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Дубна

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Č.Burdik, O.Navrátil

NONLINEAR SUPERPOSITION FORMULAS BASED ON LIE GROUP $SO(n+1, n)$

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1. Introduction

Let us consider a system of n first–order differential equations

$$\dot{x}^{\mu}(t) = \chi^{\mu}(x^1, x^2, \ldots, x^n, t), \quad \mu = 1, \ldots, n, \tag{1}$$

where the dot denotes differentiation $x^{\mu}(t)$ with respect to time t. It is known for a very long time that, in some cases which we will specify later, it is possible to express the general solution as a nonlinear function of a finite number of particular solutions; it is of the form

$$\mathbf{x} = \mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m, c_1, \ldots, c_s), \qquad \mathbf{x} \in \mathbf{R}^n, \tag{2}$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are particular solutions (1), $c_1, c_2, \ldots, c_s$ are arbitrary constants, and $\mathbf{x}(t)$ is the general solution. These relations are called nonlinear superposition formulas. Here $\mathbf{x}$ means a vector with elements $x^1, x^2, \ldots, x^n$.

An example of such systems is a homogeneous system of the first order linear differential equations, in which the general solution is expressed as a linear combination of n linearly independent particular solutions. The other known example is the Riccati equation

$$\dot{x} = a(t) + b(t)x + c(t)x^2, \tag{3}$$

where $a(t)$, $b(t)$, $c(t)$ are continuous differentiable functions with respect to t. In this case, for any four solutions $x_i(t)$, $i = 1, \ldots, 4$, the relation

$$\frac{x_1(t) - x_3(t)}{x_1(t) - x_4(t)} \cdot \frac{x_2(t) - x_4(t)}{x_2(t) - x_3(t)} = \frac{u_1 - u_3}{u_1 - u_4} \cdot \frac{u_2 - u_4}{u_2 - u_3}, \tag{4}$$

where $x_i(0) = u_i$ are initial conditions, is valid.

In the general case, such systems of differential equations are connected with the local Lie group G of transformations on a factor space $M = G/G_0$, where G_0 is a Lie subalgebra of G [1]. We recall this connection briefly.

By the local Lie group G of transformations on the M, we understand a smooth mapping $\varphi : G \times M \to M$, (we use the abbreviation $\varphi(g, u) = g \cdot u$), for which: a) $e \cdot u = u$, for any $u \in M$, where e is the unit element of the group G, b) for any two elements $g_1, g_2 \in G$ and any $u \in M$ is $g_2 \cdot (g_1 \cdot u) = (g_2 g_1) \cdot u$ and c) $g \cdot u = u$ for any $u \in M$ imply $g = e$ [2].

In the local coordinate system, we write $x = g \cdot u$ as

$$x^{\mu} = f^{\mu}(a^1, \ldots, a^N, u^1, \ldots, u^n), \quad \mu = 1, \ldots, n, \tag{5}$$

where N is a dimension of the group G and a^r, $r = 1, \ldots, N$, are their local coordinates. For $x^{\mu}(\mathrm{a}, \mathrm{u})$, we can write

$$\frac{\partial x^{\mu}}{\partial a^r} = \sum_{s=1}^{N} \xi_s^{\mu}(\mathrm{x}) v_r^s(\mathrm{a}), \tag{6}$$

where the vector fields $X_s(\mathbf{x}) = -\sum_{\mu=1}^{n} \xi_s^\mu(\mathbf{x}) \frac{\partial}{\partial x^\mu}$ obey the equations

$$[X_r, X_s] = \sum_{t=1}^{N} c_{rs}^t X_t \, ,$$

in which c_{rs}^t are structure constants of the Lie algebra of the group G.

Conversely, the vector fields $X_k(\mathbf{x})$ determine the infinitesimal action of the local Lie group on the space M uniquely [2].

Let $g(t) = (a^1(t), \ldots, a^N(t))$, $t \in \mathbf{R}$, be a curve in the Lie group G such that $g(0) = e$. This gives a curve in the space M. Derivation of equation (5) with respect to the parameter t gives, by using (6), the system of differential equations

$$\dot{x}^\mu = \sum_{r=1}^{N} \xi_r^\mu(\mathbf{x}) Z^r(t) \, , \quad \mu = 1, \ldots, n \, . \tag{7}$$

In our paper we will deal with such systems connected with the Lie group $\mathrm{SO}(n+1, n)$.

If the system of equations (1) has form (7), then there is a curve in some local coordinates on the Lie group G, which acts on the factor space M. In this case, it is possible to find the superposition formula [1]. Any particular solution of the system (7) can be written in the form

$$\mathbf{x}_k(t) = g(t) \cdot \mathbf{u}_k \, , \tag{8}$$

where $\mathbf{u}_k = \mathbf{x}_k(0)$ is the initial condition.

We express the action of the local group G by using action of this group to few points of the space M, which is supposed known. In principle, it means to find from the system of the equations

$$\begin{array}{l} \mathbf{x}_1 = \mathbf{f}(a^1, \ldots, a^N, \mathbf{u}_1) \\ \mathbf{x}_2 = \mathbf{f}(a^1, \ldots, a^N, \mathbf{u}_2) \\ \quad \cdots \\ \mathbf{x}_r = \mathbf{f}(a^1, \ldots, a^N, \mathbf{u}_r) \end{array} \tag{9}$$

the coordinates of the group a^i. To be able to find the group coordinates a^i, $i = 1, \ldots, N$, we should use the action of r points. It is evident that the number r must fulfil the inequality $nr \geq N$, where n is the dimension of M, and N is the dimension of the group G.

If we solve this, then we are able to express the elements of Lie group G by means of the known transformations of r points in the form

$$g = g(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r) \, , \tag{10}$$

therefore

$$\mathbf{x} = g(\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{u}_1, \ldots, \mathbf{u}_r) \cdot \mathbf{u} \tag{11}$$

holds. Now we see that formula (11) is invariant with respect to the action of the local group G.

If $\mathbf{x}_i(t)$, $i = 1, \ldots, r$, are known solutions of the system (7) for given functions $Z^r(t)$, then any other solution $\mathbf{x}(t)$ of that system is given by (11). Therefore the relation (11) is the superposition formula [1].

For example, the Riccati equation (3) is connected with the Lie group $G = SL(2)$ that acts on the space M as

$$g(t) \cdot u = x(t) = \frac{a_{21}(t) + u a_{22}(t)}{a_{11}(t) + u a_{12}(t)} \, ,$$

where we represented the elements of the group G by matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, with the determinant equals to 1.

In the eighties, there where a lot of papers [3]–[8] which were devoted to the systems of differential equations of this type and especially to finding the superposition formulas for these systems. For the most part, the authors study the systems, which are connected with Lie groups $\mathrm{SL}(n, \mathbf{R})$ or $\mathrm{SL}(n, \mathbf{C})$. In the general paper [1] the methods are described that are used for constructing the superposition formulas. They are demonstrated with examples, in which the Lie groups $\mathrm{SL}(n, \mathbf{R})$ and $\mathrm{O}(p+1, n-p+1)$ act on simple projective spaces.

In [3] the more general case of the projective matrix Riccati equation are studied for $\mathrm{SL}(2n, \mathbf{R})$ and $\mathrm{SP}(2n, \mathbf{R})$. In paper [4], the systems that arise from the action of $\mathrm{SL}(n, \mathbf{C})$ on the factor spaces $\mathrm{SL}(n, \mathbf{C})/\mathrm{O}(n, \mathbf{C})$ and $\mathrm{SL}(2n, \mathbf{C})/\mathrm{SP}(2n, \mathbf{C})$ are studied. Paper [5], is devoted to the superposition formulas for rectangular–matrix Riccati equations on the space $\mathrm{SL}(n+k, \mathbf{C})/P(k)$, where $P(k)$ are special maximal parabolic subgroups of the $\mathrm{SL}(n+k, \mathbf{C})$. In [6], the authors deal with systems connected with the Lie group $\mathrm{SU}(n, n)$, and in [7], th same method for $\mathrm{SO}(n, n)$ is used. Finally, in paper [8], the authors study the systems of equations that are connected with the action of Lie group $\mathrm{SL}(n, \mathbf{C})$ on the space $M = \mathrm{SL}(n, \mathbf{C})/G_0$, where G_0 is a special non–maximal parabolic subgroup.

The authors mostly used a set of special solutions for reconstructing the group action on the space M. This approach simplifies the solution of (9). But on the other hand, we are not able to use the resulting superposition formulas directly for constructing the solution of the system on the basis any set of particular solutions. By means of the action of groups elements, we must first transform our particular solutions to a special set of particular solutions used in superposition formulas.

In the next, we study the systems of equations (7) which are connected with the action of Lie group $\mathrm{SO}(n+1, n)$ on the space $M = \mathrm{SO}(n+1, n)/P$, where P is one of the maximal parabolic subgroups. To our knowledge systems of this type were not studied so far. In distinction to the papers cited above, we do not choose a special set of particular solutions for reconstructing the group action on M, or, in the terminology of paper [1], we will construct group invariants which give nonlinear superposition formulas in implicit form.

2. Lie group $\mathrm{SO}(n+1,n)$ and its Lie algebra

In this section, we fix the notation. The Lie group $\mathrm{SO}(n+1,n)$ is a group of real matrices $\mathbf{G}$ with dimension $(2n+1)\times(2n+1)$ that fulfil the equations

$$\mathbf{G}^T\cdot\sigma\cdot\mathbf{G}=\sigma\,, \qquad \text{where } \sigma=\begin{pmatrix}1&0&0\\0&0&\mathbf{I}\\0&\mathbf{I}&0\end{pmatrix}, \tag{12}$$

$\mathbf{G}^T$ denotes a transposed matrix and $\mathbf{I}$ is a unit matrix with dimension $n\times n$.

Matrix $\mathbf{G}$ is written in the form

$$\mathbf{G}=\begin{pmatrix}g_{11}&\mathbf{g}_{12}^T&\mathbf{g}_{13}^T\\ \mathbf{g}_{21}&\mathbf{G}_{22}&\mathbf{G}_{23}\\ \mathbf{g}_{31}&\mathbf{G}_{32}&\mathbf{G}_{33}\end{pmatrix}, \tag{13}$$

where $\mathbf{g}$ denotes a column vector, and $\mathbf{g}^T$ is its transposed vector represented as row. Therefore $\mathbf{g}\mathbf{g}^T$ is an $n\times n$ matrix, and $\mathbf{g}^T\mathbf{g}$ is an inner product.

If we insert (13) into (12) we obtain the following formulas for elements of the matrix $\mathbf{G}$

$$\begin{aligned}
&g_{11}^2+\mathbf{g}_{21}^T\mathbf{g}_{31}+\mathbf{g}_{31}^T\mathbf{g}_{21}=1\\
&g_{11}\mathbf{g}_{12}+\mathbf{G}_{22}^T\mathbf{g}_{31}+\mathbf{G}_{32}^T\mathbf{g}_{21}=0\\
&g_{11}\mathbf{g}_{13}+\mathbf{G}_{23}^T\mathbf{g}_{31}+\mathbf{G}_{33}^T\mathbf{g}_{21}=0\\
&\mathbf{g}_{12}\mathbf{g}_{12}^T+\mathbf{G}_{22}^T\mathbf{G}_{32}+\mathbf{G}_{32}^T\mathbf{G}_{22}=0\\
&\mathbf{g}_{13}\mathbf{g}_{13}^T+\mathbf{G}_{23}^T\mathbf{G}_{33}+\mathbf{G}_{33}^T\mathbf{G}_{23}=0\\
&\mathbf{g}_{12}\mathbf{g}_{13}^T+\mathbf{G}_{22}^T\mathbf{G}_{33}+\mathbf{G}_{32}^T\mathbf{G}_{23}=\mathbf{I}
\end{aligned} \tag{14}$$

In this realization the Lie algebra $\mathrm{so}(n+1,n)$ is given by real matrices

$$\mathbf{A}=\begin{pmatrix}0&\mathbf{x}^T&\mathbf{z}^T\\-\mathbf{z}&\mathbf{H}&\mathbf{W}\\-\mathbf{x}&\mathbf{Y}&-\mathbf{H}^T\end{pmatrix},$$

where $\mathbf{W}^T=-\mathbf{W}$ a $\mathbf{Y}^T=-\mathbf{Y}$.

In the group $\mathrm{SO}(n+1,n)$, we take a subgroup G_0 that is generated by the matrices

$$\mathbf{G}_0=\begin{pmatrix}1&0&0\\0&\mathbf{D}&0\\0&0&(\mathbf{D}^T)^{-1}\end{pmatrix}\cdot\begin{pmatrix}1&0&\mathbf{z}^T\\-\mathbf{z}&\mathbf{I}&\mathbf{Z}\\0&0&\mathbf{I}\end{pmatrix},$$

where the equality $\mathbf{Z}+\mathbf{Z}^T+\mathbf{z}\mathbf{z}^T=0$ holds. The factor space $M=G/G_0$ can be represented by the matrices

$$\Xi=\begin{pmatrix}1&\mathbf{x}^T&0\\0&\mathbf{I}&0\\-\mathbf{x}&\mathbf{X}&\mathbf{I}\end{pmatrix},$$

where $\mathbf{X}+\mathbf{X}^T+\mathbf{x}\mathbf{x}^T=0$. As coordinates on M, we choose $\mathbf{x}$ and the antisymmetric part of the matrix $\mathbf{X}$; this means that

$$\mathbf{Y}=\mathbf{X}+\frac{1}{2}\mathbf{x}\mathbf{x}^T. \tag{15}$$

The action of the group G on the factor space M in these coordinates can be obtained from the equation

$$\begin{pmatrix}g_{11}&\mathbf{g}_{12}^T&\mathbf{g}_{13}^T\\ \mathbf{g}_{21}&\mathbf{G}_{22}&\mathbf{G}_{23}\\ \mathbf{g}_{31}&\mathbf{G}_{32}&\mathbf{G}_{33}\end{pmatrix}\cdot\begin{pmatrix}1&\mathbf{u}^T&0\\0&\mathbf{I}&0\\-\mathbf{u}&\mathbf{U}&\mathbf{I}\end{pmatrix}=$$
$$=\begin{pmatrix}1&\mathbf{x}^T&0\\0&\mathbf{I}&0\\-\mathbf{x}&\mathbf{X}&\mathbf{I}\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&\mathbf{D}&0\\0&0&(\mathbf{D}^T)^{-1}\end{pmatrix}\cdot\begin{pmatrix}1&0&\mathbf{z}^T\\-\mathbf{z}&\mathbf{I}&\mathbf{Z}\\0&0&\mathbf{I}\end{pmatrix}.$$

If we compare the coefficients on both sides of this equation, we obtain the following formulas

$$\begin{aligned}
\mathbf{D}&=\mathbf{G}_{22}+\mathbf{g}_{21}\mathbf{u}^T+\mathbf{G}_{23}\mathbf{U}\\
\mathbf{x}&=(\mathbf{D}^T)^{-1}(\mathbf{g}_{12}+g_{11}\mathbf{u}+\mathbf{U}^T\mathbf{g}_{13})\\
\mathbf{X}&=(\mathbf{G}_{32}+\mathbf{g}_{31}\mathbf{u}^T+\mathbf{G}_{33}\mathbf{U})\mathbf{D}^{-1}
\end{aligned} \tag{16}$$

for the group elements. Because we restrict ourselves to the local Lie group, we can suppose that matrix $\mathbf{D}$ is invertible.

Starting with the action of this group on the space M and by using the expansion to the first order, we derive an explicit expression for vector fields, in the basis of the algebra $\mathrm{so}(n+1,n)$ in the representation $(T_gf)(m)=f(g^{-1}\cdot m)$. Specifically we get

$$\begin{aligned}
Y_{ij}&\mapsto -\frac{\partial}{\partial Y_{ij}}\\
x_i&\mapsto -\frac{\partial}{\partial x_i}-\frac{1}{2}\sum_{r=1}^{n}x_r\frac{\partial}{\partial Y_{ri}}\\
D_{ij}&\mapsto x_j\frac{\partial}{\partial x_i}-\sum_{r=1}^{n}Y_{ir}\frac{\partial}{\partial Y_{rj}}\\
z_i&\mapsto -\sum_{r=1}^{n}\left(Y_{ir}+\frac{x_ix_r}{2}\right)\frac{\partial}{\partial x_r}-\frac{1}{2}\sum_{r,s=1}^{n}Y_{ri}x_s\frac{\partial}{\partial Y_{rs}}\\
W_{ij}&\mapsto \sum_{r=1}^{n}(x_jY_{ri}-x_iY_{rj})\frac{\partial}{\partial x_r}-\sum_{r,s=1}^{n}Y_{ir}Y_{js}\frac{\partial}{\partial Y_{rs}},
\end{aligned}$$

where we define $Y_{rs}=-Y_{sr}$.

The system of differential equations (7) is in this case of the form

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{a}(t) - \mathbf{B}(t)\mathbf{x} - \mathbf{Y}\mathbf{c}(t) + \frac{1}{2}\mathbf{x}\mathbf{c}^T(t)\mathbf{x} - \mathbf{Y}\mathbf{C}(t)\mathbf{x} \\
\dot{\mathbf{Y}}(t) &= \mathbf{A}(t) + \frac{1}{2}(\mathbf{x}\mathbf{a}^T(t) - \mathbf{a}(t)\mathbf{x}^T) - (\mathbf{Y}\mathbf{B}(t) + \mathbf{B}^T(t)\mathbf{Y}) + \\
&\quad + \frac{1}{2}(\mathbf{Y}\mathbf{c}(t)\mathbf{x}^T + \mathbf{x}\mathbf{c}^T(t)\mathbf{Y}) - \mathbf{Y}\mathbf{C}(t)\mathbf{Y}
\end{aligned}
\tag{17}
$$

where $\mathbf{a}(t)$ and $\mathbf{c}(t)$ are vector functions differentiable with respect to t; $\mathbf{A}(t)$, $\mathbf{B}(t)$ and $\mathbf{C}(t)$ are differentiable matrix functions, for which $\mathbf{A}^T = -\mathbf{A}$, $\mathbf{C}^T = -\mathbf{C}$ and $\mathbf{Y}^T = -\mathbf{Y}$.

This system of differential equation is the same as in the paper [9], where is given classification of all systems of nonlinear ordinary differential equations of this type.

3. Representations of the action of the group by means of particular solutions

The system of equations (17) arises from the action of the Lie group $\mathrm{SO}(n+1,n)$ on the factor space $M = \mathrm{SO}(n+1,n)/G_0$. Therefore, one can find the superposition formula. The action of the group $\mathrm{SO}(n+1,n)$ on the space M is given by the relations (16). We try to express this action by means of any known solutions $\mathbf{x}_k = g \cdot \mathbf{u}_k$. If $\mathbf{x}_k(t)$ are solutions of the differential equations (17) with the initial conditions $\mathbf{x}_k(0) = \mathbf{u}_k$, we obtain the general solution $\mathbf{x}(t)$ with the initial condition $\mathbf{x}(0) = \mathbf{u}$ in the form

$$
\mathbf{x}(t) = G(\mathbf{x}_1(t), \ldots, \mathbf{x}_r(t), \mathbf{u}_1, \ldots, \mathbf{u}_r) \cdot \mathbf{u}\,,
$$

where $g(\mathbf{x}_1(t), \ldots, \mathbf{x}_r(t), \mathbf{u}_1, \ldots, \mathbf{u}_r) = g(t)$ is the expression of an element of the group G in terms of the known transformation.

Dimension of the group $\mathrm{SO}(n+1,n)$ is equal to $n(2n+1)$ and the dimension of the space $M = SO(n+1,n)/G_0$ is $\frac{n(n+1)}{2}$. So, to express N coordinates of the group elements by using the known solutions, we must know at least r solutions, where r fulfils the inequality $n(2n+1) \leq \frac{n(n+1)}{2} r$. Consequently, for $n = 1$, it is sufficient to known three particular solutions, and for $n > 1$, we must know at least, four particular solutions of the system (17).

By $\mathbf{x}_i$ we denote a vector, and by $\mathbf{X}_i = \mathbf{Y}_i - \frac{1}{2}\mathbf{x}_i\mathbf{x}_i^T$, a matrix for and i^{th} solution of the system of differential equations (17) with initial conditions $\mathbf{u}_i$ and $\mathbf{U}_i$; $\mathbf{D}_i$ is an invertible matrix defined for an appropriate solution by the first equation in (16). Further, we denote $\mathbf{u}_{ik} = \mathbf{u}_i - \mathbf{u}_k$, $\mathbf{x}_{ik} = \mathbf{x}_i - \mathbf{x}_k$. Similarly, we define $\mathbf{X}_{ik}$ and $\mathbf{U}_{ik}$. Temporarily we suppose that all matrices which we will use in our calculations are invertible.

From equation (16) we obtain some coordinates of elements of the group G. Readily we discover that for indices i and k, the relations

$$
\begin{aligned}
\mathbf{g}_{13} &= (\mathbf{U}_{ik}^T)^{-1} \cdot (\mathbf{D}_i^T\mathbf{x}_i - \mathbf{D}_k^T\mathbf{x}_k - g_{11}\mathbf{u}_{ik}) \\
\mathbf{G}_{23} &= (\mathbf{D}_i - \mathbf{D}_k - \mathbf{g}_{21}\mathbf{u}_{ik}^T)\cdot\mathbf{U}_{ik}^{-1} \\
\mathbf{G}_{33} &= (\mathbf{X}_i\mathbf{D}_i - \mathbf{X}_k\mathbf{D}_k - \mathbf{g}_{31}\mathbf{u}_{ik}^T) \cdot \mathbf{U}_{ik}^{-1} \\
\mathbf{g}_{12} &= \mathbf{D}_i^T\mathbf{x}_i - g_{11}\mathbf{u}_i - \mathbf{U}_i^T\mathbf{g}_{13} \\
\mathbf{G}_{22} &= \mathbf{D}_i - \mathbf{g}_{21}\mathbf{u}_i^T - \mathbf{G}_{23}\mathbf{U}_i \\
\mathbf{G}_{32} &= \mathbf{X}_i\mathbf{D}_i - \mathbf{g}_{31}\mathbf{u}_i^T - \mathbf{G}_{33}\mathbf{U}_i
\end{aligned}
$$

are valid. If we now apply these equations to (14), after a simple algebra we obtain the following formulas

$$
\begin{aligned}
&g_{11}^2 + \mathbf{g}_{21}^T\mathbf{g}_{31} + \mathbf{g}_{31}^T\mathbf{g}_{21} = 1 \\
&\mathbf{D}_i^T \cdot (g_{11}\mathbf{x}_i + \mathbf{g}_{31} + \mathbf{X}_i^T\mathbf{g}_{21}) = \mathbf{u}_i \\
&\mathbf{D}_i^T \cdot (\mathbf{x}_i\mathbf{x}_k^T + \mathbf{X}_i^T + \mathbf{X}_k) \cdot \mathbf{D}_k = \mathbf{u}_i\mathbf{u}_k^T + \mathbf{U}_i^T + \mathbf{U}_k
\end{aligned}
\tag{18}
$$

From the second equation in (18) we can express

$$
\begin{aligned}
\mathbf{g}_{21} &= -(\mathbf{X}_{ik}^T)^{-1}\left[g_{11}\mathbf{x}_i - (\mathbf{D}_i^T)^{-1}\mathbf{u}_i\right] - (\mathbf{X}_{ki}^T)^{-1}\left[g_{11}\mathbf{x}_k - (\mathbf{D}_k^T)^{-1}\mathbf{u}_k\right] \\
\mathbf{g}_{31} &= \mathbf{X}_k^T(\mathbf{X}_{ik}^T)^{-1}\left[g_{11}\mathbf{x}_i - (\mathbf{D}_i^T)^{-1}\mathbf{u}_i\right] + \mathbf{X}_i^T(\mathbf{X}_{ki}^T)^{-1}\left[g_{11}\mathbf{x}_k - (\mathbf{D}_k^T)^{-1}\mathbf{u}_k\right]
\end{aligned}
\tag{19}
$$

Next we use the notation

$$
\begin{aligned}
\Omega_{ik} &= \mathbf{x}_i\mathbf{x}_k^T + \mathbf{X}_i^T + \mathbf{X}_k \\
\omega_{ik} &= \mathbf{u}_i\mathbf{u}_k^T + \mathbf{U}_i^T + \mathbf{U}_k \\
\mathbf{h}_i &= g_{11}\mathbf{x}_i - (\mathbf{D}_i^T)^{-1}\mathbf{u}_i
\end{aligned}
\tag{20}
$$

It is easy to see that the relations

$$
\begin{aligned}
&\mathbf{x}_i\mathbf{x}_k^T + \Omega_{ik} + \Omega_{ki} = 0 \quad \text{and} \quad \Omega_{ik}^T = \Omega_{ki} \\
&\mathbf{u}_i\mathbf{u}_k^T + \omega_{ik} + \omega_{ki} = 0 \quad \text{and} \quad \omega_{ik}^T = \omega_{ki}
\end{aligned}
$$

are valid. By simple algebraic calculations from equations (18) and (19) for any i, j, and k, we derive the following formulas

$$
\mathbf{D}_i^T\Omega_{ik}\mathbf{D}_k = \omega_{ik} \tag{21}
$$

$$
\mathbf{h}_k + \mathbf{X}_{jk}^T \cdot (\mathbf{X}_{ij}^T)^{-1}\mathbf{h}_i + \mathbf{X}_{ik}^T \cdot (\mathbf{X}_{ji}^T)^{-1}\mathbf{h}_j = 0 \tag{22}
$$

$$
\begin{aligned}
&g_{11}^2 + \left[\mathbf{h}_k^T\mathbf{X}_{ik}^{-1}\mathbf{x}_k + \mathbf{h}_k^T\mathbf{X}_k i^{-1}\mathbf{x}_i\right] \cdot \left[\mathbf{x}_k^T(\mathbf{X}_{ik}^T)^{-1}\mathbf{h}_i + \mathbf{x}_i^T(\mathbf{X}_{ki}^T)^{-1}\mathbf{h}_k\right] - \\
&\quad -\mathbf{h}_i^T\mathbf{X}_{ik}^{-1}\Omega_{ik}(\mathbf{X}_{ki}^T)^{-1}\mathbf{h}_k - \mathbf{h}_k^T\mathbf{X}_{ki}^{-1}\Omega_{ki}(\mathbf{X}_{ik}^T)^{-1}\mathbf{h}_i = 1\,.
\end{aligned}
\tag{23}
$$

We found the system of equations from which it is possible to determine the matrix (13). Since we restricted ourselves to the neighbourhood of the point $t = 0$, we can suppose that matrices $\mathbf{D}_i(t)$ are invertible. From the relation (21) it follows that in this neighbourhood matrices $\Omega_{ik}(k)$ and ω_{ki} are simultaneously invertible or non-invertible. So, it is easy to see that, if matrix ω_{ik} is invertible, it is enough to know only one solution of this system $\mathbf{D}_i$ for one value of i. We can find other $\mathbf{D}_k$ from equation (21). If we know $\mathbf{D}_i$ explicitly, the equation (23) is a quadratic equation with respect to g_{11}. The odd equations can then give the matrices $\mathbf{D}_i$.

We enunciate the previous account in terms of the following theorem.

Theorem: Let $\mathbf{x}_i(t)$ and $\mathbf{Y}_i(t)$, $i = 1, 2, 3$, be three solutions of the equation (17), $\mathbf{X}_i(t) = \mathbf{Y}_i(t) - \frac{1}{2}\mathbf{x}_i\mathbf{x}_i^T$, $\mathbf{u}_i = \mathbf{x}_i(0)$, $\mathbf{U}_i = \mathbf{X}_i(0)$ and let the matrices $\mathbf{U}_{ik}$ be invertible. Then, there is a neighbourhood of the point $t = 0$, matrices $\mathbf{D}_i(t)$, $i = 1, 2, 3$, $\mathbf{D}_i(0) = \mathbf{I}$, and function $g_{11}(t)$, $g_{11}(0) = 1$ such, that, for them formulas (21), (22), and (23) are true.

4. The superposition formulas

In the previous sections, we formulated conditions (21)–(23) that are valid for any three solutions of the system (17). Although we are not able to solve the system (21)–(23) explicitly, it is possible to derive, from them, certain relations for solutions of the system of differential equations.

We suppose that we have five solutions of this system, and there exists the inverse matrix to ω_{ik} for any $i \neq k$, $i, k = 1, \ldots, 5$. Then from the equation (21), we obtain

$$\mathbf{D}_k^{-1}\Omega_{rk}^{-1}(\mathbf{D}_r^T)^{-1} = \omega_{rk}^{-1}.$$

If we now multiply the equation (21) by this equation from the left, and then, by (21) from the left for the couple (ri), we obtain equation

$$\mathbf{D}_i^T\Omega_{ik}\Omega_{rk}^{-1}\Omega_{ri}\mathbf{D}_i = \omega_{ik}\omega_{rk}^{-1}\omega_{ri}, \tag{24}$$

that is true for any i, k, and r. We multiply this equation further from the right by the inverse equation (24) for the ternary (sti). Then, we obtain

$$\mathbf{D}_i^{-1}\Omega_{ti}^{-1}\Omega_{ts}\Omega_{is}^{-1}\Omega_{ik}\Omega_{rk}^{-1}\Omega_{ri}\mathbf{D}_i = \omega_{ti}^{-1}\omega_{ts}\omega_{is}^{-1}\omega_{ik}\omega_{rk}^{-1}\omega_{ri},$$

which is true for any $(ikrst)$. If now we put $s = k$ in this equation, we obtain, for any four solutions of (17), the relation

$$\mathbf{D}_i^{-1}\Omega_{si}^{-1}\Omega_{sk}\Omega_{rk}^{-1}\Omega_{ri}\mathbf{D}_i = \omega_{si}^{-1}\omega_{sk}\omega_{rk}^{-1}\omega_{ri}. \tag{25}$$

It means that matrices $\Omega_{si}^{-1}\Omega_{sk}\Omega_{rk}^{-1}\Omega_{ri}$ and $\omega_{si}^{-1}\omega_{sk}\omega_{rk}^{-1}\omega_{ri}$ are similar. Therefore, all their invariants are identical.

Now, we will use this interesting property of the solution of differential equations (17) to obtain superposition formulas of the system (17) for small n.

For $n = 1$ we get the Lie group SO(2, 1). In this case, the vector $\mathbf{x}$ is reduced to the number x, and $\mathbf{Y} = 0$. From this we have $\mathbf{X}_i = -\frac{1}{2}x_i^2$ and $\Omega_{ik} = -\frac{1}{2}(x_i - x_k)^2$. In this case, equation (25) has after the extraction form

$$\frac{x_s - x_k}{x_s - x_i}\cdot\frac{x_r - x_i}{x_r - x_k} = \frac{u_s - u_k}{x_u - u_i}\cdot\frac{u_r - u_i}{u_r - u_k}.$$

That represents the well-known superposition formula (4) for the Riccati equation (3). This is a consequence of the isomorphism between so(2, 1) and sl(2).

Now, we will study the case $n = 2$, that is the Lie group SO(3, 2). In this case, we have

$$\mathbf{Y} = \begin{pmatrix} 0 & x_3 \\ -x_3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \frac{1}{2}\begin{pmatrix} -x_1^2 & -x_1x_2 + 2x_3 \\ -x_1x_2 - 2x_3 & -x_2^2 \end{pmatrix}.$$

If we denote the components of the i^{th} solution by $x_1^{(i)}$, $x_2^{(i)}$ a $x_3^{(i)}$, we obtain

$$\det(\mathbf{U}_{ik}) = \frac{1}{4}\left(4(u_3^{(i)} - u_3^{(k)})^2 - (u_2^{(i)}u_1^{(k)} - u_1^{(i)}u_2^{(k)})^2\right)$$

and the determinants of the matrices Ω_{ik} and ω_{ik} are

$$\det(\Omega_{ik}) = \frac{1}{4}\Delta_{ik}^2 \quad \text{and} \quad \det(\omega_{ik}) = \frac{1}{4}\delta_{ik}^2, \tag{26}$$

where

$$\Delta_{ik} = 2x_3^{(i)} - 2x_3^{(k)} + x_2^{(i)}x_1^{(k)} - x_1^{(i)}x_2^{(k)} \quad \text{and} \quad \delta_{ik} = 2u_3^{(i)} - 2u_3^{(k)} + u_2^{(i)}u_1^{(k)} - u_1^{(i)}u_2^{(k)}. \tag{27}$$

We see that the conditions for the matrices $\mathbf{U}_{ik}$ to be invertible imply the invertibility for matrices ω_{ik}.

If we now take the determinant in equation (25) we obtain for any four different solutions the equality

$$\frac{\Delta_{sk}}{\Delta_{si}}\cdot\frac{\Delta_{ri}}{\Delta_{rk}} = \frac{\delta_{sk}}{\delta_{si}}\cdot\frac{\delta_{ri}}{\delta_{rk}}. \tag{28}$$

As it was mentioned above, in this case, we can construct the general solution of the system (17) by using four particular solutions. Take now five solutions, for which from (28) we obtain independent equations

$$\frac{\Delta_{24}}{\Delta_{14}}\cdot\frac{\Delta_{13}}{\Delta_{23}} = \frac{\delta_{24}}{\delta_{14}}\cdot\frac{\delta_{13}}{\delta_{23}}, \qquad \frac{\Delta_{34}}{\Delta_{14}}\cdot\frac{\Delta_{12}}{\Delta_{23}} = \frac{\delta_{34}}{\delta_{14}}\cdot\frac{\delta_{12}}{\delta_{23}} \tag{29}$$

$$\frac{\Delta_{25}}{\Delta_{15}}\cdot\frac{\Delta_{13}}{\Delta_{23}} = \frac{\delta_{25}}{\delta_{15}}\cdot\frac{\delta_{13}}{\delta_{23}}, \qquad \frac{\Delta_{35}}{\Delta_{15}}\cdot\frac{\Delta_{12}}{\Delta_{23}} = \frac{\delta_{35}}{\delta_{15}}\cdot\frac{\delta_{12}}{\delta_{23}}, \qquad \frac{\Delta_{45}}{\Delta_{15}}\cdot\frac{\Delta_{12}}{\Delta_{24}} = \frac{\delta_{45}}{\delta_{15}}\cdot\frac{\delta_{12}}{\delta_{24}} \tag{30}$$

Equations (30) are understood as a system of linear equations for $x_1^{(5)}$, $x_2^{(5)}$, and $x_3^{(5)}$. This system has a solution when its determinant $D(t)$ is different from zero. By direct calculation for $t = 0$, we obtain

$$D(0) = (\delta_{15}-\delta_{25})(\delta_{13}-\delta_{14}+\delta_{34})-(\delta_{15}-\delta_{35})(\delta_{12}-\delta_{14}+\delta_{24})+(\delta_{15}-\delta_{45})(\delta_{12}-\delta_{13}+\delta_{23})\,. \tag{31}$$

The terms in the second parentheses do not depend on $u_3^{(i)}$ and are not identical zero. Because the terms in first parentheses depend on $u_3^{(i)} - u_3^{(k)}$, this determinant is not identical zero for any possible initial conditions $\mathbf{u}_i$ and $\mathbf{U}_i$. As the determinant $D(t)$ is given by the solutions of system (17) which are continuous and $D(0) \neq 0$, there is any neighbourhood of $t = 0$, in which the determinant $D(t) \neq 0$. We see that, in this neighbourhood we can determine, from the system of the equations (30), the solutions $x_1^{(5)}$, $x_2^{(5)}$ and $x_3^{(5)}$ of the system of differential equations (17) by using the particular solution $\mathbf{x}_i$, $\mathbf{Y}_i$ for $i = 1, \ldots, 4$. In other words, formulas (28) give the implicit form of nonlinear superposition formulas for the system of differential equations (17) which is connected with the action of the Lie group SO(3,2) on space M.

Comments: Equations (29) imply that four solutions are not fully independent. For example, if we know three solutions $\mathbf{x}_i$, $\mathbf{Y}_i$, $i = 1, 2, 3$, and from the fourth we know $x_3^{(4)}$, we can obtain $x_1^{(4)}$ and $x_2^{(4)}$ from (29). This is a consequence of the fact that the reconstruction of action of the group requires only 10 independent functions.

5. Conclusions

The main results of this paper are the following:

1. We constructed systems of first–order ordinary differential equations that arise from the infinitesimal action of the local Lie group $SO(n+1, n)$ on the factor–space $M = SO(n+1, n)/P$, where P is one of the maximal parabolic subgroups of $SO(n+1, n)$. These systems allow a superposition formula.

2. We found any set of invariants for these systems which are expressed in terms the solutions. These are invariants of matrices Ω_{ik}. The matrices Ω_{ik} play, in our case, a similar role as the matrix anharmonic ratio for projective matrix Riccati equations in [3].

3. In the case of SO(3,2), we proved that, from these invariant it is possible to find the general solution of our system on the basis of four particular solutions. Therefore, in this case, these set of invariants gives implicit nonlinear superposition formula. It is necessary to note, that, even though the local groups SO(3,2) and SP(4, $\mathbf{R}$) are isomorphic, our system of differential equations differs from the one studied in [3] for the Lie group SP(4, $\mathbf{R}$), because we use another maximal parabolic subgroup for constructing the space M.

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