strongly connected components. We say that two vertices $i, j \in D_A$ are *highly connected*, in notation: $i \equiv_d j$, if i, j are contained in a cycle σ with the maximal cycle mean value $\mu(\sigma) = \lambda(A)$ (i, j are eigennodes). The subgraphs induced by the equivalence classes of $\equiv_d$ are called highly connected components in D_A, the set of all such components is denoted by $\mathcal{P}_A$.

Definition 2 *Let* A *be a matrix and* D_A^d *with* $\lambda(A) = 0$ *be a threshold digraph. Dimension of eigenspace* $dim(s(A)$ *is defined as cardinality of* $\mathcal{P}_A$.

From definition of dimension and a complete set of generators F_A' it is clear that $\dim(s(A)) = |F_A'|$.

Theorem 2.2 *Let* A *be monotone matrix. Then* $dim(s(A)) = 1$.

Proof. W.l.o.g. suppose that A is a given monotone matrix and that $\lambda(A) = 0$. Then $D_A^d = D_A^0$. From the definition of monotone matrix follows that the zero entries of matrix A are situated in right-down corner of A as the follows.

$$\mathbf{A} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Let a_{kk} be the first diagonal element of A equal to 0. Each element of the square submatrix B which contains last $n-k+1$ columns and last $n-k+1$ rows of A is zero matrix, or by words of graph theory D_B^0 is complete digraph, whereby all nodes of it lie on the same cycle. Moreover, the others cycles of D_A^0 (if there such cycles exist) contain at least one node from the set $\{k, k+1, \ldots, n\}$. But it creates the only one highly connected components in D_A^0. □

Theorem 2.3 *There is an algorithm $\mathcal{A}$ which, for a given monotone matrix $A \in M_n$, computes an eigenproblem in $O(n^2)$ time.*

Proof. The eigenvalue $\lambda(A)$ and the matrix A_λ can be computed in $O(n^2)$ time, in view of *Theorem 2.1*. From *Theorem 2.2* is clear that to each monotone matrix and corresponding eigenvalue generates the only one eigenvector. The others eigenvectors are equivalent. The matrix A_λ is non-positive, because the maximal cycle mean in the underling graph of A_λ is zero. Every maximal-weight path in D_{A_λ} is a minimal-weight path in D_{-A_λ} and the problem can be solved by small adaptation of Dijkstra algorithm [6]. □

3 Eigenproblem for Toeplitz matrix

The results in this section are quite similar to those presented in the previous section for monotone matrices. However, the situation with Toeplitz matrices in not completely analogous, and our results concern only Toeplitz matrices with restrictive conditions.

Definition 3 *A matrix $A = (a_{ij})$ is a Toeplitz matrix (generated by element $a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}$), if there is a function $a : \{-n+1, \ldots, n-1\} \to R$ such that $a_{ij} = a(i-j)$, i.e. if denote*

$a(i-j) = a_{i-j}$ the matrix A has the form

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ldots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{-n+2} & a_{-n+3} & a_{-n+4} & \ldots & a_0 & a_1 \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \ldots & a_{-1} & a_0 \end{pmatrix}$$

A Toeplitz matrix is known if its first row and first column are known. The function a essentially contains this information.

Example 2 *The following 4×4 matrix A is a (symmetric) Toeplitz matrix generated by elements $0,4,0,2,0,4,0$:*

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$$

Here is the first condition which ensures the better computational complexity for solvability of the eigenproblem. We denote

$$h^+(A) := \max\{a_i;\ 0 \le i < n\},\quad h^-(A) := \max\{a_i;\ -n < i \le 0\}$$
$$h(A) := \min\{h^+(A),\ h^-(A)\}$$

$$I^+(A) := \{i;\ 0 \le i < n \wedge a_i \ge h(A)\}$$
$$I^-(A) := \{i;\ -n < i \le 0 \wedge a_i \ge h(A)\}$$

Definition 4 *Let A be a Toeplitz matrix generated by elements $a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}$. We define conditions of the matrix A*

$$\begin{array}{ll} & h^+(A) = h^-(A) \\ (C) & \text{if } i \in I^+(A) \wedge a_i = h(A) \text{ then } a_{i+1} = h(A),\ 0 \le i \le n-2 \\ & \text{if } i \in I^-(A) \wedge a_i = h(A) \text{ then } a_{i-1} = h(A),\ -n+2 \le i \le 0 \end{array}$$

It can be easily seen that for any Toeplitz matrix fulfilling the conditions (C) the eigenvalue $\lambda(A)$ is the maximal element in A, because

$$\max_{i,j\in N}\{a_{ij}\} = \frac{a_{1n}+a_{n1}}{2} \le \lambda(A) \le \max_{i,j\in N}\{a_{ij}\}.$$

Such we have.

Theorem 3.1 *Let $A = (a_{ij})$ be a Toeplitz matrix fulfilling the conditions (C). Then*

$$\lambda(A) = \max_{i,j\in N}\{a_{ij}\}.$$

By the consideration from first section, the computation of one eigenvector can be done in $O(n^2)$ time, if matrix is monotone. We shall show that the same amount of time is sufficient for solvability of the eigenproblem for Toeplitz matrix with conditions (C).

Theorem 3.2 *Let A be a Toeplitz matrix fulfilling the conditions (C). Then*

$$\dim(s(A)) = 1.$$

Proof. W.l.o.g. suppose that A is a Toeplitz matrix fulfilling the conditions (C) and $\lambda(A) = 0$. Then $D_A^d = D_A^0$. From the definition of a Toeplitz matrix and conditions (C) follows that the zero entries of matrix A are situated in right-upper corner and in left-down corner of A as follows.

$$\mathbf{A} = \begin{pmatrix} \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ddots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots \end{pmatrix}$$

Let a_{1k} and $a_{l1}, k \le l$ be the first element equal to 0 in first row and first column of A, respectively. The structure of D_A^0 is as the following. Since there exist 2-cycles

$$(1i),\ k \le i \le n;\ (2,i),\ k+1 \le i \le n; \ldots;\ (k,n),$$

i.e.

$$1 \equiv_0 i,\ k \le i \le n;\ 2 \equiv_0 i,\ k+1 \le i \le n; \ldots;\ k \equiv_0 n,$$

each node from the set $\{1,2,\ldots,n-k+1,k,k+1,\ldots,n\}$ lies on at least one 2-cycle. It is clear that digraph D_A^0 is connected and each optimal cycle in D_A^0 contains at least one element from the set $\{1,2,\ldots,n-k+1,k,k+1,\ldots,n\}$. It creates the only one highly connected component and the assertion follows. □

Theorem 3.3 *There is an algorithm $\mathcal{A}$ which, for a given Toeplitz matrix $A \in M_n$ fulfilling the conditions (C), computes an eigenproblem in $O(n^2)$ time.*

Proof. The eigenvalue $\lambda(A)$ and the matrix A_λ can be computed in $O(n^2)$ time, in view of *Theorem 3.1*. From *Theorem 3.2* is clear that each Toeplitz matrix fulfilling the conditions (C) and corresponding eigenvalue generate the only one eigenvector. The others eigenvectors are equivalent. The remaining part of the proof runs similarly as proof of *Theorem 2.3*. □

Remark 1 A matrix $A = (a_{ij})$ is called a *circulant* if there is a function $g : \{0,1,\ldots,n-1\} \to R$ such that $a_{ij} = g(i-j)$ mod n. The similar results ($O(n^2)$-algorithm) were received for the eigenproblem of circulant matrix [12].

4 Conclusion

We have enlarged the class of matrices which allow a faster solution of the eigenproblem. Concerning future research, we would like to identify other classes of Toeplitz matrices, properties of which offer faster algorithms for solvable of it.

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