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ON AN EQUIVALENCE OF THE CENTERED GAUSSIAN MEASURE IN L_2 WITH THE CORRELATION OPERATOR $\left(-\frac{d^2}{dx^2}\right)^{-1}$ AND THE CONDITIONAL WIENER MEASURE

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1 Introduction

In this paper, we prove the equivalence of the conditional Wiener measure w, defined in $C_0([0,1])$, and the centered Gaussian measure μ, defined $L_2(0,1)$, with the correlation operator $(-\frac{d^2}{dx^2})^{-1}$, taken with zero Dirichlet boundary conditions, so that for an arbitrary functional φ in $L_2(0,1)$ integrable over the measure μ the following integral

$$\int\limits_{C_0([0,1])} \varphi(x)dw(x)$$

is determined and it coincides with

$$\int\limits_{L_2(0,1)} \varphi(x)d\mu(x).$$

In addition, some properties of the measure w are proved. The present investigation is mainly stimulated by the fact that recently a number of papers was published devoted to the construction of invariant measures for nonlinear partial differential equations, such as a nonlinear Schrödinger equation (see, for example, [1-7]), in part of which the measure w is used [1-4] and in part the measure μ [6,7]. Therefore, it follows from the results of the present paper that in the indicated papers the same measure is in fact used for constructing invariant measures.

A number of papers is devoted to constructing the conditional Wiener and Gaussian measures (see, for example, [8-12]). In addition, in [10-12], the equivalence in a sense of the usual (unconditional) Wiener measure and a Gaussian measure is shown. The author of the present paper must remark that generally the opinion of specialists is known, according to which the measures w and μ are in fact equivalent; however, he does not know a paper where the questions studied here are considered.

2 Definitions. Formulation of the main result

We begin with some <u>Definitions</u>. In the present article, all quantities (variables, spaces, and so on) are real. Let $C = C_0([0,1])$ be the standard space of functions continuous on $[0,1]$ and becoming zero at the ends of this segment, with the norm $||g||_C = \max_{x\in[0,1]} |g(x)|$, and let $L_2 = L_2(0,1)$ be the standard Lebesque space consisting of functions square integrable over $(0,1)$, with the scalar product $(g,h)_{L_2} = \int_0^1 g(x)h(x)dx$ and the norm $||g||^2_{L_2} = (g,g)_{L_2}$; as it is well-known, L_2 is a Hilbert space. Let $C_0^\infty(0,1)$ be the linear space of functions infinitely differentiable in $[0,1]$ and becoming zero at the ends of this segment, and let Δ be the closure in L_2 of the operator $-\frac{d^2}{dx^2}$, taken with the domain $C_0^\infty(0,1)$. Then, it is well-known that Δ is a self-adjoint positive operator in L_2. Let $e_n(x) = \sqrt{2}\sin\pi(n+1)x$ and $\lambda_n = [\pi(n+1)]^2$ where $n = 0,1,2,\dots$. Then, λ_n and e_n are eigenvalues and corresponding eigenfunctions of the operator Δ; in addition, $\{e_n\}_{n=0,1,2,\dots}$ is an orthonormal basis in L_2. Finally, let $H_0^1 = H_0^1(0,1)$ be the standard Sobolev space being the completion of the space $C_0^\infty(0,1)$ taken with the norm $||g||^2_{H_0^1} = \int_0^1 [g'(x)]^2dx$; clearly, H_0^1 is a Hilbert space.

Now we briefly recall definitions of the measures μ and w. The measure μ can be constructed as follows. Let a positive integer N and a Borel set $F \subset R^{N+1}$ be arbitrary. The set $M \subset L_2$ of the kind

$$M = \{u \in L_2 : \ [(u,e_0)_{L_2},\dots,(u,e_N)_{L_2}] \in F\}$$

is called *cylindrical* in L_2. For the cylindrical set of the above kind we set

$$\mu(M) = (2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{k=0}^{N}\lambda_k x_k^2} dx_0 \dots dx_N.$$

Clearly, the family of all cylindrical subsets of the space L_2 is an algebra (it will be denoted by $\mathcal{A}_L$), on which, as one can easily verify, μ is an additive measure. Furthermore, since the operator Δ^{-1} is of trace class in L_2 (i. e. all its eigenvalues λ_n^{-1} are positive and $\sum_{n=0}^{\infty}\lambda_n^{-1} < \infty$), the measure μ is countably additive on $\mathcal{A}_L$ (see [9]), therefore it can be uniquely extended onto the minimal sigma-algebra $\mathcal{B}_L$ of subsets of L_2 containing $\mathcal{A}_L$, and this sigma-algebra $\mathcal{B}_L$ will be proved to be the Borel sigma-algebra in L_2. The measure μ defined on $\mathcal{B}_L$ is called the centered Gaussian measure in L_2 with the correlation operator Δ^{-1}.

Now we recall the definition of the measure w (a careful construction of this measure is presented in the next section). Let $p(x,t)$ be the fundamental solution of the heat equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t), \quad x \in R, \ t > 0,$$

i. e. $p(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$, where $x \in R$ and $t > 0$, so that

$$\int_{-\infty}^{\infty} p(x,t)dx \equiv 1 \tag{1}$$

and for $0 < t_1 < t_2$

$$\int_{-\infty}^{\infty} p(y,t_1)p(x-y,t_2-t_1)dy = p(x,t_2). \tag{2}$$

A set $M \subset C$ is called *cylindrical* in C if

$$M = \{u \in C : \ [u(t_1),u(t_2),\dots,u(t_N)] \in F\}$$

for some positive integer N, $0 < t_1 < t_2 < \dots < t_N < 1$, and a Borel set $F \subset R^N$. For the indicated cylindrical set M we set

$$w(M) = \sqrt{2\pi}\int_F e^{-\sum_{k=1}^{N+1} p(x_k - x_{k-1}, t_k - t_{k-1})} dx_1 \dots dx_N,$$

where $x_0 = x_{N+1} = 0$, $t_0 = 0$ and $t_{N+1} = 1$. Using (1) and (2), one can easily verify that the set $\mathcal{A}_C$ of all cylindrical subsets of C is an algebra, on which w is an additive measure; in addition $w(C) = 1$. It is known that the measure w is countably additive on $\mathcal{A}_C$, too (see the next section), therefore it can be uniquely extended onto the minimal sigma-algebra $\mathcal{B}_C$ of subsets of C containing $\mathcal{A}_C$, and this sigma-algebra $\mathcal{B}_C$ will be proved to be the Borel sigma-algebra in C. The constructed measure w is called the *conditional Wiener measure*. For the convenience of readers, in what follows we present proofs of some of the indicated facts and, in addition, briefly consider some properties of w.

In the theorem below we formulate main results of the present paper.

Theorem.
(a) *For any* $A \in \mathcal{B}_L$, *the following takes place:* $A \cap C \in \mathcal{B}_C$ *and* $\mu(A) = w(A \cap C)$;
(b) *for any functional* φ *in* L_2 *integrated over the measure* μ, *the functional* φ *is also integrable in* C *over the measure* w *and*

$$\int_{L_2} \varphi(x) d\mu(x) = \int_{C} \varphi(x) dw(x).$$

Remark. One may interpret the statements (a) and (b) of this theorem as the equivalence of the measures μ and w.

This theorem will be proved in Section 4; in Section 3 we establish proofs to the construction of the measure w and prove some of its properties.

3 Auxiliary results. Constructing the measure w

The fact that $\mathcal{B}_L$ is the Borel sigma-algebra in L_2 follows from the following.

Lemma 1. *Let* $B_R(a) = \{u \in L_2 : \ ||u - a||_{L_2} \leq R\}$, *where* $R > 0$ *and* $a \in L_2$. *Then* $B_R(a) \in \mathcal{B}_L$.

Proof follows from the representation $B_R(a) = \bigcap_{n=1}^{\infty} M_n$ where

$$M_n = \left\{u \in L_2 : \ (u, e_0)^2_{L_2} + ... + (u, e_n)^2_{L_2} \leq R^2\right\} \square$$

In what follows in this section, we construct the measure w and consider some of its properties. We exploit methods introduced in [11] for studying the usual (unconditional) Wiener measure. So, w is an additive measure on the algebra $\mathcal{A}_C$. Let $w^\star$ be the outer measure corresponding to w, i. e. for any $A \subset C$ $w^\star(A) = \inf_{A \subset \bigcup_k M_k} \sum_k w(M_k)$ where the infimum is taken over all finite and countable coverings of the set A by sets M_k from $\mathcal{A}_C$. Let also

$$I_{\alpha,a,k,n} = \left\{u \in C : \ \left|u\left(\frac{k}{2^n}\right) - u\left(\frac{k-1}{2^n}\right)\right| > a 2^{-\alpha n}\right\},$$

where $k = 1, 2, ..., 2^n$, and

$$H_\alpha(a) = \left\{u \in C : \ \exists\ s_1 = \frac{k}{2^n}, s_2 = \frac{l}{2^m} \in [0,1] \text{ such that } s_1 \neq s_2 \text{ and } |u(s_1) - u(s_2)| > a|s_1 - s_2|^\alpha\right\},$$

where in both the cases, k, l, m and n are positive integer and $\alpha, a > 0$.

Lemma 2. *Let* $\alpha \in (0, \frac{1}{2})$ *and* $a > 0$. *Then* $w(I_{\alpha,a,k,n}) \leq 2^{\frac{n}{2}} e^{-\frac{1}{2}a^2 2^{n(1-2\alpha)}}$.

Proof. Clearly $I_{\alpha,a,k,n} \in \mathcal{A}_C$ so that the quantity $w(I_{\alpha,a,k,n})$ is determined. Further, $w(I_{\alpha,a,k,n}) =$

$$= \frac{1}{2\pi\left[\frac{k-1}{2^n}\frac{1}{2^n}\left(1 - \frac{k}{2^n}\right)\right]^{\frac{1}{2}}} \int_{|x-y| > a2^{-n\alpha}} e^{-\frac{1}{2}\left[\frac{2^n x^2}{k-1} + 2^n(x-y)^2 + \frac{y^2}{1-k2^{-n}}\right]} dx\, dy \leq$$

$$\leq \frac{2^{\frac{n}{2}} e^{-\frac{1}{2}a^2 2^{n(1-2\alpha)}}}{2\pi\left[\frac{k-1}{2^n}\left(1 - \frac{k}{2^n}\right)\right]^{\frac{1}{2}}} \int_{R^2} e^{-\frac{1}{2}\left[\frac{2^n x^2}{k-1} + \frac{y^2}{1-k2^{-n}}\right]} dx\, dy = 2^{\frac{n}{2}} e^{-\frac{1}{2}a^2 2^{n(1-2\alpha)}}. \square$$

Let $\alpha \in (0, \frac{1}{2})$. For any $a > 0$ we have $H_\alpha\left(\frac{2a}{1-2^{-\alpha}}\right) \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n} I_{\alpha,a,k,n}$ (for the proof, see [11]), therefore

$$w^\star\left(H_\alpha\left(\frac{2a}{1-2^{-\alpha}}\right)\right) \leq \sum_{n=1}^{\infty} 2^{n+\frac{n}{2}} e^{-\frac{1}{2}a^2 2^{n(1-2\alpha)}} < \infty$$

and, in addition,

$$w^\star\left(H_\alpha\left(\frac{2a}{1-2^{-\alpha}}\right)\right) \to 0 \text{ as } a \to \infty. \tag{3}$$

Also, it is clear that for any $a > 0$ $C \setminus H_\alpha(a)$ is a compact subset of C so that in particular $H_\alpha(a) \in \mathcal{B}_C$.

Lemma 3. *The measure w is countably additive on the algebra $\mathcal{A}_C$.*

Proof. It suffices to prove that, if $\{M_n\}_{n=1,2,3,...}$ is a sequence of cylindrical sets in $\mathcal{A}_C$, $M_1 \supset M_2 \supset ... \supset M_n \supset ...$ and $\bigcap_{n=1}^{\infty} M_n = \emptyset$, then $w(M_n) \to 0$ as $n \to \infty$. Take an arbitrary $\epsilon > 0$. By the known property of Borel measures, for any n there exists a closed cylindrical set $K_n \subset M_n$ such that $w(M_n \setminus K_n) < \epsilon 2^{-n-1}$. Let $L_n = \bigcap_{l=1}^{n} K_l$. Then for any n $w(M_n \setminus L_n) \leq \sum_{l=1}^{n} \epsilon 2^{-l-1} < \frac{\epsilon}{2}$, because $M_n \setminus L_n = M_n \setminus \left(\bigcap_{l=1}^{n} K_l\right) = \bigcup_{l=1}^{n} (M_n \setminus K_l) \subset \bigcup_{l=1}^{n} (M_l \setminus K_l)$. Now, it suffices to prove the existence of $n_0 > 0$ such that $w(L_{n_0}) < \frac{\epsilon}{2}$, because if this occurs, then $w(M_n) < \epsilon$ for $n \geq n_0$.

Fix an arbitrary $\alpha \in (0, \frac{1}{2})$ and take $a > 0$ such that $w^*(H_\alpha(a)) < \frac{\epsilon}{2}$. Then we obviously have, for any cylindrical set M satisfying $M \cap (C \setminus H_\alpha(a)) = \emptyset$, $w(M) \leq w^*(H_\alpha(a)) < \frac{\epsilon}{2}$. Let us prove that there exists a number $n_0 > 0$ such that $I_n := L_n \cap (C \setminus H_\alpha(a)) = \emptyset$ for $n \geq n_0$. Suppose the opposite. We have

$$I_1 \supset I_2 \supset ... \supset I_n \supset ... \text{ and } \bigcap_{n=1}^{\infty} I_n = \emptyset.$$

By the supposition, for each number $n \geq 1$ there exists $u_n \in I_n$. Since the set $C \setminus H_\alpha(a)$ is compact in C, there exists a subsequence $\{u_{n_k}\}_{k=1,2,3,...}$ of the sequence $\{u_n\}_{n=1,2,3,...}$ converging in C to some u. But then $u \in C \setminus H_\alpha(a)$ because $C \setminus H_\alpha(a)$ is closed.

Take an arbitrary number $l > 0$. Then $u_{n_k} \in I_l$ for all $n_k \geq l$. Since I_l is closed, we have $u \in I_l$. But then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ because $l > 0$ is arbitrary, i. e. we get a contradiction. Thus, it is proved that $I_n = \emptyset$ for all sufficiently large numbers n, which implies $w(L_n) < \frac{\epsilon}{2}$ for all sufficiently large n. Lemma 3 is proved.□

Lemma 4. *$\mathcal{B}_C$ is the Borel sigma-algebra in C.*

Proof. Let $R > 0$ and $a \in C$ be arbitrary. It suffices to prove that

$B_R(a) := \{u \in C : \|u - a\|_C \leq R\}$ belongs $\mathcal{B}_C$. But this follows by the relation $B_R(a) = \bigcap_{n=1}^{\infty} M_n$ where

$$M_n = \left\{u \in C : \left|u(k2^{-n}) - a(k2^{-n})\right| \leq R \;\forall\, k = 1, 2, ..., 2^n\right\}.□$$

So, the measure w is constructed. In conclusion of this section, we shall prove the following well-known important property of this measure. Let for $\alpha > 0$ $C^\alpha = \{u \in C : \exists\, a > 0$ such that $|u(x) - u(y)| \leq a|x - y|^\alpha \;\forall\, x, y \in [0, 1]\}$.

Proposition. *$C^\alpha \in \mathcal{B}_C$ for any $\alpha > 0$; in addition, $w(C^\alpha) = 1$ if $0 < \alpha < \frac{1}{2}$ and $w(C^\alpha) = 0$ if $\alpha > \frac{1}{2}$.*

Proof. Since $C^\alpha = \bigcup_{a=1}^{\infty} [C \setminus H_\alpha(a)]$ for $\alpha > 0$, the set C^α belongs to $\mathcal{B}_C$. In addition, $w(C^\alpha) = 1 - \lim_{a \to +\infty} w(H_\alpha(a)) = 1$ for $\alpha \in (0, \frac{1}{2})$.

Let now $\alpha > \frac{1}{2}$ and let

$$J_{\alpha,a,n} = \left\{u \in C : \left|u(k2^{-n}) - u((k-1)2^{-n})\right| \leq a2^{-n\alpha} \;\forall\, k = 1, 2, ..., 2^n\right\}$$

where $a > 0$. We have $C \setminus H_\alpha(a) \subset J_{\alpha,a,n}$ for any $n = 1, 2, 3, ...$ Estimating $e^{-\frac{1}{2}\frac{(x_{2^n-1} - x_{2^n-2})^2}{2^{-n}}}$ from above by 1 and making the change of variables $y_1 = x_1$, $y_2 = x_2 - x_1$, $y_3 = x_3 - x_2$, ... $y_{2^n-2} = x_{2^n-2} - x_{2^n-3}$, $y_{2^n-1} = x_{2^n-1}$, we obtain ($x_{2^n} = 0$):

$$w(J_{\alpha,a,n}) = \left[2\pi)^{2^n-1} 2^{-n2^n}\right]^{-\frac{1}{2}} \times$$

$$\times \int\limits_{\substack{|x_k - x_{k-1}| \leq a2^{-\alpha n} \\ k=1,2,...,2^n}} e^{-\frac{1}{2}\left[\frac{x_1^2}{2^{-n}} + \frac{(x_2 - x_1)^2}{2^{-n}} + ... + \frac{(x_{2^n-1} - x_{2^n-2})^2}{2^{-n}} + \frac{x_{2^n-1}^2}{2^{-n}}\right]} dx_1 ... dx_{2^n-1} \leq$$

$$\leq \left[(2\pi)^{2^n-1} 2^{-n2^n}\right]^{-\frac{1}{2}} \int\limits_{\substack{|y_k| \leq a2^{-\alpha n} \\ k=1,2,...,2^n-1}} e^{-\frac{1}{2}\sum_{k=1}^{2^n-1} \frac{y_k^2}{2^{-n}}} dy_1 ... dy_{2^n-1} =$$

$$= \left[\frac{1}{\sqrt{2\pi}} 2^{\frac{n}{2(2^n-1)}} \int\limits_{|z| \leq a2^{-\alpha n + \frac{n}{2}}} e^{-\frac{z^2}{2}} dz\right]^{2^n-1} \to 0 \text{ as } n \to \infty$$

because the integral in the expression in square brackets from the right-hand side of these relations tends to zero as $n \to \infty$. Consequently, since $C^\alpha = \bigcup_{a=1}^{\infty} [C \setminus H_\alpha(a)]$ and, as it is proved $w(C \setminus H_\alpha(a)) \le \lim_{n\to\infty} w(J_{\alpha,a,n}) = 0$ for any $a > 0$, we get $w(C^\alpha) = 0$ for $\alpha > \frac{1}{2}$.□

4 Proof of Theorem

Lemma 5. *$A \cap C \in \mathcal{B}_C$ for any $A \in \mathcal{B}_L$.*

Proof. Clearly, if $A \subset L_2$ is open, then $A \cap C$ is open in C. Therefore, if $A \subset L_2$ is closed, then $A \cap C = C \setminus [C \cap (L_2 \setminus A)]$ is closed C. Suppose the existence of $D \in \mathcal{B}_L$ such that $D \cap C \notin \mathcal{B}_C$. Consider the family S of all Borel subsets A of the space L_2 such that $A \cap C \in \mathcal{B}_C$. Then, in view of the above-described facts, S contains all open and closed subsets of L_2. Let us prove that S is a sigma-algebra.

Let $\{A_n\}_{n=1,2,3,...} \subset S$. It suffices to prove that $A = \bigcap_{n=1}^{\infty} A_n \in S$. But indeed $A \cap C = \bigcap_{n=1}^{\infty} (A_n \cap C) \in \mathcal{B}_C$. Hence, indeed S is a sigma-algebra containing all open and closed subsets of L_2. But $D \notin S$, i. e., S is contained in $\mathcal{B}_L$, and it does not coincide with $\mathcal{B}_L$. The obtained contradiction implies the statement of Lemma 5.□

Take an arbitrary positive integer N and consider on the segment $[0,1]$ the network S_N with the mesh width $h = 2^{-N}$ and nodes $x_k = kh,\ k = 0,1,...,2^N$. Consider the linear space L_N of all broken lines on S_N being linear functions between arbitrary two neighboring nodes of the network, continuous on $[0,1]$ and becoming zero at $x = 0$ and $x = 1$. On the linear space L_N, we consider the normalized to 1 nonnegative countably additive measure w_N defined by the rule: for any Borel set $F \subset R^{2^N-1}$ and $M = \{u \in L_N : [u(2^{-N}),...,u((2^N-1)\cdot 2^{-N})] \in F\}$ we set

$$w_N(M) = \left[(2\pi)^{2^N-1} 2^{-N2^N}\right]^{-\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{k=1}^{2^N} \frac{(x_k - x_{k-1})^2}{2^{-N}}} dx_1 ... dx_{2^N-1}$$

where $x_0 = x_{2^N} = 0$; clearly, for any fixed N the family of all subsets of L_N of the indicated kind is a sigma-algebra, on which w_N is a countably additive measure, and $w_N(L_N) = 1$.

Lemma 6. *The extension of the measure w_N on the space C, defined by the rule: $w_N(A) = w_N(A \cap L_N)$ for any $A \in \mathcal{B}_C$, is a well-defined nonnegative normalized to 1 Borel measure in C.*

Proof. It suffices to prove that if $A \in \mathcal{B}_C$, then $F_N = F_N(A) := \{[u(2^{-N}),...,u((2^N-1)\cdot 2^{-N})] : u \in A \cap L_N\}$ is a Borel subset of R^{2^N-1}. It is also clear that F_N is a Borel subset of R^{2^N-1} if and only if $A_N = A \cap L_N$ is a Borel set as a subset of L_N where the linear space L_N is equipped with the topology of the uniform convergence. Suppose the opposite, i. e., that there exists $A \in \mathcal{B}_C$ such that $A \cap L_N$ is not a Borel subset of L_N. It is easy to verify, as in the proof of Lemma 5, that all Borel subsets of the space C, the intersection of each of which with L_N is a Borel subset of this space, form a sigma-algebra $\mathcal{B}'$ in the space C containing all open and closed subsets of C. But according to our assumption, there exists $A \in \mathcal{B}_C$ not belonging to $\mathcal{B}'$ which is contradictorily.□

Lemma 7. *Let $w_N(A) = w_N(A \cap L_N)$ where $A \in \mathcal{B}_L$. Then, w_N becomes a nonnegative normalized to 1 Borel measure in L_2.*

Proof follows from Lemmas 5 and 6.□

In what follows, measures w_N, where $N = 1,2,3,...$, are considered as Borel measures in C or L_2, in the dependence on the context. Now we also recall that, in a complete separable metric space P, a sequence $\{\nu_n\}_{n=1,2,3,...}$ of nonnegative normalized to 1 Borel measures ν_n is called a weakly converging to a nonnegative normalized to 1 Borel measure ν if

$$\lim_{n\to\infty} \int_P \varphi(x) d\nu_n(x) = \int_P \varphi(x) d\nu(x)$$

for an arbitrary continuous and bounded functional φ in P.

Lemma 8. *The sequence of measures $\{w_N\}_{N=1,2,3,...}$ weakly converges to w in C.*

Proof. Fix arbitrary $\epsilon > 0$ and $\alpha \in (0, \frac{1}{2})$. Let us prove the existence of $a > 0$ such that for the set $K := C \setminus H_\alpha\left(\frac{2a}{1-2^{-\alpha}}\right) = \{u \in C^\alpha : |u(x) - u(y)| \le \frac{2a}{1-2^{-\alpha}} |x-y|^\alpha\ \forall\ x, y \in [0,1]\}$ the following takes place:

$$w_N(C \setminus K) < \epsilon,\ N = 1,2,3,... \qquad (4)$$

(the fact that the set K of the indicated kind is a Borel set in C is proved in Section 3). Choose $a > 0$ such that

$$\sum_{N=1}^{\infty} 2^{\frac{3N}{2}} e^{-\frac{1}{2}a^2 2^{N(1-2\alpha)}} < \epsilon \text{ for all } N = 1, 2, 3, \ldots \tag{5}$$

Take an arbitrary positive integer N and let K be the set introduced above corresponding to $a > 0$ which obeys (5). Consider the set $K_N = K \cap L_N$. As one can easily verify, for any positive integer $n > N$ and any $k = 1, 2, \ldots, 2^n$ the following takes place: $(I_{\alpha,a,k,n} \cap L_N) \subset \bigcup_{k=1}^{2^N} [I_{\alpha,a,k,N} \cap L_N]$.

Then, since as it is noted in Section 3, $(C \setminus K) \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n} I_{\alpha,a,k,n}$, we have:

$$((C \setminus K) \cap L_N) \subset \bigcup_{n=1}^{N} \bigcup_{k=1}^{2^n} (I_{\alpha,a,k,n} \cap L_N),$$

therefore

$$w_N(C \setminus K) = w_N((C \setminus K) \cap L_N) \leq \sum_{n=1}^{N} \sum_{k=1}^{2^n} w_N(I_{\alpha,a,k,n} \cap L_N).$$

One can prove as in the proof of Lemma 2 that for any $n = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, 2^n$ the following

$$w_N(I_{\alpha,a,k,n}) \leq 2^{\frac{n}{2}} e^{-\frac{1}{2}a^2 2^{n(1-2\alpha)}}$$

is valid. But this together with (5) yields that $w_N(L_N \setminus K_N) < \epsilon$, and the existence of the above-described set K satisfying (4) is proved.

It is clear that for any $\epsilon > 0$ the corresponding set K is compact in C. Hence, by the Prokhorov theorem [13] the sequence of measures $\{w_N\}_{N=1,2,3,\ldots}$ is weakly compact in C. Let w' be an arbitrary its limit point (in the sense of the weak convergence). Let us prove that $w' = w$. Suppose the opposite. Then in view of the uniqueness of the extension of a measure from an algebra onto the minimal sigma-algebra containing this algebra and since the minimal sigma-algebra containing the algebra of cylindrical subsets of C of the kind $\{u \in C : [u(2^{-N}), u(2 \cdot 2^{-N}), \ldots, u((2^N - 1) \cdot 2^{-N})] \in F\}$, where N and F are arbitrary positive integer and Borel subset of R^{N+1}, is the Borel sigma-algebra in C (see Lemma 4), there exists a cylindrical set $M = \{u \in C : [u(2^{-N}), u(2 \cdot 2^{-N}), \ldots, u((2^N - 1) \cdot 2^{-N})] \in F\}$ such that $w'(M) \neq w(M)$. In view of the known property of Borel measures, according to which a measure of a Borel set is equal to the infimum of measures of open sets containing this Borel set, we can accept that the set M is open.

For an arbitrary sufficiently small $\epsilon > 0$, consider a functional φ_ϵ in C continuous and such that

1. $\varphi_\epsilon(x) = 1$ if $x \in M$ and dist $(x, \partial M) \geq \epsilon$;
2. $0 \leq \varphi_\epsilon(x) \leq 1$ for any $x \in C$;
3. $\varphi_\epsilon(x) = 0$ for $x \notin M$;
4. $\varphi_\epsilon(x)$ depends only on $x(2^{-N}), x(2 \cdot 2^{-N}), \ldots, x((2^N - 1) \cdot 2^{-N})$.

We have for those subsequence $\{w_{N_k}\}_{k=1,2,3,\ldots}$ of the sequence $\{w_N\}_{N=1,2,3,\ldots}$, which weakly converges to w' in C:

$$\lim_{k \to \infty} \int_M \varphi_\epsilon(x) dw_{N_k}(x) = \int_M \varphi_\epsilon(x) dw'(x). \tag{6}$$

At the same time, according to the definition of an integral by integral sums,

$$\int_M \varphi_\epsilon(x) dw_N(x) = \int_M \varphi_\epsilon(x) dw(x) \tag{7}$$

for all sufficiently large N. But it is clear that the integrals in the right-hand sides of (6) and (7) are arbitrary close to $w'(M)$ and $w(M)$, respectively, for sufficiently small $\epsilon > 0$, i. e. they are different for sufficiently small $\epsilon > 0$. This contradiction proves the lemma.□

Lemma 9. *The sequence of measures $\{w_N\}_{N=1,2,3,\ldots}$ weakly converges to the measure μ in L_2.*

Proof. Take an arbitrary $\epsilon > 0$. It follows from the proof of Lemma 8 that there exists a compact set $K_\epsilon \subset L_2$ such that $w_N(L_2 \setminus K_\epsilon) < \epsilon$ for all $N = 1, 2, 3, \ldots$ (this set simply coincides with the set K from the proof of Lemma 8 corresponding to our ϵ). Let us also prove that for any cylindrical set

$$M = \{u \in L_2 : [(u, e_0)_{L_2}, \ldots, (u, e_N)_{L_2}] \in F\},$$

such that the set $F \subset R^{N+1}$ is bounded and the Lebesque $(N + 1)$-dimensional measure of its boundary ∂F is equal to zero, the following occurs:

$$\lim_{n \to \infty} w_n(M) = \mu(M). \tag{8}$$

Let P_n be the orthogonal projector in the space H_0^1 onto the subspace L_n and $e_i^n = P_n e_i$; clearly, for any i

$$e_i^n = e_i + \alpha_i^n \quad \text{where} \quad ||\alpha_i^n||_{H_0^1} \to 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

Also, for any n the set $M_n := M \cap L_n$ can be represented as follows:

$$M_n = \{u \in L_n : \ [(u, e_0)_{L_2}, ..., (u, e_N)_{L_2}] \in F\} =$$

$$= \{u \in L_n : \ [\lambda_0^{-1}(u, e_0)_{H_0^1}, ..., \lambda_N^{-1}(u, e_N)_{H_0^1}] \in F\} =$$

$$= \{u \in L_n : \ [\lambda_0^{-1}(u, e_0^n)_{H_0^1}, ..., \lambda_N^{-1}(u, e_N^n)_{H_0^1} \in F\}. \tag{10}$$

By L_n^N we denote the orthogonal complement, again in the sense of the space H_0^1, in the space L_n of the subspace span $\{e_0^n, ..., e_N^n\}$. Clearly, $\dim L_n = 2^n - 1$, therefore, since by (9) the vectors $e_0^n, ..., e_N^n$ are linearly independent for all sufficiently large n, we have $\dim L_n^N = 2^n - N - 2$ for the same n. For each sufficiently large n denote by $e_{N+1}^n, ..., e_{2^n-2}^n$ an arbitrary basis in the space L_n^N orthonormal in the sense of the scalar product of H_0^1. Then, for all sufficiently large n vectors $\{e_k^n\}_{k=0,1,...,2^n-2}$ form a basis in L_n, therefore for the same n and any $u \in L_n$

$$u = \sum_{k=0}^{n-2} x_k e_k^n. \tag{11}$$

In addition, it is easy to see that in L_n the transition from coordinates $[u(2^{-n}), u(2 \cdot 2^{-n}), ..., u((2^n - 1) \cdot 2^{-n})$ to coordinates $x = (x_1, x_2, ..., x_{2^n-1})$ from (11) of a broken line $u \in L_n$ for all sufficiently large n is made by a nondegenerate $(2^n - 1) \times (2^n - 1)$ matrix that is constant in L_n (it does not depend on u).

Let Λ be the diagonal $(N+1) \times (N+1)$ matrix with the principal diagonal $(\lambda_0, ..., \lambda_N)$. For any $\epsilon > 0$ by $\delta = \delta(\epsilon) > 0$ we denote a constant such that

$$(2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int\limits_{\{y \in R^{N+1}: \ \mathrm{dist}\ (y, \partial F) \le \delta\}} e^{-\frac{1}{2}\sum_{k=0}^{N} \lambda_k y_k^2} dy_0 ... dy_N < \epsilon$$

(this $\delta > 0$ exists due to the fact that the Lebesque measure of the boundary ∂F is equal to zero). Also, introduce the $(N+1) \times (N+1)$ matrix $Q_n = ((e_i^n, e_j^n)_{H_0^1})_{i,j=0,1,...,N}$. In view of (9) $Q_n \to \Lambda$ as $n \to \infty$.

Let $x^N = (x_0, ..., x_N) \in R^{N+1}$. Finally, let $F_\delta^1 = \{y \in F : \ \mathrm{dist}(y, \partial F) \ge \delta\}$, $F_\delta^2 = \{y \in R^{N+1} : \ \mathrm{dist}(y, F) \le \delta\}$, where $\delta > 0$, and $F^n =$

$$= \left\{ x^N \in R^{N+1} : \ \left[\lambda_0^{-1} \sum_{k=0}^{N} x_k (e_k^n, e_0^n)_{H_0^1}, ..., \lambda_N^{-1} \sum_{k=0}^{N} x_k (e_k^n, e_N^n)_{H_0^1} \right] \in F \right\}.$$

It is clear, first, that for all sufficiently large n the function $u \in L_n$ with coordinates $x \in R^{2^n-1}$ belongs to M_n if and only if $x^N \in F^n$, and, second, that, in view of (9) and (10), for any $\delta > 0$ and all sufficiently large n the following takes place:

$$F_\delta^1 \subset F^n \subset F_\delta^2. \tag{12}$$

Substituting the expansion (11) into the expression for $w_n(M_n)$, we obtain

$$w_n(M_n) = c_n (2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int\limits_{F^n} e^{-\frac{1}{2}(Q_n x^N, x^N)} dx_0 ... dx_N$$

for all sufficiently large n, where $c_n > 0$. From this, taking at first $M_n = L_n$, $n = 1, 2, 3, ...$, in view of the facts that $\lim_{n\to\infty} Q_n = \Lambda$ and $w_n(L_n) = 1$, we get

$$1 = \lim_{n \to \infty} \left\{ c_n (2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int\limits_{R^{N+1}} e^{-\frac{1}{2}\sum_{k=0}^{N} \lambda_k y_k^2} dy_0 ... dy_N \right\},$$

hence, $\lim_{n\to\infty} c_n = 1$. Further, taking an arbitrary $\epsilon > 0$ and choosing for M_n the above-described cylindrical set with a bounded F, the Lebesque measure of the boundary of which is equal to zero, in view of the above arguments, (12) and the property $Q_n \to \Lambda$ as $n \to \infty$, we derive

$$(2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int\limits_{F_{\delta(\epsilon)}^1} e^{-\frac{1}{2}\sum_{k=0}^{N} \lambda_k x_k^2} dx_0 ... dx_N \le \liminf_{n\to\infty} w_n(M_n) \le$$

$$\le \limsup_{n\to\infty} w_n(M_n) \le (2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int\limits_{F_{\delta(\epsilon)}^2} e^{-\frac{1}{2}\sum_{k=0}^{N} \lambda_k x_k^2} dx_0 ... dx_N.$$

By construction, the absolute value of the difference between the left-hand and right-hand side of these inequalities is smaller than 2ϵ, therefore, since $\epsilon > 0$ is arbitrary, we deduce that

$$\lim_{n\to\infty} w_n(M_n) = (2\pi)^{-\frac{N+1}{2}} \prod_{k=0}^{N} \lambda_k^{\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{k=0}^{N}\lambda_k x_k^2} dx_0...dx_N = \mu(M),$$

and the property (8) we need in is proved.

Taking now into account the fact that, as one can easily verify, the minimal sigma-algebra containing the algebra of all cylindrical sets from $\mathcal{A}_L$ with bounded sets F, the Lebesque measures of the boundaries of which are equal to zero, is the Borel sigma-algebra in L_2 (see Lemma 1), the further proof of the present lemma is analogous to the proof of Lemma 8.□

As a corollary to Lemmas 8 and 9, we establish the following well-known result (see [13]).

Corollary 1. $\liminf_{n\to\infty} w_n(A) \geq \mu(A)$ *for any open* $A \subset L_2$ *and* $\liminf_{n\to\infty} w_n(A) \geq w(A)$ *for any open* $A \subset C$. $\limsup_{n\to\infty} w_n(A) \leq \mu(A)$ *for any closed* $A \subset L_2$ *and* $\limsup_{n\to\infty} w_n(A) \leq w(A)$ *for any closed* $A \subset C$.

Lemma 10. $\mu(A) = w(A \cap C)$ *for any open* $A \subset L_2$.

Proof. Let $A \subset L_2$ be an open set. Then, $A \cap C$ is open in C. As earlier, for each $\epsilon > 0$, there exist a set $K \subset A$ compact in L_2 such that $\mu(A \setminus K) < \epsilon$ and a set $K_1 \subset A \cap C$ compact in C such that $w((A\cap C)\setminus K_1) < \epsilon$. Let $K_\epsilon = K \cup K_1$. Then, it is clear that K_ϵ is compact in L_2, $\mu(A \setminus K_\epsilon) < \epsilon$, and that $(K_\epsilon \cap C) \subset (A\cap C)$. Clearly, there exists a covering of K_ϵ in L_2 by open balls $B_{R_i}(a_i)$, $R_i > 0$, $a_i \in L_2$, $i = 1, 2, ..., l$, such that $\overline{B} = \bigcup_{i=1}^{l} \overline{B_{R_i}(a_i)} \subset A$ where $\overline{D}$ is the closure of a set D (here in L_2). Then, $\mu(A \setminus B) < \epsilon$, $(\overline{B}\cap C) \subset A\cap C$ and $w((A\cap C)\setminus(B\cap C)) < \epsilon$; in addition, $B \cap C$ is open and $\overline{B} \cap C$ is closed in C. In view of Corollary 1, we have:

$$\mu(A) - \epsilon < \mu(B) \leq \liminf_{n\to\infty} w_n(B) \leq$$

$$\leq \liminf_{n\to\infty} w_n(\overline{B} \cap C) \leq w(\overline{B} \cap C) \leq w(A \cap C)$$

and

$$w(A \cap C) - \epsilon < w(B \cap C) \leq \liminf_{n\to\infty} w_n(B \cap C) \leq$$

$$\leq \liminf_{n\to\infty} w_n(\overline{B}) \leq \mu(\overline{B}) \leq \mu(A),$$

which implies that $|\mu(A) - w(A\cap C)| < \epsilon$ and, in view of the arbitrariness of $\epsilon > 0$, we have $\mu(A) = w(A \cap C)$.□

Corollary 2. $\mu(A) = w(A \cap C)$ *for any* $A \subset L_2$ *closed in* L_2.

Let $A \in \mathcal{B}_L$ be arbitrary. Then, by the known property of Borel measures for any $\epsilon > 0$ there exists a set $B \supset A$, open in L_2, and a set $D \subset A$, closed in L_2, such that $\mu(B\setminus D) < \epsilon$. Hence, $(D\cap C) \subset (A\cap C) \subset (B \cap C)$ and

$$w((B \cap C) \setminus (D \cap C)) = \mu(B \setminus D) < \epsilon;$$

in addition, according to Lemma 10 and Corollary 2 $\mu(B) = w(B \cap C)$ and $\mu(D) = w(D \cap C)$. In view of the arbitrariness of $\epsilon > 0$, this yields that $\mu(A) = w(A \cap C)$. Thus, the statement (a) of Theorem is proved. The statement (b) follows from the definition of the Lebesque integral by integral sums. Theorem is completely proved.□

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