объединенный
институт
ядерных
исследований
дубна

E5 - 13019

T.D.Todorov

EXTENDED ASYMPTOTIC FUNCTIONS - SOME EXAMPLES*

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1. SOME EXAMPLES OF EXTENDED ASYMPTOTIC FUNCTIONS

The first example is very simple. It will illustrate the material of ref. /8/ Sec. 3. We shall discuss it in detail. The remaining examples are a bit more complicated, but the methods of their obtaining are the same as in the first one. That is why, there are no many explanations about them. Several examples of functions of two variables are given. They are treated, in fact, by the corresponding generalizations of the lemmas (ref. /8/ (3.8), (3.12),(3.19)). These examples will be used for the construction of asymptotic functions similar, in a certain sense, to the distributions, which is, by the way, our final aim /8/.

(1.1) EXAMPLE: $\phi(x)=1/x$, $x \in R \setminus \{0\}$. In this example $X = R\setminus\{0\}$, $= R$ and

$$X_{as} = (R \setminus \{0\})_{as} = A \setminus \Theta. \tag{1.2}$$

In fact, it is clear that $\phi_{as}(a) = 1/a$, $a \in A \setminus \Theta$. But we will obtain this result with the help of Lemmas (3.8), (3.12) and (3.19) of ref. /8/ in order to illustrate their use: Corresponding to Lemma (3.12) (for $\bar{x} = 0$, $m=-2$), we obtain

$$\phi_{as}(h) \subseteq o^{-2\mu_h}, \tag{1.3}$$

where μ_h is the power of h. In particular, for $h = s + o^1$ we have

$$\phi_{as}(s + o^1) \subseteq o^{-2}. \tag{1.4}$$

The expansion (/8/ (3.15)) reduces to

$$\phi(t) = 1/t, \quad t \in R, \ t \neq 0 \tag{1.5}$$

and the formula (/8/ (3.17)) reduces to:

$$\phi_{as}(h) = 1/h, \quad h \in \Omega_0 \setminus \Theta. \tag{1.6}$$

Lemma (3.19) of ref. /8/ gives (for $m_{\pm} = 0$):

$$\phi_{as}(a) \subseteq o^{\circ}, \qquad a \in \Omega_{\infty} \setminus \Theta. \tag{1.7}$$

In fact, we have

$$\phi_{as}(a) = 1/a, \qquad a \in \Omega_{\infty} \setminus \Theta. \tag{1.8}$$

Summarizing (1.3) and (1.8) we obtain:

$$\phi_{as}(a) = 1/a, \qquad a \in A \setminus \Theta. \qquad (1.9)$$

So, in the case under consideration, the domain D of ϕ coincides with X_{as}, i.e., $D = X_{as} = A \setminus \Theta$. For the power $\mu(a)$ the order $\nu(a)$ of $\phi_{as}(a)$ we have (see ref.[5] Theorem 12, (ii)):

$$\mu(a) = -\mu_a, \qquad a \in A \setminus \Theta, \qquad (1.10)$$

$$\nu(a) = \nu_a - 2\mu_a, \qquad a \in A \setminus \Theta, \qquad (1.11)$$

where μ_a and ν_a are the power and the order of a respectively.

<u>(1.12) EXAMPLE</u>: Let $\phi \in \mathfrak{L}$ or $\phi \in S$, where $\mathfrak{L}$ and S are the well-known spaces of test-functions[10]. Then $\phi_{as}(a)$ exists for all $a \in A$ and

$$\phi_{as}(a) = \begin{cases} \sum_{k=0}^{\infty} \frac{\phi^{(k)}(x)}{k!} h^k, & a = x + h,\ x \in R,\ h \in \Omega_0, \\ 0, & a \in \Omega_\infty. \end{cases} \qquad (1.13)$$

Remind once again that the convergence of the above series is in the sense of the interval topology of A (ref.[6] Sec. 5) and the series of the type

$$\sum_{k=\mu}^{\infty} \alpha_k h^k \qquad h \in \Omega_0 \qquad (1.14)$$

is always (for any $\alpha_k \in C$) convergent on the set of infinitesimals Ω_0 (ref.[6] Theorem 41).

<u>(1.15) EXAMPLE</u>: If $\phi \in C^{(\infty)}$, then $\phi_{as}(a)$ exists for all finite numbers, i.e., for all $a \in \Omega$, and

$$\phi_{as}(a) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(x)}{k!} h^k, \qquad a = x + h,\ x \in R,\ h \in \Omega_0. \qquad (1.16)$$

It could happen, of course, that some of the functions from $C^{(\infty)}$ to be extended on the whole A. For example, $\sin a \equiv (\sin)_{as}(a)$ exists for all $a \in A$ (but not only for $a \in \Omega$) and

$$\sin a = \begin{cases} \sum_{k=0}^{\infty} \frac{\sin^{(k)} x}{k!} h^k, & a = x + h,\ x \in R,\ x \in \Omega_0, \\ o^{-1}, & a \in \Omega_\infty. \end{cases} \qquad (1.17)$$

On the contrary, the asymptotic extension $e^a = (e^x)_{as}(a)$ of the function $e^x, x \in R$ exists for all finite numbers, i.e., for $a \in \Omega$, as well as for all infinitely large negative asymptotic numbers, but it does not exist for positive infinitely large numbers, i.e.,

$$e^a = \begin{cases} \sum_{k=0}^{\infty} \frac{e^x}{k!} h^k, & a = x + h,\ x \in R,\ h \in \Omega_0 \\ 0, & a \in \Omega_\infty^- . \end{cases} \qquad (1.18)$$

<u>(1.19) EXAMPLE</u>: Let

$$\Delta(x, \epsilon) = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}, \qquad x \in R,\ \epsilon \in (0,1). \qquad (1.20)$$

As it is known, (1.20) is a δ-sequence, i.e.,

$$\Delta(x, \epsilon) \xrightarrow[\epsilon \to 0]{\mathfrak{L}'} \delta(x), \qquad (1.21)$$

where $\delta(x)$ is the Dirac's δ-function. The asymptotic extension $\Delta_{as}(a, b)$ exists for all $a \in A$ and $b \in (0,1)_{as}$[8] (ref.[8] (2.18)). But we are going to give the values of $\Delta_{as}(a,b)$ only for the fixed $b = s + o^1$ (recall ref.[8] (2.26)) that the asymptotic number $s + o^1$ is a positive infinitesimal belonging to $(0,1)_{as}$) and all finite asymptotic numbers, i.e., for $a \in \Omega$:

$$\Delta_{as}(a, s + o^1) = \begin{cases} (\frac{1}{s} + o^{-1}) \frac{1}{\pi} \frac{1}{x^2 + 1}, & a = sx + sh,\ x \in R,\ h \in \Omega_0, \\ o^{-2}, & a = o^0, \\ (s + o^1) \frac{1}{\pi} \frac{1}{x^2}, & a = x + h,\ x \in R,\ x \neq 0,\ h \in \Omega_0. \end{cases} \qquad (1.22)$$

For the power $\mu(a)$ and the order $\nu(a)$ of $\Delta_{as}(a, s + o^1)$ we have:

$$\mu(a) = \nu(a) = \begin{cases} -1, & a = sx + sh \\ -2, & a = o^0 \\ 1, & a = x + h,\ x \in R,\ x \neq 0,\ h \in \Omega_0. \end{cases} \qquad (1.23)$$

Recall once again (ref.[8] (3.28)) that "s" in (1.22) and (1.27) is the asymptotic number "$s+o^{\infty}$", i.e., $s \equiv s+o^{\infty}$ (ref.[8] (3.27)). We can denote $\Delta_{as}(a, s+o^1)$ as a δ-function:

$$\delta(a) = \Delta_{as}(a, s+o^1), \qquad a \in \Omega. \tag{1.24}$$

Then (1.22) will imply

$$\delta(a) = \begin{cases} \text{is infinitely large when } a \text{ is an infinitesimal,} \\ \text{is infinitesimal when } a \text{ is finite and different} \\ \text{from infinitesimal} \end{cases} \tag{1.25}$$

which coincides with our intuitive idea about Dirac's δ-function. Let us find the n-th power of δ, where $n \in N$. (Do not forget that every two asymptotic functions can be added and multiplied (ref.[8] (1.12)):

$$\delta^n(a) = \begin{cases} (s^{-n}+o^{-n})\left(\frac{1}{\pi}\frac{1}{x^2+1}\right)^n, & a = sx+sh,\ x \in R,\ h \in \Omega_0, \\ o^{-2n}, & a = o^{\circ}, \\ (s^n+o^n)\left(\frac{1}{\pi}\frac{1}{x^2}\right)^n, & a = x+h,\ x \in R,\ x \neq 0,\ h \in \Omega_0. \end{cases} \tag{1.26}$$

After we introduce differentiation and integration for the asymptotic functions, we shall come back to these examples once again. You will see that δ given by (1.24) is a realization of Dirac delta-function (ref.[7] Sec. 9) and (1.26) is its n-th power.

<u>(1.27) EXAMPLE</u>: Consider the function

$$\Delta^{(n)}(x,\epsilon) = \frac{1}{\epsilon^{n+1}}\rho^{(n)}\left(\frac{x}{\epsilon}\right), \quad x \in R, \quad \epsilon \in (0,1), \tag{1.28}$$

where $n \in \{0,1,2,\dots\}$ and

$$\rho(x) = \frac{1}{\pi}\frac{1}{x^2+1}, \qquad x \in R. \tag{1.29}$$

It is clear that (1.28) is the n-th derivative with respect to x of the δ-sequence (1.20) from the previous example. It is not difficult to calculate: $\rho^{(n)}$:

$$\rho^{(n)}(x) = \left(\frac{x}{x^2+1}\right)^{n+1}\frac{(-1)^n n!}{\pi}\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}\frac{(-1)^k}{x^{2k+1}}, \quad x \in R. \tag{1.30}$$

The asymptotic extension of (1.28): $(\Delta^{(n)})_{as} = (a,b)$ exists for every $a \in A$ and every $b \in (0,1)_{as}$. But, as in the previous example, we will give the values of it only for fixed $b = s+o^1$ and finite a, i.e., for $a \in \Omega$:

$$(\Delta^{(n)})_{as}(a, s+o) = \begin{cases} (s^{-n-1}+o^{-n-1})\,\rho^{(n)}(x), & a = sx+sh,\ x \in R,\ h \in \Omega, \\ o^{-n-2}, & a = o^{\circ}, \\ (s+o^1)\frac{(-1)^n(n+1)!}{\pi}\frac{1}{x^{n+2}}, & \begin{cases} a = x+h,\ x \in R,\ x \neq 0, \\ h \in \Omega_0. \end{cases} \end{cases} \tag{1.31}$$

If we denote the power and the order of $(\Delta^{(n)})_{as}(a, s+o^1)$ by $\mu^{(n)}(a)$ and $\nu^{(n)}(a)$ respectively, then we shall obtain:

$$\mu^{(n)}(a) = \nu^{(n)}(a) = \begin{cases} -n-1, & a = sx+sh,\ x \in R,\ h \in \Omega_0, \\ -n-2, & a = o^{\circ}, \\ 1, & a = x+h,\ x \in R,\ x \neq 0,\ h \in \Omega_0. \end{cases} \tag{1.32}$$

<u>(1.33) EXAMPLE</u>: Let $n \in N$ and let us consider the (ordinary) functions:

$$P_n(x,\epsilon) = \left(\frac{x}{x^2+\epsilon^2}\right)^n \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\left(\frac{\epsilon}{x}\right)^{2k}, \quad x \in R,\ \epsilon \in (0,1). \tag{1.34}$$

As it is known[10] the limit of (1,34) by $\epsilon \to 0$ with respect to the topology of $\mathfrak{D}'$ exists and it defines the distribution $P(1/x^n)$, i.e.,

$$P_n(\cdot,\epsilon) \xrightarrow[\epsilon \to 0]{\mathfrak{D}'} P(1/x^n) \in \mathfrak{D}'. \tag{1.35}$$

Let us set (for the sake of convenience):

$$P_n(x,\epsilon) = \frac{1}{\epsilon^n}\psi_n\left(\frac{x}{\epsilon}\right), \qquad x \in R, \quad \epsilon \in (0,1), \tag{1.36}$$

where

$$\psi_n(x) = \left(\frac{x}{x^2+1}\right)^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \frac{1}{x^{2k}}, \qquad x \in R. \tag{1.37}$$

As in the previous two examples the asymptotic extension:

$$(P_n)_{as}(a,b) = \frac{1}{b}(\psi_n)_{as}\left(\frac{a}{b}\right) \tag{1.38}$$

exists for every $a \in A$ and $b \in (0,1)_{as}$ where $(\psi_n)_{as}$ is the asymptotic extension of ψ_n. We shall write down its value for $a \in \Omega$ and fixed $b = s + o^1$. We shall set first:

$$(x^{-n})(a) \stackrel{def}{=} (P_n)_{as}(a, s+o^1), \qquad a \in \Omega. \tag{1.39}$$

For the values of (1.39) we obtain:

$$x^{-n}(a) = \begin{cases} (s^{-n}+o^{-n})\psi_n(x), & a = sx + sh, \quad x \in R, \quad h \in \Omega_0, \\ o^{-n-1}, & a = o^\circ, \\ x^{-n} + o^\circ, & a = x + o^\circ, \quad x \in R, \quad x \neq 0, \\ x^{-n} - x^{-n-1} nys + o^1, & a = x + sy + o^1, \quad x, y \in R, \ x \neq 0, \\ x^{-n} - x^{-n-1} nys + \omega(a)s^2 + o^2, & a = x + sy + s^2 z + s^2 h, \ x, y, z \in R, \ x \neq 0, \ h \in \Omega_0, \end{cases} \tag{1.40}$$

where

$$\omega(a) = \frac{n(n-1)}{2}\left(\frac{y}{x^n} - \frac{1}{x^2}\right) - \frac{n}{x^{n-1}}\left(\frac{z}{x^2} - \frac{y}{x^3} + \frac{1}{x^3}\right),$$
$$a = x + ys + zs^2 + s^2 h, \quad x, y, z \in R, \ x \neq 0, \ h \in \Omega_0. \tag{1.41}$$

For the power $\mu_n(a)$ and the order $\nu_n(a)$ of $x^{-n}(a)$ we obtain:

$$\mu_n(a) = \begin{cases} -n, & a = sx + sh, \quad x \in R, \quad h \in \Omega_0, \\ -n-1, & a = o^\circ, \\ 0 & a = x + h, \ x \in R, \ x \neq 0 \end{cases} \tag{1.42}$$

and

$$\nu_n(a) = \begin{cases} -n, & a = sx + sh, \quad x \in R, \ h \in \Omega_0, \\ -n-1, & a = o^\circ, \\ 0, & a = x + o^\circ, \quad x \in R, \ x \neq 0, \\ 1, & a = x + sy + o^1, \ x, y \in R, \ x \neq 0, \\ 2, & a = x + sy + s^2 z + s^2 h, \ x, y, z \in R, \ x \neq 0, \ h \in \Omega_0. \end{cases} \tag{1.43}$$

After introducing differentiation and integration for the asymptotic functions, we shall see that the asymptotic function (1.39) (or, which is the same, (1.40)) is a realization (in the sense of ref. [7] Sec. 9) of the distribution $P(1/x^n)$.

<u>(1.44) EXAMPLE:</u> Let

$$H(x,\epsilon) = \frac{1}{2\pi}[\arg(-x + i\epsilon) - \arg(-x - i\epsilon)], \quad x \in R, \quad \epsilon \in (0,1) \tag{1.45}$$

where, as usually,

$$\arg(x+iy) = \begin{cases} \mathrm{arctg}(y/x), & x, y > 0, \quad x^2+y^2 \neq 0, \\ \pi + \mathrm{arctg}|y/x|, & x \leq 0, \ y \geq 0, \ x^2+y^2 \neq 0, \\ -\pi + \mathrm{arctg}(y/x), & x \leq 0, \ y < 0, \\ \mathrm{arctg}|y/x|, & x \geq 0, \ y \leq 0, \ x^2+y^2 \neq 0, \end{cases} \tag{1.46}$$

and $\mathrm{arctg}(\pm\infty) = \pm\pi/2$. Notice [10] that:

$$H(\cdot,\epsilon) \xrightarrow[\epsilon \to 0]{\mathfrak{L}'} H \in \mathfrak{L}', \tag{1.47}$$

where H is the Heavyside distribution. The asymptotic extension $H_{as}(a,b)$ exists for all $a \in A$ and all $b \in (0,1)_{as}$. The values of the asymptotic function (for fixed $b = s + o^1$):

$$H(a) \stackrel{def.}{=} H_{as}(a, s + o^1), \qquad a \in \Omega \tag{1.48}$$

is given by:

$$H(a)=\begin{cases} 1-\frac{s}{\pi x}+o^1, & a=x+h,\quad x\in R,\quad x>0,\quad h\in\Omega_0,\\ 1-\frac{1}{\pi}\operatorname{arctg}(1/x), & a=xs+sh,\ x\in R,\ x>0,\ h\in\Omega_0,\\ \frac{1}{2}+o^{\circ}, & a=s^n x+s^n h, x\in R, x\neq 0, h\in\Omega_0, n=2,3,\dots\\ o^{-1}, & a=o^{\nu},\ \ \nu=0,1,\dots,\infty,\\ -\frac{s}{\pi x}+o^{1}, & a=x+h,\ x\in R,\ x<0,\ h\in\Omega_0,\\ -\frac{1}{\pi}\operatorname{arctg}(1/x)+o^{\circ}, & a=sx+sh,\ x\in R,\ x<0,\ h\in\Omega_0. \end{cases} \tag{1.49}$$

The power $\mu(a)$ and the order $\nu(a)$ of $H(a)$ is given by:

$$\mu(a)=\begin{cases} 1 & a=x+h,\ x\in R,\ x<0,\ h\in\Omega_0,\\ -1, & a=o^{\nu},\ \ \nu=0,1,\dots,\infty,\\ 0, & \text{for all other } a\in\Omega; \end{cases} \tag{1.50}$$

$$\nu(a)=\begin{cases} 1, & a=x+h,\ x\in R,\ x\neq 0,\ h\in\Omega,\\ -1, & a=o^{\nu},\ \ \nu=0,1,\dots,\infty,\\ 0, & \text{for all other } a\in\Omega. \end{cases} \tag{1.51}$$

We shall see in future that the asymptotic function just defined is a realization (in the sense of ref.[7] Sec. 9) of the H-distribution.

2. THE SET OF THE EXTENDED ASYMPTOTIC FUNCTIONS

The extended asymptotic functions introduced in the previous paper[8] offer us interesting examples of asymptotic functions (Sec. 1). We will completely persuade the reader in the next pages of this paper (or in a next one, maybe). But the collection of these functions is not a good collection because it is not closed with respect to the addition and multiplication. The example given in (1.1):

$$\left(\frac{1}{x}\right)_{as}(a)=\frac{1}{a}, \qquad a\in A\setminus\Theta \tag{2.1}$$

can illustrate that defect. Indeed,

$$\left(\frac{1}{x}-\frac{1}{x}\right)_{as}(a)=(o)_{as}(a)=0, \qquad a\in A\setminus\Theta \tag{2.2}$$

in contrast to

$$\frac{1}{a}-\frac{1}{a}=o^{\nu(a)}, \qquad a\in A\setminus\Theta, \tag{2.3}$$

where $\nu(a)$ is given in (1.11). In other words,

$$\left(\frac{1}{x}-\frac{1}{x}\right)_{as}(a)\neq\left(\frac{1}{x}\right)_{as}(a)-\left(\frac{1}{x}\right)_{as}(a), \qquad a\in A\setminus\Theta. \tag{2.4}$$

What is more, the asymptotic function (2.3) is not an extended asymptotic function, i.e., there does not exist an ordinary function ϕ such that

$$o^{\nu(a)}=\phi_{as}(a), \qquad a\in A\setminus\Theta.$$

In other words, the set of all extended asymptotic functions is not closed with respect to the addition (and subtraction, of course). This set is not closed under the multiplication, too. Indeed, we can verify that

$$\delta^{n}(a)=(\Lambda^{n})_{as}(a, s+o^{1}), \qquad \text{for } a\in\Omega,\ a\neq o^{\circ} \tag{2.5}$$

but

$$o^{-2n}=\delta^{n}(a)\neq(\Lambda^{n})_{as}(a, s+o)=o^{-n-1}, \text{ for } a=o^{\circ}, \tag{2.6}$$

where Λ is the delta-sequence given in Example (1.19), Λ^n is its n-th power and δ is the asymptotic function given by (1.24). Strictly speaking,

$$\delta^{n}(a)\neq(\Lambda^{n})_{as}(a, s+o^{1}), \tag{2.7}$$

on Ω and what is more, δ^n is not an extended asymptotic function, i.e., it is not an asymptotic extension of any ordinary function. The question arises: What is the difference between $(\phi\pm\psi)_{as}$ and $\phi_{as}\pm\psi_{as}$, between $(\phi,\psi)_{as}$ and ϕ_{as},ψ_{as}, and between $(\phi/\psi)_{as}$ and ϕ_{as}/ψ_{as} for any two ordinary functions ϕ and ψ. The answer is: The difference is always Θ-valued function, i.e., a function of the type

$$o^{\nu(a)}, \qquad a\in A \tag{2.8}$$

(where ν is a mapping of the type $\nu: A\to Z\cup\{\infty\}$). The following theorem deals with this problem.

(2.9) THEOREM: Let X and Y be two open subsets of R and let $X\cap Y\neq 0$. Let ϕ and ψ be two continuous ordinary functions defined on X and Y, respectively, and let, finally, $\phi_{as}(a), a\in D$ and $\psi_{as}(a)$, $a\in E$ be the asymptotic extensions of ϕ and ψ, respectively. Then:

$$\phi_{as}(a) \pm \psi_{as}(a) = (\phi \pm \psi)_{as}(a) + o^{\nu(a)}, \qquad a \in D \cap E, \qquad (2.10)$$

$$\phi_{as}(a) \cdot \psi_{as}(a) = (\phi, \psi)_{as}(a) + o^{\nu_o(a)}, \qquad a \in D \cap E, \qquad (2.11)$$

$$\phi_{as}(a) / \psi_{as}(a) = (\phi/\psi)_{as}(a) + o^{\bar{\nu}(a)}, \qquad a \in D \cap E \setminus E_0, \qquad (2.12)$$

where $\nu(a)$, $\nu_0(a)$ and $\bar{\nu}(a)$ are the orders of $\phi_{as}(a) \pm \psi_{as}(a)$, $\phi_{as}(a) \cdot \psi_{as}(a)$ and $\phi_{ss}(a)/\psi_{as}(a)$, respectively, and E_0 is the set of all a from E such that $\psi_{as}(a) \in \Theta$.

<u>(2.13) REMARK:</u> As usually, $\phi \pm \psi$ and ϕ, ψ are defined on $X \cap Y$ and ϕ/ψ is defined on $X \cap Y \setminus Y_0$, where

$$Y_0 = \{x : x \in Y,\ \psi(x) = 0\}.$$

On the other hand, $\phi_{as} \pm \psi_{as}$ and $\phi_{as} \cdot \psi_{as}$ are defined on $D \cap E$ and ϕ_{as}/ψ_{as} is defined on $D \cap E \setminus E_0$, corresponding to ref. /8/ Definition (1.7). Moreover, $D \cap E \subseteq (X \cap Y)_{as}$ as corresponding to ref. /8/ Theorem (2.24)). So, the statement of the above theorem is correctly formulated.

<u>(2.14) REMARK:</u> Corresponding to ref. /5/ Theorem 14, (2.10), (2.11) and (2.12) are equivalent to

$$\phi_{as}(a) \pm \psi_{as}(a) - (\phi \pm \psi)_{as}(a) = o^{\nu(a)}, \qquad (2.15)$$

$$\phi_{as}(a) \cdot \psi_{as}(a) - (\phi \cdot \psi)_{as}(a) = o^{\nu_o(a)}, \qquad (2.16)$$

$$\phi_{as}(a)/\psi_{as}(a) - (\phi/\psi)_{as}(a) = o^{\bar{\nu}(a)}, \qquad (2.17)$$

respectively.

<u>(2.18) REMARK:</u> The theorem holds, of course, not only for functions of one variable but also for functions of many variables. For the sake of convenience we shall continue to consider the case $n = 1$ only, corresponding to our agreement in ref. /8/ (3.6).

<u>PROOF:</u> Let $a \in D \cap E$. Corresponding to ref. /8/ Definition (3.1)

$$\phi_{as}(a) = \mathrm{as}\{\phi(\alpha) : \alpha \in a\}, \qquad (2.19)$$

$$\psi_{as}(a) = \mathrm{as}\{\psi(\alpha) : \alpha \in a\}, \qquad (2.20)$$

$$(\phi + \psi)_{as}(a) = \mathrm{as}\{\phi(\alpha) + \psi(\alpha) : \alpha \in a\}, \qquad (2.21)$$

$$\phi_{as}(a) + \psi_{as}(a) = \{\gamma + \delta : \gamma \in \phi_{as}(a),\ \delta \in \psi_{as}(a)\}, \qquad (2.22)$$

where (2.22) expresses the addition of the values of $\phi_{as}(a)$

and $\psi_{as}(a)$ respectively, corresponding to ref. /5/ Definition 6. Consequently, we have

$$(\phi + \psi)_{as}(a) \subseteq \phi_{as}(a) + \psi_{as}(a). \qquad (2.23)$$

Bearing in mind ref. /5/ Theorem 3, we obtain

$$\phi_{as}(a) + \psi_{as}(a) - (\phi + \psi)_{as}(a) = o^{\nu(a)}, \qquad (2.24)$$

where $\nu(a)$ is the order of the right-hand side of (2.23). But (2.24) implies (2.10). The case of multiplication and division can be treated in the same way. Of course (2.22) must be replaced by:

$$\phi_{as}(a) \cdot \psi_{as}(a) = \{\gamma \cdot \delta : \gamma \in \phi_{as}(a),\ \delta \in \psi_{as}(a)\}, \qquad (2.25)$$

$$\phi_{as}(a)/\psi_{as}(a) = \{\gamma/\delta : \gamma \in \phi_{as},\ \delta \in \psi_{as}(a)\}, \qquad (2.26)$$

respectively (ref. /5/ Definition 6). The proof is finished.

<u>(2.27) COROLLARY:</u> (Transfer Principle). Let $F(z_1, \ldots, z_n)$, $z_k \in C$, $k = 1, \ldots, n$ be a rational complex-valued function of n complex variables and let $\phi_k(x)$, $x \in X$, $k = 1, \ldots, n$ be complex-valued functions defined on the open subset X of R such that

$$F[\phi_1(x), \ldots, \phi_n(x)] = 0, \qquad x \in X. \qquad (2.28)$$

Then

$$F_{as}[\phi_{1as}(a), \ldots, \phi_{nas}(a)] \in \Theta, \quad a \in X_{as}. \qquad (2.29)$$

<u>(2.30) EXAMPLE:</u> Let $F(x,y) = x^2 + y^2 - 1$ and $\phi_1(x) = \sin x$, $x \in R$, $\phi_2(x) = \cos x$, $x \in R$. We obtain

$$\sin^2 a + \cos^2 a - 1 = o^{\nu_a}, \qquad a \in A, \qquad (2.31)$$

where ν_a is the order of a. But (2.31) is equivalent to the identity

$$\sin^2 a + \cos^2 a = 1^{\nu_a}, \qquad a \in A \qquad (2.32)$$

which is the analogue of $\sin^2 x + \cos^2 x = 1, x \in R$ in A. Why did we call the above corollary "Transfer Principle?" Because with its help we can transfer some identities from R to A (as well as from C to A^*) just we did in Example (2.30).

(2.33) REMARK: (Our Plan for the Next Paper): By means of Theorem (2.9) we are going to "close" the set of the extended asymptotic functions with respect to the addition and multiplication. More strictly, we shall introduce another class F of asymptotic functions called "quasi-extended asymptotic functions" with the following properties: (i) F contains all extended asymptotic functions (in particular, all examples given in Sec. 1); (ii) F is close with respect to the addition and multiplication; (iii) The analytic operations (differentiation, integration and so on) can be naturally defined in F; (iv) The Scwartz distributions have realizations in F in a certain sense (see ref. [7] Sec. 9).

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